Axisymmetric Motion of Two Spherical Particles with Slip Surfaces

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The slow motion of two rigid spherical particles along the line through their centers in an unbounded viscous fluid is considered. The fluid is allowed to slip at the surfaces of the particles. Also, the particles may differ in radius, in slip coefficient, and in migration velocity (or in applied force). Using spherical bipolar coordinates, the creeping flow equations are solved in the quasi-steady situation, and the interaction effects between the particles are evaluated for various cases. The interaction between particles is found to be more significant when the slip coefficients at the particle surfaces become smaller. In general, the influence of the interaction on the smaller particle is stronger than on the larger one. The creeping motion of a spherical particle in the direction perpendicular to a plane wall is also studied for the case in which the solid–fluid interfaces may have different slip coefficients. The retarding effect of the plane wall on the motion of the particle can be very significant when the surface-to-surface distance gets close to zero. Our results for the particle–particle and particle–wall interaction parameters at any separation distance agree very well with the existing solutions for the limiting situations of no slip and perfect slip at the solid–fluid interfaces. © 1995 Academic Press, Inc.

Key Words: aerosol spheres; particle interactions; boundary effects.

INTRODUCTION

The application of hydrodynamic theory to the behavior of solid particles or fluid drops moving in a continuous medium at very small Reynolds numbers has continued to receive much attention from investigators in the fields of chemical, biochemical, and environmental engineering and science. The majority of these particle or drop phenomena are fundamental in nature, but permit one to develop rational understanding of many practical systems and industrial processes such as sedimentation, flotation, spray drying, agglomeration, and the motion of blood cells in an artery or vein.

The theoretical treatment of this subject has grown out of the work of Stokes (1) for a translating rigid sphere in a viscous fluid. Hadamard (2) and Rybczynski (3) extended independently this result to the translation of a fluid sphere.

Assuming continuous velocity and continuous tangential shearing stress across the interface of fluid phases, they found that the force exerted on the spherical drop of radius a by the surrounding fluid of viscosity η is

$$\mathbf{F}^{(0)} = -6\pi\eta a \frac{3\eta^* + 2}{3\eta^* + 3} \mathbf{U}. \quad [1]$$

Here $\mathbf{U}$ is the migration velocity of the drop and η^* is the internal-to-external viscosity ratio. Since the fluid properties are arbitrary, Eq. [1] degenerates to the case of motion of a solid sphere (Stokes' law) when $\eta^* \rightarrow \infty$ and to the case of motion of a gas bubble in the limit $\eta^* \rightarrow 0$.

In most practical applications, particles or drops are not isolated. So, it is important to determine if the presence of neighboring particles and/or boundaries significantly affects the movement of particles. Problems of the hydrodynamic interactions between two or more particles and between particles and boundaries have been treated extensively in the past. Summaries for the current state of knowledge in this area and some informative references can be found in Kim and Karrila (4) and Keh and Tseng (5).

When one tries to solve the Navier–Stokes equations, it is usually assumed that no slippage arises at the solid–fluid interfaces. Actually, this is an idealization of the transport processes occurring. That the adjacent fluid (especially if the fluid is a rarefied gas) can slip over a solid surface has been confirmed, both experimentally and theoretically (6). Presumably any such slipping would be proportional to the local velocity gradient next to the solid surface (7), at least so long as this gradient is small. The constant of proportionality, $1/\beta$, may be termed a “slip coefficient.” Basset (8) derived the following expression for the force acting on a translating rigid sphere with a slip–flow boundary condition at its surface (e.g., a sedimenting aerosol sphere):

$$\mathbf{F}^{(0)} = -6\pi\eta a \frac{\beta a + 2\eta}{\beta a + 3\eta} \mathbf{U}. \quad [2]$$

In the particular case of $\beta \rightarrow \infty$, there is no slip at the particle surface and Eq. [2] degenerates to Stokes' law. When $\beta =$

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0, there is a perfect slip at the particle surface and Eq. [2] is consistent with Eq. [1] (taking $\eta^* = 0$). Note that, as can be seen from Eqs. [1] and [2], the flow field caused by the migration of a "slip" solid sphere is the same as the external flow field generated by the same motion of a fluid drop with a value of η^* equal to $\beta a/3\eta$.

The slip coefficient has been determined experimentally and found to agree with the general kinetic theory of gases. It can be calculated from the formula

$$\frac{\eta}{\beta} = C_m l, \quad [3]$$

where l is the mean free path of a gas molecule and C_m is a dimensionless constant related to the momentum accommodation coefficient at the solid surface. Although C_m surely depends upon the nature of the surface, examination of the experimental data suggests that it will be in the range 1.14–1.5 (9, 10). Note that the slip-flow boundary condition is not only applicable in the continuum regime ($l/a \ll 1$), but also appears to be valid for some cases even into the molecular flow regime ($l/a \geq 1$).

The interaction between two solid particles with finite values of $\beta a/\eta$ is different, both physically and mathematically, from that between two fluid drops of finite viscosities. Through an exact representation in spherical bipolar coordinates, Davis (9) and Reed and Morrison (11) examined the creeping motion of two rigid spheres with slip surfaces along their line of centers with equal velocities and with equal and opposite velocities. Numerical results to correct the Basset equation [2] were calculated by Reed and Morrison for the case of equal radii and equal slip coefficients, which disagree with some of Davis' calculations. On the other hand, Goren (12) and Reed and Morrison (11) investigated the slow motion of a rigid sphere normal to an infinite plane wall, where the fluid may slip at the solid surfaces. The numerical results obtained by Reed and Morrison for the case of equal slip coefficients disagree partially with those presented by Goren due to the smaller number of terms retained in the numerical analysis.

