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Analysis of Torsional Vibration Systems by the Extended Transfer Matrix Method

This study applies the extended transfer matrix method and Newton-Raphson technique with complex numbers for torsional vibration analysis of damped systems. The relationships of the vibratory amplitude, the vibratory torque, the derivatives of the vibratory angular displacement and the vibratory torque between components at the left end and the right end of the torsional vibration system are derived. The derivatives of the vibratory angular displacement and the vibratory torque are used directly in the Newton-Raphson technique to determine the eigensolutions of systems that are compared and show good agreement with the available data.

Introduction

The well known Holzer method [1] that is a simple and systematic approach to determine the frequencies and mode shapes of systems was used originally for analysis of an undamped system. Den Hartog and Li [2] improved the Holzer method by using complex numbers. Pestel and Leckie [3] introduced the transfer matrix method with the point and field transfer matrices. Eshleman [4] developed a technique for the solution of torsional vibration problems encountered in design, development and diagnosis of engine systems. Sankar [5] used the extended transfer matrix method to analyze an undamped system. Doughty and Vafaei [6] used complex numbers in the transfer matrix with the Newton-Raphson technique to determine the eigensolutions of a damped system. Dawson and Davies [7] used the extended transfer matrix with the Newton-Raphson technique to analyze the torsional vibration for undamped systems. Doughty [8] considered the systems of properly counting for the shafting inertia in the torsional vibration analysis by using a Rayleigh-type approximation for the displacement between stations. Neilson and Barr [9] studied imbalance effects in rotor systems under high angular accelerations. Yun and Lee [10] studied the dynamic analysis of flexible rotors subjected to the torque and the force. Lu [11] analyzed the transfer matrix method for the nonlinear vibration of the rotor-bearing system. Kang and Chung [12] studied the transfer matrix method for the general periodic torsional vibration of rotor systems. Huang [13] analyzed the engine excitation on a transmission with a damper for the converter operation. Doughty [14] analyzed the steady state torsional response with the viscous damping.

Although Doughty and Vafaei [6] used complex numbers with the Newton-Raphson technique to calculate the eigensolutions of damped systems, they used the conventional transfer matrix method. A complicated multiplication should be performed before the Newton Raphson technique is applied. The multiplication results in a nominal equation that is well known as the eigenvalue equation. The term number of the equation increases and the terms become complicate if the number of rotors in the system becomes huge. Dawson and Davies [7] used the extended transfer matrix with the Newton-Raphson technique. Nevertheless, their study was limited to undamped systems. The damping effect exists in the most of real systems. It is important to include the damping effect to better predict the torsional vibration of systems. Therefore, this study uses

complex numbers to extend the extended transfer matrix method with the Newton-Raphson technique to analyze the torsional vibration for damped systems. This method eliminates the operation of the inverse matrix because the derivatives of angular displacement and the torque are used directly with the Newton-Raphson technique to determine the eigenvalues of torsional vibration systems.

Method of Approach

A mass elastic system with both ends free is shown in Fig. 1. The symbols I , k , c and d are the mass moment of inertia of a disk, the stiffness of a shaft, the internal damping coefficient and the external damping coefficient, respectively.

It is assumed that disks are rigid bodies with infinite stiffnesses, that shafts are slender with negligible mass moments of inertia, that the damping effect is proportional to the relative vibratory angular speed, and that the damping coefficient is constant. The equations of motion for a disk and a shaft as shown in Fig. 2, respectively, are

$$\Theta_i^L = \Theta_i^R$$

$$T_i^R - T_i^L = I_i \frac{d^2}{dt^2} \Theta_i + d_i \frac{d\Theta_i}{dt} \quad (1)$$

and

$$T_i^R = T_{i+1}^L$$

$$T_i^R = k_i (\Theta_{i+1}^L - \Theta_i^R) + c_i \frac{d}{dt} (\Theta_{i+1}^L - \Theta_i^R) \quad (2)$$