The object of this paper is to obtain an exact solution to the quasi-steady problem of axisymmetric motion of two rigid spherical particles with slip surfaces. The particles may differ in radius, in slip coefficient, and in translational velocity (or in applied force). The slow motion of a spherical particle in the direction normal to a plane wall is also investigated for the case where the solid-fluid interfaces may have different slip coefficients. The creeping flow equations applicable to the systems are solved by using bipolar coordinates, and the fluid streamlines for various cases are presented. Our numerical results for the particle-particle and particle-wall interaction parameters compare favorably with the existing solutions for the limiting cases of no slip and perfect slip at the

solid-fluid interfaces and agree with those obtained by Reed and Morrison (11) in the situations considered by them.

ANALYSIS FOR THE MOTION OF TWO SPHERES

We consider the slow motion of two spherical particles along their line of centers in an unbounded quiescent fluid, which is Newtonian and incompressible. The particles will maintain their spherical shape against the fluid stresses, and the fluid may slip at the surfaces of the particles. Also, the particles are allowed to be different in radius, in slip coefficient, and in migration velocity (or in applied force). Our purpose here is to determine the correction to the Basset equation [2] for each particle due to the presence of the other in the flow field.

For convenience in satisfying the boundary conditions at the particle-fluid interfaces, an orthogonal curvilinear coordinate system (ψ, ξ, ϕ) , known as spherical bipolar coordinates (shown in Fig. 1), is utilized to solve this problem. This coordinate system is related to cylindrical coordinates (z, ρ, ϕ) in any meridian plane $\phi = \text{constant}$ by the relation (7, 13)

$$\rho = \frac{c \sin \psi}{\cosh \xi - \cos \psi} \quad [4a]$$

and

$$z = \frac{c \sinh \xi}{\cosh \xi - \cos \psi}, \quad [4b]$$

where c is a characteristic length in the bipolar coordinate system which is positive. The coordinate surfaces $\xi = \text{con-}$

FIG. 1. Geometrical sketch of spherical bipolar coordinates.

stant correspond to a family of nonintersecting spheres whose centers lie along the z -axis. The value $\xi = 0$ is a sphere of infinite radius and equivalent to the entire plane $z = 0$. Two spheres external to each other are chosen to be $\xi = \xi_1$ (with $\xi_1 > 0$) and $\xi = \xi_2$ (with $\xi_2 < 0$), and the sphere radii a_1 and a_2 as well as the distances of their centers from the origin d_1 and d_2 are given by

$$a_i = c \operatorname{cosech} |\xi_i| \quad [5]$$

and

$$d_i = c \coth |\xi_i|, \quad [6]$$

for $i = 1$ or 2 . The center-to-center distance between the particles, r_{12} , equals $(d_1 + d_2)$.

Fluid Velocity Distribution

When the migration velocities of two spheres are not identical, the transport of momentum is inherently unsteady. However, the problem can be considered quasi-steady if Reynolds numbers are small. The fluid velocity field is governed by the fourth-order differential equation for viscous axisymmetric flows,

$$E^4 \Psi = E^2(E^2 \Psi) = 0, \quad [7]$$

where Ψ is the Stokes stream function for the fluid. The operator E^2 assumes the following form in bipolar coordinates:

$$E^2 = \frac{\cosh \xi - \mu}{c^2} \left\{ \frac{\partial}{\partial \xi} \left[(\cosh \xi - \mu) \frac{\partial}{\partial \xi} \right] + (1 - \mu^2) \frac{\partial}{\partial \mu} \left[(\cosh \xi - \mu) \frac{\partial}{\partial \mu} \right] \right\}. \quad [8]$$

Here, we have put $\mu = \cos \psi$ for brevity. The stream function Ψ is related to the components of velocity field $\mathbf{v}$ by

$$v_\xi = - \frac{(\cosh \xi - \mu)^2}{c^2} \frac{\partial \Psi}{\partial \mu} \quad [9a]$$

and

$$v_\psi = - \frac{(\cosh \xi - \mu)^2}{c^2(1 - \mu^2)^{1/2}} \frac{\partial \Psi}{\partial \xi}. \quad [9b]$$

Since the tangential velocity of the fluid relative to each sphere at a point on its surface is proportional to the tangential stress prevailing at that point and the fluid is motionless far from the particles, the boundary conditions for the velocity field are

$$\xi = \xi_i: \quad \mathbf{v} = \mathbf{U}_i - \frac{\xi_i}{|\xi_i|} \frac{1}{\beta_i} (1 - \mathbf{e}_\xi \mathbf{e}_\xi) \mathbf{e}_\xi: \tau, \quad [10a]$$

$$(\rho^2 + z^2)^{1/2} \rightarrow \infty: \quad \mathbf{v} \rightarrow \mathbf{0}, \quad [10b]$$

for $i = 1$ or 2 . Here, $\tau (= \eta[\nabla \mathbf{v} + (\nabla \mathbf{v})^T])$ is the viscous stress tensor for the fluid; $\mathbf{e}_\xi$ is the unit vector in the ξ direction; $\mathbf{I}$ is the unit dyadic; $1/\beta_1$ and $1/\beta_2$ are the slip coefficients about the two particle-fluid interfaces; and $\mathbf{U}_1 (= U_1 \mathbf{e}_z)$, where $\mathbf{e}_z$ is the unit vector in the z direction) and $\mathbf{U}_2 (= U_2 \mathbf{e}_z)$ are the instantaneous translational velocities of the two particles to be determined.

A general solution of Eq. [7] satisfying the boundary condition [10b] is (7, 14)

$$\begin{aligned} \Psi = c^2 (\cosh \xi - \mu)^{-3/2} \sum_{n=1}^{\infty} & \left[A_n \cosh \left(n - \frac{1}{2} \right) \xi \right. \\ & + B_n \sinh \left(n - \frac{1}{2} \right) \xi + C_n \cosh \left(n + \frac{3}{2} \right) \xi \\ & \left. + D_n \sinh \left(n + \frac{3}{2} \right) \xi \right] G_{n+1}^{-1/2}(\mu). \quad [11] \end{aligned}$$

Here, $G_{n+1}^{-1/2}(\mu)$ is the Gegenbauer polynomial of order $n+1$ and degree $-\frac{1}{2}$, which is related to Legendre polynomials by

$$G_{n+1}^{-1/2}(\mu) = \frac{P_{n-1}(\mu) - P_{n+1}(\mu)}{2n+1}. \quad [12]$$

The unknown coefficients A_n , B_n , C_n , and D_n are to be determined from boundary conditions [10a] using the recurrence relations of the Legendre polynomials and utilizing the expansion, which can be derived using the generating function of the Legendre polynomials,

$$\begin{aligned} & \frac{\cosh \xi}{(\cosh \xi - \mu)^{1/2}} - \frac{\sinh^2 \xi}{(\cosh \xi - \mu)^{3/2}} \\ & = \sqrt{2} \sum_{n=0}^{\infty} \exp \left[- \left(n + \frac{1}{2} \right) |\xi| \right] \\ & \quad \times [\cosh \xi - (2n+1) \sinh |\xi|] P_n(\mu). \quad [13] \end{aligned}$$

The procedure is straightforward but tedious, and the results, which consist of four linear algebraic recursion formulas, are

$$\begin{aligned} & A_n \cosh \left(n - \frac{1}{2} \right) \xi_i + B_n \sinh \left(n - \frac{1}{2} \right) \xi_i + C_n \cosh \left(n + \frac{3}{2} \right) \xi_i \\ & + D_n \sinh \left(n + \frac{3}{2} \right) \xi_i = - \frac{\sqrt{2}}{2} U_i n(n+1) \\ & \quad \times \left\{ \frac{\exp[-(n-\frac{1}{2})|\xi_i|]}{2n-1} - \frac{\exp[-(n+\frac{3}{2})|\xi_i|]}{2n+3} \right\}, \quad [14a] \end{aligned}$$

$$\begin{aligned}
& \frac{n}{2n+3} \left\{ \left(n + \frac{1}{2} \right)^2 \left[A_{n+1} \cosh \left(n + \frac{1}{2} \right) \xi_i \right. \right. \\
& \quad \left. \left. + B_{n+1} \sinh \left(n + \frac{1}{2} \right) \xi_i \right] + \left(n + \frac{5}{2} \right)^2 \right. \\
& \quad \left. \times \left[C_{n+1} \cosh \left(n + \frac{5}{2} \right) \xi_i + D_{n+1} \sinh \left(n + \frac{5}{2} \right) \xi_i \right] \right\} \\
& - \cosh \xi_i \left\{ \left(n - \frac{1}{2} \right)^2 \left[A_n \cosh \left(n - \frac{1}{2} \right) \xi_i \right. \right. \\
& \quad \left. \left. + B_n \sinh \left(n - \frac{1}{2} \right) \xi_i \right] + \left(n + \frac{3}{2} \right)^2 \left[C_n \cosh \left(n + \frac{3}{2} \right) \xi_i \right. \right. \\
& \quad \left. \left. + D_n \sinh \left(n + \frac{3}{2} \right) \xi_i \right] \right\} - \frac{\xi_i}{|\xi_i|} \frac{c\beta_i}{\eta} \left\{ \left(n - \frac{1}{2} \right) \right. \\
& \quad \times \left[A_n \sinh \left(n - \frac{1}{2} \right) \xi_i + B_n \cosh \left(n - \frac{1}{2} \right) \xi_i \right] \\
& \quad \left. + \left(n + \frac{3}{2} \right) \left[C_n \sinh \left(n + \frac{3}{2} \right) \xi_i + D_n \cosh \left(n + \frac{3}{2} \right) \xi_i \right] \right\} \\
& + \frac{n+1}{2n-1} \left\{ \left(n - \frac{3}{2} \right)^2 \left[A_{n-1} \cosh \left(n - \frac{3}{2} \right) \xi_i \right. \right. \\
& \quad \left. \left. + B_{n-1} \sinh \left(n - \frac{3}{2} \right) \xi_i \right] + \left(n + \frac{1}{2} \right)^2 \left[C_{n-1} \cosh \left(n + \frac{1}{2} \right) \xi_i \right. \right. \\
& \quad \left. \left. + D_{n-1} \sinh \left(n + \frac{1}{2} \right) \xi_i \right] \right\} = \frac{\sqrt{2}}{4} U_i n(n+1) \\
& \times \left(\frac{3}{2} \sinh |\xi_i| - \frac{c\beta_i}{\eta} \right) \left\{ \exp \left[- \left(n - \frac{1}{2} \right) |\xi_i| \right] \right. \\
& \quad \left. - \exp \left[- \left(n + \frac{3}{2} \right) |\xi_i| \right] \right\} - \frac{\sqrt{2}}{4} U_i n(n+1) \cosh \xi_i \\
& \times \left\{ \frac{\exp \left[- \left(n - \frac{1}{2} \right) |\xi_i| \right]}{2n-1} - \frac{\exp \left[- \left(n + \frac{3}{2} \right) |\xi_i| \right]}{2n+3} \right\}, \quad [14b]
\end{aligned}$$

where $i = 1$ or 2 . Because the coefficients A_n , B_n , C_n , and D_n become small with large n , simultaneous solution of these four recursion equations for the first m sets yields $4m$ coefficients, thereby determining the Stokes stream function for the fluid according to Eq. [11].