The angular displacements and torques of stations can be expressed as [6]

$$\{\Theta(t)\} = \begin{Bmatrix} \theta_1^r + j\theta_1^i \\ \theta_2^r + j\theta_2^i \\ \vdots \\ \theta_{n-1}^r + j\theta_{n-1}^i \\ \theta_n^r + j\theta_n^i \end{Bmatrix} e^{(-\sigma + j\omega)t} = \{\theta(t)\} e^{(-\sigma + j\omega)t} \quad (3)$$

and

$$\{T(t)\} = \begin{Bmatrix} \tau_1^r + j\tau_1^i \\ \tau_2^r + j\tau_2^i \\ \vdots \\ \tau_{n-1}^r + j\tau_{n-1}^i \\ \tau_n^r + j\tau_n^i \end{Bmatrix} e^{(-\sigma + j\omega)t} = \{\tau(t)\} e^{(-\sigma + j\omega)t} \quad (4)$$

where $-\sigma$ and ω are the real part and the imaginary part of the

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Fig. 1 Mass elastic system with both ends free

Fig. 2 Schematic diagrams of a disk and a shaft

eigenvalue, λ , respectively. Substituting Eqs. (3) and (4) into Eqs. (1) and (2) yields

$$\theta_i^L = \theta_i^R$$

$$\tau_i^R - \tau_i^L = I_i(-\sigma + j\omega)^2 \theta_i^L + j d_i(-\sigma + j\omega) \theta_i^L \quad (5)$$

and

$$\tau_i^L = \tau_i^R$$

$$\tau_i^R = k_i(\theta_{i+1}^L - \theta_i^R) + c_i(-\sigma + j\omega)(\theta_{i+1}^L - \theta_i^R) \quad (6)$$

The state vector of a component is defined as

$$Z = \begin{Bmatrix} Z^r \\ Z^i \end{Bmatrix} = \begin{Bmatrix} \theta^r \\ \tau^r \\ \theta^i \\ \tau^i \end{Bmatrix} \quad (7)$$

which is a column matrix of the vibratory angular displacement

and the vibratory torque. The superscripts r and i indicate the real part and the imaginary part, respectively. The extended state vector of a component can be expressed as

$$Z = \left\{ \theta^r \quad \tau^r \quad \theta^i \quad \tau^i \quad \frac{\partial \theta^r}{\partial \sigma} \quad \frac{\partial \tau^r}{\partial \sigma} \quad \frac{\partial \theta^i}{\partial \sigma} \quad \frac{\partial \tau^i}{\partial \sigma} \quad \frac{\partial \theta^r}{\partial \omega} \quad \frac{\partial \tau^r}{\partial \omega} \quad \frac{\partial \theta^i}{\partial \omega} \quad \frac{\partial \tau^i}{\partial \omega} \right\}^T \quad (8)$$

The field transfer matrix, F , and the point transfer matrix, P , are defined as functions of $I, k, c, d, \sigma, \omega$ [3]. Differentiating the state vector with respect to σ and ω , respectively, and combining with Eq. (7), the relationship of the extended state vectors for a shaft and a disk, respectively, are

$$Z_{i+1}^L = \begin{Bmatrix} Z \\ \frac{\partial Z}{\partial \sigma} \\ \frac{\partial Z}{\partial \omega} \end{Bmatrix}_{i+1}^L = \begin{bmatrix} F & 0 & 0 \\ \frac{\partial F}{\partial \sigma} & F & 0 \\ \frac{\partial F}{\partial \omega} & 0 & F \end{bmatrix}_i \times \begin{Bmatrix} Z \\ \frac{\partial Z}{\partial \sigma} \\ \frac{\partial Z}{\partial \omega} \end{Bmatrix}_i^R \quad (9)$$

and

$$Z_i^R = \begin{Bmatrix} Z \\ \frac{\partial Z}{\partial \sigma} \\ \frac{\partial Z}{\partial \omega} \end{Bmatrix}_i^R = \begin{bmatrix} P & 0 & 0 \\ \frac{\partial P}{\partial \sigma} & P & 0 \\ \frac{\partial P}{\partial \omega} & 0 & P \end{bmatrix}_i \times \begin{Bmatrix} Z \\ \frac{\partial Z}{\partial \sigma} \\ \frac{\partial Z}{\partial \omega} \end{Bmatrix}_i^L \quad (10)$$