The Resistance Problem

The resistance problem is defined as that in which the forces acting on the particles are to be determined for a specified particle's motion in the ambient fluid. The drag forces exerted by the fluid on the two spheres can be calculated by the formulas (14)

$$F_1 = -2\sqrt{2} \pi \eta c \sum_{n=1}^{\infty} (A_n + B_n + C_n + D_n) \quad [15a]$$

and

$$F_2 = -2\sqrt{2} \pi \eta c \sum_{n=1}^{\infty} (A_n - B_n + C_n - D_n). \quad [15b]$$

The result of drag forces can be written as

$$F_i = \sum_{j=1}^2 K_{ij} F_j^{(0)}, \quad i = 1 \text{ or } 2, \quad [16]$$

with

$$F_j^{(0)} = -\frac{1}{m_j} U_j, \quad [17]$$

which is the drag force on particle j in the absence of the other particle. Obviously, from Eq. [2],

$$m_j = \frac{1}{6\pi\eta a_j} \frac{\beta_j a_j + 3\eta}{\beta_j a_j + 2\eta}. \quad [18]$$

The dimensionless resistance parameters K_{ij} are functions of the slip coefficients, sizes, and separation distance of the particles. When the two spheres are separated by an infinite distance,

$$K_{11} = K_{22} = 1 \quad [19a]$$

and

$$K_{12} = K_{21} = 0. \quad [19b]$$

The Mobility Problem

The mobility problem is defined as that in which the forces acting on the particles are prescribed and the particles' motion in the surrounding fluid is to be determined. The presentation of the mobility problem is rather awkward since the boundary conditions involve the unknowns, but in many physical problems the forces are the prescribed quantities and the particles must move accordingly. In a mobility problem, Eq. [15] still gives the drag forces exerted by the fluid on the two spheres, which are equal in magnitude but opposite in direction to the corresponding applied forces on the particles in the quasi-steady limit. However, the coefficients A_n , B_n , C_n , and D_n obtained from Eq. [14] contain the unknown translational velocities of the two particles. These unknown velocities can be determined by solving Eq. [15] for given values of F_1 and F_2 .

The result of the particles' motion can be expressed as

$$U_i = \sum_{j=1}^2 M_{ij} U_j^{(0)}, \quad i = 1 \text{ or } 2, \quad [20]$$

FIG. 2. Streamlines for the motion of two identical spheres along the line through their centers with $\beta a/\eta = 1$ and $2a/r_{12} = 0.5$: (a) the case of $U_1 = U_2$, (b) the case of $U_1 = -U_2$, (c) the case of $U_2 = 0$, and (d) the case of $U_1 = U_2$ in a reference frame traveling with the particles. The value on each streamline indicates its dimensionless stream function, $\Psi/c^2 U_1$.

with

$$U_j^{(0)} = -m_j F_j, \quad [21]$$

which is the translational velocity of particle j subject to an applied force $-F_j$ in the absence of the other particle. Similar to the resistance parameters K_{ij} , the dimensionless mobility parameters M_{ij} are functions of the slip coefficients, sizes, and separation distance of the particles as well. Also, when the two particles are separated by an infinite distance,

$$M_{11} = M_{22} = 1 \quad [22a]$$

and

$$M_{12} = M_{21} = 0. \quad [22b]$$

Using Eqs. [16], [17], [20], and [21], it can be shown that the following relation between the dimensionless resistance and mobility parameters always holds for two arbitrary spheres:

$$K_{11} = [M_{11} - M_{12}M_{21}/M_{22}]^{-1}, \quad [23a]$$

$$K_{12} = \frac{m_2}{m_1} [M_{21} - M_{11}M_{22}/M_{12}]^{-1}, \quad [23b]$$

or

$$M_{11} = [K_{11} - K_{12}K_{21}/K_{22}]^{-1}, \quad [24a]$$

$$M_{12} = \frac{m_1}{m_2} [K_{21} - K_{11}K_{22}/K_{12}]^{-1}. \quad [24b]$$

The corresponding expressions for K_{22} , K_{21} , M_{22} , and M_{21} can be obtained from the above formulas by interchanging subscripts 1 and 2. Thus, one can solve either the resistance problem or the mobility problem first and then use relation [23] or [24] to obtain the solution for the other.

RESULTS FOR MOTION OF TWO SPHERES

The coefficients A_n , B_n , C_n , and D_n of the stream function given by Eq. [11] have been calculated for various values of $\beta_1 a_1/\eta$, $\beta_2 a_2/\eta$, a_2/a_1 , and $(a_1 + a_2)/r_{12}$ using a DEC VAX 6520 digital computer. For the difficult case of $a_2/a_1 = 10$ and $(a_1 + a_2)/r_{12} = 0.9999$, m equal to about 2000 was sufficiently large so that the $(m + 1)$ th terms of these coefficients are negligible and increases in m do not alter the calculated values appreciably.

Streamlines

The distortion of the fluid flow due to interactions between two spherical particles is illustrated graphically in Figs. 2 and 3. In each case, "contour-type" streamlines in the meridian section are depicted. Figure 2 shows the situation of two identical spheres ($\beta_1 = \beta_2 = \beta$ and $a_1 = a_2 = a$) when the distance between their surfaces is equal to the sum of their radii ($2a/r_{12} = 0.5$) and $\beta a/\eta = 1$. Streamlines for the system in which the particles are traveling in the same direction with zero relative velocity (exerted by the same force) are shown in Fig. 2a, while a similar drawing for the case where the particles are approaching with equal velocities (exerted by forces equal in magnitude but opposite in direction) is given in Fig. 2b. Since the inertial effects in the equation of fluid motion have been neglected, the fore-and-aft symmetry can be observed in these drawings. Figure 2c displays streamlines for the case of a translating particle approaching

a stationary particle. The fore-and-aft symmetry is destroyed in this case. Also, the streamline pattern corresponding to the case considered in Fig. 2a in a reference frame riding with the particles is depicted in Fig. 2d. In this reference frame, the two particles are stationary and the fluid streams past them in the opposite direction, with the stream function including a uniform stream in addition to that given by Eq. [11] for Fig. 2a. Again, the fore-and-aft symmetry is obtained in Fig. 2d. When the parameter $\beta a/\eta$ increases (decreases), the curvature of each corresponding streamline shown in Fig. 2 will be smaller (larger) and the spacing between two corresponding streamlines will be narrower (broader).