The relationship between the right side of the $(i+1)$ th component and the left side of the i th component is

$$\begin{Bmatrix} Z \\ \frac{\partial Z}{\partial \sigma} \\ \frac{\partial Z}{\partial \omega} \end{Bmatrix}_{i+1}^R = \begin{bmatrix} T & 0 & 0 \\ \frac{\partial T}{\partial \sigma} & T & 0 \\ \frac{\partial T}{\partial \omega} & 0 & T \end{bmatrix}_i \times \begin{Bmatrix} Z \\ \frac{\partial Z}{\partial \sigma} \\ \frac{\partial Z}{\partial \omega} \end{Bmatrix}_i^L \quad (11)$$

where T is the extended transfer matrix. The elements in T are the functions of $I, k, c, d, \sigma, \omega$. Using the transfer matrix method, the relationship between the right side of the n th component and the left side of the first component can be obtained as

Nomenclature

A = matrix product
 a, b, e, g, h = element in the matrix
 c = internal damping coefficient
 d = external damping coefficient
 I = mass moment of inertia
 k = stiffness
 T = torque matrix; transposed transfer matrix or torque
 t = time
 Z = state vector
 $\bar{Z}$ = extended state vector

σ = absolute value of real part of eigenvalue
 λ = eigenvalue of matrix
 Θ = angular displacement vector
 θ = angular displacement magnitude
 τ = torque magnitude
 α = phase angle
 δ = external forced circular frequency
 ω = imaginary part of eigenvalue

Subscripts

e = external
 i = i th component

ij = i th row and j th column element of matrix

Superscripts

c = cosine component
 i = imaginary part
 L = left side of component
 R = right side of component
 r = real part
 s = sine component

$$\begin{Bmatrix} \theta^r \\ \tau^r \\ \theta^i \\ \tau^i \\ \frac{\partial \theta^r}{\partial \sigma} \\ \frac{\partial \tau^r}{\partial \sigma} \\ \frac{\partial \theta^i}{\partial \sigma} \\ \frac{\partial \tau^i}{\partial \sigma} \\ \frac{\partial \theta^r}{\partial \omega} \\ \frac{\partial \tau^r}{\partial \omega} \\ \frac{\partial \theta^i}{\partial \omega} \\ \frac{\partial \tau^i}{\partial \omega} \end{Bmatrix}^R = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0 \\ a_{91} & a_{92} & a_{93} & a_{94} & 0 & 0 & 0 & 0 & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{101} & a_{102} & a_{103} & a_{104} & 0 & 0 & 0 & 0 & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{111} & a_{112} & a_{113} & a_{114} & 0 & 0 & 0 & 0 & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{121} & a_{122} & a_{123} & a_{124} & 0 & 0 & 0 & 0 & a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} \theta^r \\ \tau^r \\ \theta^i \\ \tau^i \\ \frac{\partial \theta^r}{\partial \sigma} \\ \frac{\partial \tau^r}{\partial \sigma} \\ \frac{\partial \theta^i}{\partial \sigma} \\ \frac{\partial \tau^i}{\partial \sigma} \\ \frac{\partial \theta^r}{\partial \omega} \\ \frac{\partial \tau^r}{\partial \omega} \\ \frac{\partial \theta^i}{\partial \omega} \\ \frac{\partial \tau^i}{\partial \omega} \end{Bmatrix}^L \quad (12)$$

or

$$\bar{Z}_n^R = A \bar{Z}_1 \quad (13)$$

The elements of the matrix are functions of I, k, c, d, σ and ω . If the extended state vector at the left side of the first component is

$$\bar{Z}_1^L = \{1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0\}^T \quad (14)$$

and Eq. (14) is substituted into Eq. (12), we can get

$$\begin{aligned} \tau_n^r &= a_{21} \\ \tau_n^i &= a_{41} \\ \frac{\partial \tau_n^r}{\partial \sigma} &= \frac{\partial a_{21}}{\partial \sigma} = a_{61} \\ \frac{\partial \tau_n^i}{\partial \sigma} &= \frac{\partial a_{41}}{\partial \sigma} = a_{81} \\ \frac{\partial \tau_n^r}{\partial \omega} &= \frac{\partial a_{21}}{\partial \omega} = a_{101} \\ \frac{\partial \tau_n^i}{\partial \omega} &= \frac{\partial a_{41}}{\partial \omega} = a_{121} \end{aligned} \quad (15)$$

If the right side of the n th component is free, the vibratory torque of the n th component is zero. Therefore, it can be written

$$\tau_n^r(\sigma, \omega) = 0$$

and

$$\tau_n^i(\sigma, \omega) = 0 \quad (16)$$

Equation (16) is known as the eigenvalue equation of a damped system.

Using the Newton-Raphson method with

$$\begin{Bmatrix} \sigma \\ \omega \end{Bmatrix}^{\text{new}} = \begin{Bmatrix} \sigma \\ \omega \end{Bmatrix}^{\text{old}} + \begin{bmatrix} \frac{\partial \tau_n^r}{\partial \sigma} & \frac{\partial \tau_n^r}{\partial \omega} \\ \frac{\partial \tau_n^i}{\partial \sigma} & \frac{\partial \tau_n^i}{\partial \omega} \end{bmatrix} \times \begin{Bmatrix} -\tau_n^r \\ -\tau_n^i \end{Bmatrix}^{\text{old}} \quad (17)$$

and substituting Eq. (15) into Eq. (17) yields

$$\begin{Bmatrix} \sigma \\ \omega \end{Bmatrix}^{\text{new}} = \begin{Bmatrix} \sigma \\ \omega \end{Bmatrix}^{\text{old}} + \begin{bmatrix} a_{61} & a_{101} \\ a_{81} & a_{121} \end{bmatrix} \times \begin{Bmatrix} -a_{21} \\ -a_{41} \end{Bmatrix}^{\text{old}} \quad (18)$$

By using the trial and error method with assumed initial values of σ, ω , the accurate eigenvalues can be obtained when τ_n^r and τ_n^i in Eq. (17) converge.

If the system shown in Fig. 1 is the equivalent mass elastic system of a multi-cylinder engine, δ stands for the excitation frequency and α_i stands for the phase angle of the cylinder with respect to the first firing cylinder of an engine, the external torque can be expressed as

$$\begin{aligned} T_{ei} &= \tau_{ei} e^{j(\delta t - \alpha_i)} \\ &= (\tau_{ei} \cos \alpha_i \cos \delta t - \tau_{ei} \sin \alpha_i \sin \delta t) \\ &\quad + j(\tau_{ei} \cos \alpha_i \sin \delta t + \tau_{ei} \sin \alpha_i \cos \delta t) \end{aligned} \quad (19)$$

The real part of the torque is

$$\begin{aligned} R[(\tau_{ei}^r + j\tau_{ei}^i)e^{j\delta t}] &= \tau_{ei} \cos \alpha_i \cos \delta t - \tau_{ei} \sin \alpha_i \sin \delta t \\ &= \tau_{ei}^c \cos \delta t + \tau_{ei}^s \sin \delta t \\ &= R[(\tau_{ei}^c - j\tau_{ei}^s)e^{j\delta t}] \end{aligned}$$

Hence, we have

$$\tau_{ei}^c = \tau_{ei} \cos \alpha_i = \tau_{ei}^r$$

and

Table 1 Mass moments of inertia of disks and stiffnesses of shafts for the first undamped system

station no.	I (N-m-s ²)	k (MN-m/rad)
1	71.882	1.3614
2	41.387	1.3614
3	41.387	1.3614
4	41.387	1.3614
5	41.387	1.3614
6	71.822	2.1523
7	4401.1	1.4789
8	1763.2	0.8417
9	162.28	-

$$\tau_{ei}^r = -\tau_{ei} \sin \alpha_i = -\tau_{ei}^i \quad (21)$$

similarly, the real part of the angular displacement is

$$R[(\theta_{ei}^r + j\theta_{ei}^i)e^{j\delta t}] = \theta_{ei}^c \cos \delta t - \theta_{ei}^i \sin \delta t \quad (22)$$