The streamlines for two situations of different spheres with $(a_1 + a_2)/r_{12} = 0.5$ are drawn in Fig. 3. Figure 3a displays the features of the fluid flow when the two spheres having identical radii ($a_1 = a_2 = a$) but unequal slip coefficients, with $\beta_1 \rightarrow \infty$ and $\beta_2 = 0$, are approaching with equal velocities. It can be seen that the two balanced recirculation regions illustrated in Fig. 2b are distorted if the two equal-sized spheres differ in the slip coefficient. The fluid recirculation generated by the particle with a smaller slip coefficient (or larger $\beta a/\eta$) is stronger. The situation of two spheres with the same slip coefficient ($\beta_1 = \beta_2 = \beta$) but unequal radii is illustrated in Fig. 3b, which corresponds to the case of $a_2/a_1 = 2$, $F_2/F_1 = 8$, and $\beta a_1/\eta = 1$. It can be seen that the fluid flow field is dominated by the larger particle and the greater interaction effect is experienced by the smaller particle.

Mobility Parameters

A number of numerical solutions of the mobility parameters M_{11} , M_{12} , M_{21} , and M_{22} , defined by Eq. [20], for the motion of two spheres along their line of centers with various values of $\beta_1 a_1/\eta$, $\beta_2 a_2/\eta$, a_2/a_1 , and $(a_1 + a_2)/r_{12}$ are presented in Tables 1–3. All of the listed results converge to at least five significant figures. As expected, these results illustrate that the particles' interaction (indicated by the parameters M_{12} and M_{21}) decreases rapidly, for all values of $\beta_1 a_1/\eta$, $\beta_2 a_2/\eta$, and a_2/a_1 , with increasing gap thickness between them (i.e., decreasing $(a_1 + a_2)/r_{12}$). However, the interaction between particles can be very strong when the surface-to-surface spacing gets close to zero.

Numerical results of the mobility parameters for the translation of two identical spheres are given in Table 1. For this symmetric case, one has $M_{11} = M_{22}$, $K_{11} = K_{22}$, $M_{12} = M_{21}$, $K_{12} = K_{21}$, $(K_{11} + K_{12}) = (M_{11} + M_{12})^{-1}$, and $(K_{11} - K_{12}) = (M_{11} - M_{12})^{-1}$. It can be seen that the interaction effect between particles will be more significant when the slip coefficient at the particle surfaces becomes smaller (or when the value of $\beta a/\eta$ becomes larger). Reed and Morrison (11) calculated the values of $(K_{11} + K_{12})$ for the translation of two identical spheres with equal velocities and of $(K_{11} - K_{12})$ for the motion of two identical spheres with equal and opposite velocities. Our solutions in Table 1 are in excellent

FIG. 3. Streamlines for the motion of two spheres along their line of centers with $(a_1 + a_2)/r_{12} = 0.5$: (a) the case of $a_1 = a_2 = a$, $\beta_1 \rightarrow \infty$, $\beta_2 \rightarrow 0$, and $U_1 = -U_2$; and (b) the case of $\beta_1 = \beta_2 = \beta$, $a_2/a_1 = 2$, $F_2/F_1 = 8$, and $\beta a_1/\eta = 1$. The value on each streamline indicates its dimensionless stream function, $\Psi/c^2 U_1$.

agreement with their calculations. The results in Table 1 also include the interaction effects between two no-slip solid spheres ($\beta \rightarrow \infty$) and between two perfect-slip gas bubbles ($\beta = 0$), which can be found to agree very well with the results obtained by Keh and Tseng (5) for the motion of two fluid spheres (taking $\eta^* \rightarrow \infty$ or $\eta^* = 0$) using a boundary collocation method. Note that the interaction between two slip spheres with a finite value of $\beta a/\eta$ is different (having larger values in $M_{11} + M_{12}$ and $M_{11} - M_{12}$ at the same separation parameter $2a/r_{12}$) from that between two liquid drops with a value of $3\eta^*$ equal to $\beta a/\eta$, although the flow field induced by an isolated slip sphere is equivalent to the external flow field caused by an isolated liquid drop under this condition.

In Table 2, results of mobility parameters for the migration of two unequal-sized spheres with $\beta_1 a_1/\eta = \beta_2 a_2/\eta = 1$ are given. In the case of $\beta_1 a_1/\eta = \beta_2 a_2/\eta$, the mobility parameters M_{11} , M_{12} , M_{21} , and M_{22} for the situation of $a_2/a_1 = \alpha$ are equal to M_{22} , M_{21} , M_{12} , and M_{11} for the situation of $a_2/a_1 = 1/\alpha$, respectively. So, only the data with $a_2/a_1 > 1$ are listed in Table 2. On the other hand, results of the mobility parameters for the translation of two unequal-sized spheres with the same slip coefficient for various values of a_2/a_1 are presented in Table 3 for the case of $\beta a_1/\eta = 1$. It can be seen from Tables 2 and 3 that the effect of particle interaction, in general, is far greater on the smaller of the two spheres than on the larger one (M_{12} increases and M_{21} decreases significantly with increasing a_2/a_1 for any given value of $(a_1 + a_2)/r_{12}$). The larger particle is hardly influenced by the presence of the smaller one for the situation of $a_2/a_1 = 10$. Using the reciprocal theorem of Lorentz (7), one can obtain that $M_{21}/M_{12} = K_{12}/K_{21} = m_2/m_1$ for the motion of any two spheres. Examination of the numerical data in