If a system is subjected to an external torque, the relationship between the right side of the n th component and the left side of the first component can be expressed as

$$\begin{Bmatrix} \theta^r \\ \tau^r \\ \theta^i \\ \tau^i \\ 1 \end{Bmatrix}_n^R = \begin{bmatrix} a_1 & b_1 & e_1 & g_1 & f_1 \\ a_2 & b_2 & e_2 & g_2 & f_2 \\ a_3 & b_3 & e_3 & g_3 & f_3 \\ a_4 & b_4 & e_4 & g_4 & f_4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta^r \\ \tau^r \\ \theta^i \\ \tau^i \\ 1 \end{Bmatrix}_1^L \quad (23)$$

where the elements in the matrix are functions of I , k , c , d , δ and τ_e . If both ends of a system are free, τ_n^r and τ_n^i are all equal to zero. Hence, we have

$$\begin{aligned} a_2\theta_1^r + e_2\theta_1^i + h_2 &= 0 \\ a_4\theta_1^r + e_4\theta_1^i + h_4 &= 0 \end{aligned} \quad (24)$$

and

$$\begin{aligned} a_1\theta_1^r + e_1\theta_1^i + h_1 &= \theta_n^r \\ a_3\theta_1^r + e_3\theta_1^i + h_3 &= \theta_n^i \end{aligned} \quad (25)$$

The vibratory angular displacement of the first component can be determined from Eq. (24). Therefore, the vibratory angular displacements of all components can be determined.

Results

The first system with the mass moments of inertia of disks and the stiffnesses of shafts of a six-cylinder engine with a flywheel and a converter listed in Table 1 is used for analysis. It is assumed that the system is undamped with both ends free.

There is a rigid mode with an eigenvalue equal to zero for a system with both ends free. The calculated natural circular frequencies of the top four modes are 1001, 1312, 2353 and 3677 rad/s. The first mode and the second mode obtained by using the Holzer method are 1030 and 1330 rad/s, respectively. The top four natural circular frequencies obtained by using the MATLAB software program are 1003, 1336, 2354 and 3811 rad/s, respectively. The results of the relative vibratory angular displacements for the first and the second mode shapes are shown in Fig. 3.

The mass moments of inertia of disks and the stiffnesses of shafts of the second undamped system with both ends free are shown in Table 2. The results of eigensolutions of the second undamped system obtained from this analysis and from the MATLAB software program are shown in Table 3.

The mass moments of inertia, the stiffnesses and the damping coefficients of the third damped system obtained from Doughty

Fig. 3 Relative vibratory angular displacements of the first undamped system

Table 2 Mass moments of inertia and stiffnesses of the second undamped system

station no.	I (N-m-s ²)	k (MN-m/rad)
1	0.680	0.170
2	0.680	0.290
3	1.360	0.079
4	2.260	0.090
5	0.136	--

Table 3 Natural frequencies and relatively vibratory angular displacements of the second undamped system

station no.	1st mode		2nd mode		3rd mode		4th mode	
	current	MATLAB	current	MATLAB	current	MATLAB	current	MATLAB
ω	235	235	512	512	839	839	956	956
	rad/s	rad/s	rad/s	rad/s	rad/s	rad/s	rad/s	rad/s
relative vibratory angular displacement								
1	1	1	1	1	1	1	1	1
2	0.78	0.78	-0.05	-0.06	-1.82	-1.82	-2.65	-2.65
3	0.55	0.57	-0.63	-0.63	-0.47	-0.47	0.89	0.88
4	-0.81	-0.83	0.09	0.06	10.16	10.21	-0.04	-0.04
5	-0.89	-0.91	0.15	0.13	-160.1	-161	0.11	0.11

Table 4 Mass moments of inertia and stiffnesses of the third undamped system

station no.	I (N-m-s ²)	k (MN-m/rad)	c (N-m-s/rad)	d (N-m-s/rad)
1	0.3	8000	10	0
2	0.5	10000	0	0
3	0.4	---	---	6

and Vafaei [6] with both ends free are shown in Table 4. The calculated results of the eigensolutions and the data obtained from Doughty and Vafaei [6] are shown in Table 5.

The fourth damped system with the mass moments of inertia, the stiffnesses and the damping coefficients of an engine with a generator obtained from Doughty and Vafaei [6] are shown in Table 6. Both ends of the damped system are free. The results of the eigensolutions for the system are shown in Table 7.

The mass moments of inertia and the external damping coefficients of disks, and the stiffnesses and the internal damping

Table 5 Results of the eigensolutions for the third damped system

station no.	current calculated results				Doughty and Vafae's results			
	first mode		second mode		first mode		second mode	
	$\lambda =$		$\lambda =$		$\lambda =$		$\lambda =$	
	-12.1+j161.3		-19.6+j246.3		-12.1+j161.3		-19.6+j246.3	
	rad/s		rad/s		rad/s		rad/s	
relative vibratory angular displacement								
	real	imag	real	imag	real	imag	real	imag
1	1.0		1.00		1.0		1.00	
2	0.03	0.05	-1.21	0.33	0.03	0.05	-1.21	0.33
3	-0.77	-0.14	0.79	-0.36	-0.77	-0.14	0.79	-0.36

Table 6 Mass moments of inertia and stiffnesses of the fourth damped system

station no.	I (N-m-s ²)	k (MN-m/rad)	c (N-m-s/rad)	d (N-m-s/rad)
1	6	16	-	-
2	4.5	16	-	-
3	4.5	16	-	-
4	6	13	-	-
5	6	16	-	-
6	4.5	16	-	-
7	4.5	16	-	-
8	6	25	-	-
9	3.5	150	-	-
10	100	0.5	330	-
11	20	10	-	-
12	350	-	-	550

Table 7 Results of the eigensolutions for the fourth damped system

station no.	current calculated results				Doughty and Vafae's results			
	1st mode		2nd mode		3rd mode		3rd mode	
	$\lambda =$		$\lambda =$		$\lambda =$		$\lambda =$	
	-1.6+j67.4		-0.3+j376.7		-8.0+j743.2		-8.0+j743.2	
	(rad/s)		(rad/s)		(rad/s)		(rad/s)	
relative vibratory angular displacement								
	real	imag	real	imag	real	imag	real	imag
1	1.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00
2	1.00	-0.00	0.95	-0.00	0.79	-0.00	0.79	-0.00
3	1.00	-0.00	0.86	-0.00	0.46	-0.01	0.46	-0.01
4	0.99	-0.00	0.73	-0.00	0.06	-0.02	0.06	-0.02
5	0.98	-0.00	0.53	-0.00	-0.45	-0.02	-0.45	-0.02
6	0.98	-0.00	0.34	-0.00	-0.77	-0.02	-0.77	-0.02
7	0.97	-0.00	1.31	-0.00	-0.97	-0.01	-0.97	-0.01
8	0.96	-0.00	-0.08	-0.00	-1.02	0.00	-1.02	0.00
9	0.95	-0.00	-0.21	-0.00	-0.92	0.02	-0.92	0.02
10	0.95	-0.00	-0.23	-0.00	-0.89	0.02	-0.89	0.02
11	-0.32	-0.01	-0.01	-0.00	86.2	-42.1	86.2	-42.1
12	-0.38	-0.01	0.02	0.00	-4.65	2.39	-4.65	2.39