TABLE 1
The Mobility Parameters for the Motion
of Two Identical Spheres

$\frac{\beta a}{\eta}$	$\frac{2a}{r_{12}}$	M_{11}	M_{12}	$M_{11} + M_{12}$
0	0.2	0.9999	0.1000	1.0999
	0.4	0.9983	0.2000	1.1984
	0.6	0.9911	0.3006	1.2917
	0.8	0.9683	0.4060	1.3743
	0.9	0.9422	0.4681	1.4103
	0.95	0.9181	0.5089	1.4269
	0.99	0.8744	0.5652	1.4396
	0.999	0.8354	0.6070	1.4424
	0.9999	0.8118	0.6309	1.4427
0.1	0.2	0.9999	0.1016	1.1015
	0.4	0.9983	0.2030	1.2013
	0.6	0.9908	0.3046	1.2953
	0.8	0.9676	0.4106	1.3782
	0.9	0.9412	0.4729	1.4141
	0.95	0.9170	0.5137	1.4307
	0.99	0.8735	0.5699	1.4433
	0.999	0.8347	0.6114	1.4461
	0.9999	0.8114	0.6349	1.4464
1	0.2	0.9998	0.1122	1.1121
	0.4	0.9977	0.2230	1.2208
	0.6	0.9884	0.3316	1.3201
	0.8	0.9619	0.4421	1.4040
	0.9	0.9336	0.5059	1.4395
	0.95	0.9087	0.5471	1.4558
	0.99	0.8658	0.6024	1.4682
	0.999	0.8295	0.6414	1.4709
	0.9999	0.8086	0.6626	1.4712
10	0.2	0.9997	0.1377	1.1374
	0.4	0.9958	0.2709	1.2667
	0.6	0.9800	0.3970	1.3770
	0.8	0.9407	0.5217	1.4624
	0.9	0.9040	0.5930	1.4970
	0.95	0.8750	0.6377	1.5127
	0.99	0.8331	0.6914	1.5245
	0.999	0.8060	0.7212	1.5271
	0.9999	0.7936	0.7338	1.5274
∞	0.2	0.9996	0.1490	1.1486
	0.4	0.9944	0.2922	1.2866
	0.6	0.9738	0.4268	1.4006
	0.8	0.9234	0.5627	1.4861
	0.9	0.8755	0.6447	1.5202
	0.95	0.8378	0.6978	1.5356
	0.99	0.7916	0.7557	1.5472
	0.999	0.7768	0.7729	1.5497
	0.9999	0.7752	0.7748	1.5500

Tables 2 and 3 indicates that this relation is very well satisfied for all the cases considered. This agreement also manifests the accuracy of our results. Note that $m_2/m_1 = a_1/a_2$ for the case of $\beta_1 a_1/\eta = \beta_2 a_2/\eta$, as can be shown using Eq. [18].

MOTION OF A SPHERE NORMAL
TO A PLANE WALL

In this section, we consider the motion of a spherical particle of radius a normal to an infinite plane wall located at a distance d from the sphere center. The fluid is allowed to slip at both the particle surface and the plane wall. The particle velocity and the drag force on the particle are denoted by Ue_z and Fe_z , respectively. For this case, the mathematical formulation is the same as that for the case of axisymmetric motion of two spheres when the radius of one sphere (say, sphere 2) is taken to be infinite. Thus, Eqs. [4]–[15a] are still applicable to the sphere–plane system if we let $a = a_1$, $d = d_1$, $U = U_1$, $F = F_1$, $\xi_2 = 0$, and $U_2 = 0$. Now, $1/\beta_1$ and $1/\beta_2$ represent the slip coefficients associated with the particle surface and the plane wall, respectively.

In the resistance problem, U is given and the resistance coefficient $F/F^{(0)}$ is to be determined. Here, $F^{(0)}$ is the drag force on the particle translating with a velocity U in the absence of the plane wall and can be calculated using Eq. [2]. In contrast, in the mobility problem, F is prescribed and the mobility coefficient $U/U^{(0)}$ is to be determined. In this sit-

TABLE 2
The Mobility Parameters for the Motion of Two Spheres
with $\beta_1 a_1/\eta = \beta_2 a_2/\eta = 1$

$\frac{a_2}{a_1}$	$\frac{a_1 + a_2}{r_{12}}$	M_{11}	M_{12}	M_{21}	M_{22}
2	0.2	0.9998	0.1496	0.0748	0.9999
	0.4	0.9963	0.2971	0.1485	0.9991
	0.6	0.9799	0.4409	0.2204	0.9958
	0.8	0.9276	0.5844	0.2922	0.9873
	0.9	0.8672	0.6644	0.3322	0.9791
	0.95	0.8125	0.7143	0.3571	0.9721
	0.99	0.7192	0.7789	0.3894	0.9599
	0.999	0.6434	0.8227	0.4114	0.9492
	0.9999	0.6013	0.8459	0.4230	0.9430
5	0.2	0.9998	0.1869	0.0374	1.0000
	0.4	0.9962	0.3702	0.0740	0.9999
	0.6	0.9780	0.5465	0.1093	0.9994
	0.8	0.9102	0.7146	0.1429	0.9983
	0.9	0.8189	0.7992	0.1598	0.9975
	0.95	0.7286	0.8466	0.1693	0.9968
	0.99	0.5698	0.9005	0.1801	0.9955
	0.999	0.4464	0.9324	0.1865	0.9944
	0.9999	0.3827	0.9477	0.1895	0.9936
10	0.2	0.9998	0.2038	0.0204	1.0000
	0.4	0.9972	0.4030	0.0403	1.0000
	0.6	0.9834	0.5932	0.0593	0.9999
	0.8	0.9259	0.7705	0.0771	0.9997
	0.9	0.8356	0.8547	0.0855	0.9996
	0.95	0.7334	0.8982	0.0898	0.9995
	0.99	0.5337	0.9420	0.0942	0.9994
	0.999	0.3759	0.9644	0.0964	0.9992
	0.9999	0.2985	0.9741	0.0974	0.9991

TABLE 3

The Mobility Parameters for the Motion of Two Spheres with the Same Slip Coefficient ($\beta_1 = \beta_2 = \beta$ and $\beta a_1/\eta = 1$)