Table 8 Mass moments of inertia, stiffnesses and damping coefficients of the forced system

station no.	I (N-m-s ²)	k (MN-m/rad)	c (N-m-s/rad)	d (N-m-s/rad)
1	1.7	0.30	5600	0.0
2	2.5	0.30	8200	0.0
3	2.3	0.35	12000	0.0
4	1.4	0.22	4000	9.9
5	3.7	0.40	-	-

coefficients of shafts in a system subjected to external torques are shown in Table 8. If the disks of the system are subjected to the external excitation torques,

$$T_e(t) = \begin{Bmatrix} 0 \\ 20 \\ 25 \\ 0 \\ 0 \end{Bmatrix} \cos 27t + \begin{Bmatrix} 0 \\ 15 \\ -20 \\ 0 \\ 0 \end{Bmatrix} \sin 54t \quad (26)$$

it is known from Eq. (21) that the original coefficients of $\cos 27t$ and the sign change of the original coefficients of $\sin 54t$ in Eq. (26) should be used in the matrix. The angular displacement of a components is

$$\theta(t) = U_1 \cos 27t + V_1 \sin 27t + U_2 \cos 54t + V_2 \sin 54t \quad (27)$$

The results of the vibratory angular displacements of the components and the data obtained by Doughty [14] are shown in Table 9.

Discussion

The derivatives of the vibratory angular displacements and the vibratory torques are used directly in the Newton-Raphson technique to calculate the eigenvalues. The results are based on the first order of approximation in the Newton-Raphson method. If τ_n^r and τ_n^i in Eq. (17) are less than 10^{-6} N.m, the values are assumed to be converged.

This method eliminates the operation of the inverse matrix because the derivatives of angular displacement and the torque are used directly with the Newton-Raphson technique to calculate the eigenvalues of systems. In addition to the zero elements, there are 80 elements in the transfer matrix in Eq. (12). Nevertheless, only six elements, a_{21} , a_{41} , a_{61} , a_{81} , a_{101} and a_{121} , are needed for Eq. (18) to calculate the eigenvalues. It is not necessary to calculate all elements of the transfer matrix. Therefore, using the Newton-Raphson technique with complex numbers in the extended transfer matrix method should be better than using the conventional transfer matrix method and then applying the

Table 9 Results of the coefficients of trigonometric functions

station no.	Angular Displacement (10^{-3}) rad							
	current calculated results				Doughty's results			
	U_1	U_2	V_1	V_2	U_1^*	U_2^*	V_1^*	V_2^*
1	2.47	-1.48	1.33	4.33	2.47	-1.48	1.33	4.33
2	1.93	-0.16	1.03	0.50	1.93	-0.16	1.03	0.50
3	-1.31	0.89	0.60	-4.39	-1.31	0.89	0.60	-4.39
4	-5.42	1.10	0.22	-3.61	-5.42	1.10	0.22	-3.61
5	-16.3	-0.18	-1.69	2.23	-16.3	-0.18	-1.69	2.23

Newton's equation to determine the eigenvalues of a system. However, there is no computer running time available from the conventional transfer matrix method. In the meantime no computer time can be recorded in the personal computer for comparison.

There are some disadvantages of using this methodology. An initial eigenvalue is required to calculate one eigenvalue of a system. Improperly assumed initial values of σ and ω may result in the overshooting that leads to difficulty in determining some eigenvalues. For an undamped system, the residual torque converges quickly. Since the deviation of eigenvalues for the damped system from the system with the damping coefficients assumed to be zero is small for the most cases, the eigenvalues obtained for the system with the damping coefficients assumed to be zero are the best assumed initial values for damped systems. Only one eigenvalue can be obtained at one time. If there are three eigenvalues for a system, it is necessary to assume three different initial eigenvalues. However, if only the natural frequency near the operation frequency is concerned for a steady running system, such as a turbine generator system, this methodology is sufficient and adequate to analyze the torsional vibration of the system.

For the first and the second undamped systems, the imaginary parts of the eigenvalues are zero, and no phase angle exists. The calculated results of the first and the second systems agree very well with the results obtained by using both the Holzer method and the MATLAB software program. The calculated results of the third and fourth damped systems show very good agreement with those obtained by Doughty and Vafaei [6] and Doughty [14]. Therefore, the methodology can be used for not only undamped systems but also damped systems subjected to external torques. It can also be used to analyze the torsional vibration for both steady systems and dynamics systems. The application range of the extended transfer matrix method with the Newton-Raphson technique is improved.

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