$\frac{a_2}{a_1}$	$\frac{a_1 + a_2}{r_{12}}$	M_{11}	M_{12}	M_{21}	M_{22}
0.1	0.2	1.0000	0.0184	0.2038	0.9998
	0.4	1.0000	0.0364	0.4031	0.9975
	0.6	0.9999	0.0536	0.5934	0.9850
	0.8	0.9998	0.0696	0.7708	0.9326
	0.9	0.9997	0.0772	0.8548	0.8493
	0.95	0.9996	0.0811	0.8977	0.7536
	0.99	0.9995	0.0849	0.9404	0.5615
	0.999	0.9993	0.0869	0.9620	0.4027
	0.9999	0.9992	0.0878	0.9718	0.3208
0.5	0.2	1.0000	0.0713	0.1496	0.9998
	0.4	0.9992	0.1416	0.2973	0.9964
	0.6	0.9962	0.2103	0.4416	0.9806
	0.8	0.9884	0.2790	0.5858	0.9292
	0.9	0.9810	0.3171	0.6658	0.8688
	0.95	0.9746	0.3408	0.7157	0.8134
	0.99	0.9635	0.3715	0.7801	0.7177
	0.999	0.9537	0.3925	0.8242	0.6387
	0.9999	0.9479	0.4038	0.8479	0.5942
2	0.2	0.9997	0.1594	0.0747	0.9999
	0.4	0.9958	0.3156	0.1480	0.9991
	0.6	0.9774	0.4661	0.2185	0.9958
	0.8	0.9198	0.6135	0.2876	0.9880
	0.9	0.8542	0.6938	0.3252	0.9809
	0.95	0.7955	0.7431	0.3483	0.9751
	0.99	0.6970	0.8055	0.3776	0.9653
	0.999	0.6190	0.8466	0.3969	0.9569
	0.9999	0.5764	0.8680	0.4069	0.9520
10	0.2	0.9998	0.2494	0.0203	1.0000
	0.4	0.9958	0.4849	0.0394	1.0000
	0.6	0.9757	0.6927	0.0563	0.9999
	0.8	0.9002	0.8594	0.0698	0.9998
	0.9	0.7923	0.9241	0.0751	0.9998
	0.95	0.6783	0.9523	0.0774	0.9998
	0.99	0.4704	0.9761	0.0793	0.9998
	0.999	0.3188	0.9862	0.0801	0.9998
	0.9999	0.2485	0.9903	0.0805	0.9998

uation, $U^{(0)}$ is the velocity of the particle subject to an applied force $-F$ in the absence of the plane wall. Obviously, the relation $F/F^{(0)} = (U/U^{(0)})^{-1}$ holds between the two problems.

The flow patterns for the migration of a sphere normal to a plane wall are illustrated in Fig. 4. Here, streamlines in a meridian plane are drawn for the case where the distance of the particle center from the plane is equal to the particle diameter and $\beta_1 a/\eta = 1$. By the linearity of the problem, the streamline patterns are independent of the direction of travel of the sphere. Figures 4a and 4b correspond to the cases of a plane wall with a slip coefficient equal to that of the particle and with a no-slip surface, respectively. The streamlines for the situation when the particle is moving

toward a perfect-slip plane (or a free-surface plane) are the same as those illustrated in the upper or lower half of Fig. 2b. It can be found from Figs. 2b and 4 that a plane wall with a larger slip coefficient ($1/\beta_2$) will exert less drag force on the particle, which results in a stronger recirculation of the fluid.

Numerical results of the mobility coefficient $U/U^{(0)}$ of a sphere translating in the direction perpendicular to a plane wall for various values of $\beta_1 a/\eta$, $\beta_2 a/\eta$, and a/d are presented in Table 4. As expected, the particle moves with the velocity that would exist in the absence of the wall as $a/d \rightarrow 0$. Then, the mobility coefficient of the particle decreases steadily as it approaches the wall (with increasing a/d), going to zero in the limit. Note that the mobility coefficient in general increases when the value of $\beta_1 a/\eta$ or $\beta_2 a/\eta$ is decreased, if the other factors are fixed.

Reed and Morrison (11) evaluated the values of $F/F^{(0)}$ for the migration of a sphere normal to a plane wall for the case of identical slip coefficients ($\beta_1 = \beta_2$) over the range $a/d < 0.957$. Our results in Table 4 are consistent with theirs. The results in Table 4 also include the interaction effects between a no-slip solid sphere and a no-slip solid wall ($\beta_1 \rightarrow \infty, \beta_2 \rightarrow \infty$), between a no-slip sphere and a perfect-slip free surface ($\beta_1 \rightarrow \infty, \beta_2 = 0$), between a perfect-slip

FIG. 4. Streamlines for the motion of a sphere normal to a plane wall with $\beta_1 a/\eta = 1$ and $a/d = 0.5$: (a) the case of $\beta_2 = \beta_1$, and (b) the case of $\beta_2 \rightarrow \infty$. The value on each streamline indicates its dimensionless stream function, $\Psi/c^2 U$.

TABLE 4
The Mobility Coefficient of a Sphere Moving Normal
to an Infinite Plane Wall

$\beta_1 a/\eta$	a/d	$U/U^{(0)}$		
		$\beta_2 = 0$	$\beta_2 = \beta_1$	$\beta_2 \rightarrow \infty$
0	0.2	0.8999	0.8999	0.8498
	0.4	0.7983	0.7983	0.6963
	0.6	0.6905	0.6905	0.5303
	0.8	0.5623	0.5623	0.3286
	0.9	0.4741	0.4741	0.1949
	0.95	0.4092	0.4092	0.1100
	0.99	0.3093	0.3093	0.0253
	0.999	0.2284	0.2284	0.0026
	0.9999	0.1809	0.1809	0.0003
0.1	0.2	0.8983	0.8890	0.8475
	0.4	0.7953	0.7847	0.6924
	0.6	0.6862	0.6751	0.5260
	0.8	0.5569	0.5459	0.3259
	0.9	0.4683	0.4575	0.1940
	0.95	0.4033	0.3930	0.1101
	0.99	0.3036	0.2945	0.0255
	0.999	0.2234	0.2158	0.0027
	0.9999	0.1765	0.1701	0.0003
1	0.2	0.8876	0.8513	0.8319
	0.4	0.7747	0.7208	0.6655
	0.6	0.6568	0.5936	0.4956
	0.8	0.5197	0.4539	0.3050
	0.9	0.4277	0.3644	0.1852
	0.95	0.3616	0.3024	0.1086
	0.99	0.2634	0.2140	0.0270
	0.999	0.1881	0.1494	0.0030
	0.9999	0.1460	0.1144	0.0003
10	0.2	0.8620	0.7985	0.7948
	0.4	0.7249	0.6128	0.5998
	0.6	0.5830	0.4413	0.4155
	0.8	0.4190	0.2756	0.2350
	0.9	0.3110	0.1873	0.1392
	0.95	0.2373	0.1360	0.0844
	0.99	0.1416	0.0777	0.0255
	0.999	0.0848	0.0461	0.0034
	0.9999	0.0597	0.0325	0.0004
∞	0.2	0.8506	0.7782	0.7782
	0.4	0.7021	0.5694	0.5694
	0.6	0.5470	0.3746	0.3746
	0.8	0.3607	0.1885	0.1885
	0.9	0.2308	0.0958	0.0958
	0.95	0.1400	0.0486	0.0486
	0.99	0.0359	0.0099	0.0099
	0.999	0.0039	0.0010	0.0010
	0.9999	0.0004	0.0001	0.0001

gas bubble and a no-slip wall ($\beta_1 = 0, \beta_2 \rightarrow \infty$), and between a perfect-slip bubble and a perfect-slip plane ($\beta_1 = \beta_2 = 0$). Our solutions agree very well with the results of the resistance coefficient $F/F^{(0)}$ obtained by Brenner (15) for the migration of a no-slip sphere normal to a no-slip or perfect-slip plane

surface and by Bart (16) for the motion of a fluid sphere (taking $\eta^* = 0$) toward a no-slip or perfect-slip plane boundary. However, the boundary effect on a slip sphere with a finite value of $\beta a/\eta$ is different from that on a liquid drop with a value of $3\eta^*$ equal to $\beta a/\eta$.

SUMMARY

In this work the creeping motion of two slip spheres (e.g., spherical aerosol particles) along their line of centers has been studied. The spheres may be different in translational velocity for a resistance problem or in applied force for a mobility problem. The fluid flow field is solved using bipolar coordinates, and the particles' interaction parameters are obtained for various values of the slip coefficients, sizes, and separation distance of the particles. It is found that the interaction effect between particles decreases with the increase of the slip coefficients at the particle surfaces. This interaction effect can be very significant when the distance between particle surfaces approaches zero. The influence of the interaction between particles, in general, is far greater on the smaller one than on the larger one. Our results include the interaction effects between two no-slip rigid spheres and between two perfect-slip gas bubbles as limiting cases.

The slow motion of a slip sphere in the direction normal to a slip plane boundary is also examined for various cases of the slip coefficients at the interfaces between phases. The mobility of the particle in general increases with the increase of the slip coefficients. The effect of retardation generated by the plane boundary on the particle increases steadily as it approaches the plane and becomes infinity when the surfaces are in contact. Our solutions also include the interaction effects for the migration of a no-slip or perfect-slip sphere toward a no-slip or perfect-slip plane boundary as special cases.

APPENDIX: NOMENCLATURE

a, a_i	radius of a spherical particle (m)
A_n, B_n, C_n, D_n	coefficients in the expression of the flow field (m/s)
c	characteristic length of the bipolar coordinate system (m)
d	distance of the particle center from the plane boundary (m)
d_i	distance of the center of particle i from the origin (m)
$\mathbf{e}_z$	axial unit vector in the cylindrical coordinate system
$\mathbf{e}_\xi$	a unit vector in the bipolar coordinate system
F, F_i	drag force exerted on a particle by the fluid (N)
$F^{(0)}, F_i^{(0)}$	drag force exerted on an isolated particle (N)

$G_{n+1}^{-1/2}$	the Gegenbauer polynomial of order $n + 1$ and degree $-\frac{1}{2}$	τ	viscous stress dyadic for the fluid (N/m^2)
$\mathbf{l}$	unit dyadic	ψ, ξ, ϕ	spherical bipolar coordinates
K_{ij}	dimensionless resistance parameters defined by Eq. [16]	Ψ	Stokes' stream function for the fluid flow (m^3/s)
l	mean free path of the gas molecules (m)		
m_j	mobility defined by Eq. [18] (s/kg)		
M_{ij}	dimensionless mobility parameters defined by Eq. [20]		
P_n	Legendre polynomial function of the first kind		
r_{12}	center-to-center distance between two particles (m)		
U, U_i	velocity of a particle (m/s)		
$U^{(0)}, U_i^{(0)}$	velocity of an isolated particle (m/s)		
$\mathbf{v}$	fluid velocity field (m/s)		
v_ξ, v_ψ	components of $\mathbf{v}$ in the bipolar coordinate system (m/s)		
z	axial cylindrical coordinate (m)		

Greek Symbols

β, β_i	reciprocal of the slip coefficient at a solid-fluid interface ($\text{kg/m}^2/\text{s}$)
η	viscosity of fluid (kg/m/s)
η^*	internal-to-external viscosity ratio of a liquid drop
μ	equal to $\cos \psi$
ξ_i	value of ξ representing the surface of particle i
ρ	radial cylindrical coordinate (m)

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