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Weight Functions and Stress Intensity Factors for Axial Cracks in Hollow Cylinders

In this study, stress intensity factors for axial cracks in hollow cylinders subjected to mechanical and thermal loadings are determined by using the weight function method. The weight function is a universal function for a given cracked body and can be obtained from any arbitrary loading system. The weight function may be thought of as Green's function for the stress intensity factor of cracked bodies. Once the weight function for a cracked body is determined, the stress intensity factor for any arbitrary loading can be simply and efficiently evaluated through the integration of the product of the loading and weight function. A numerical method for the determination of weight functions relevant to cracked bodies with finite dimensions is used. Results for weight functions covering a wide range of hollow cylinder geometries are presented in functional or graphical form. The explicit crack face weight functions for applying mechanical loadings are obtained by using the least-squares fitting procedure. As a demonstration, some examples of special loading problems are solved by the weight function method, and the results are compared with available results in the published literature.

1 Introduction

Hollow circular cylinders are widely used as pressure vessels, for example, in the nuclear and chemical industries. In the manufacturing process and during service life, a crack may initiate on an internal or external boundary in a circular cylinder. The magnitude of the stress intensity factor determines whether or not the crack will propagate. Considerable effort has been devoted to the computation of the stress intensity factors to assess whether structural failure will occur or not. Stress intensity factors are now available for a wide range of crack configurations and loadings and have been summarized in well-known handbooks of Tada et al. (1973), Rooke and Cartwright (1976), and Sih (1973). These are inadequate with regard to the needs in practical applications.

In the classical study of thermoelastic crack problem, the theoretical solutions are available only for very few problems in which cracks are contained in infinite media under special thermal loading conditions, such as Sih (1962), Florence and Goodier (1960, 1963), Olesiak and Sneddon (1959), Kassir and Sih (1968), and Kassir and Bregman (1971). For cracked bodies of finite dimension, exact solutions are very difficult to obtain; hence, Wilson and Yu (1979), Hellen and Cesari (1979) employed the finite element method to deal with these problems. The method is usually combined with the modified J -integral

theory provided by Wilson and Yu (1979). The other prevailing method employed by Emery et al. (1969), Nied (1983), Oliveira and Wu (1987), and Bahr and Balke (1987) is based on the concept of superposition; that is, the thermal loading is replaced by mechanical traction forces which are the same as equivalent internal force at the prospective crack face in the absence of crack.

Previous methods have a common disadvantage that the complicated finite element calculation must be repeated for the same cracked body subjected to different mechanical or thermal loading. Particularly, when these methods are applied in the transient thermal loading situation, a large amount of numerical calculation will be involved. The weight function method provides an alternative, yet more efficient methodology in the analysis of cracked bodies.

The weight function method, which was first proposed by Bueckner (1970), is a powerful and efficient method for determining the stress intensity factor for the mechanical loading system. In his formulation the weight function is the displacement of fundamental state induced by a self-equilibrating loading for which the singularity is one order higher than normal state. Rice (1972) proposed a convenient formulation for the determination of weight function, which then became the most prevailing form. The weight function is a universal function for a given crack geometry and composition and is independent of applied loading. If the weight function is obtained from a simple loading case, it can then be used to calculate stress intensity factors for any other complicated loading system of the same cracked geometry.

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An edge crack in a circular disk subjected to an arbitrarily distributed loading was analyzed using the weight function method by Wu (1991). For the single-edge cracked circular disk, the exact solution of the stress intensity factor for load case of uniform crack face loading obtained by Gregory (1977, 1979, 1989) was used as the reference solution for deriving the mechanical weight function. Analytical formulation for the computation of stress intensity factors for arbitrary loading have been presented by Wu (1991); the results are very accurate for any relative crack length ranging from zero to unity. The crack face weight function was also applied to determine the transient stress intensity factors in a thermal shock condition by using the known result of the thermal stress in a circular cylinder by Wu (1991).

In recent years, the finite element method applied to fracture mechanics has been well developed. There are several studies that have sought to improve the calculation technique and provide a possible and efficient way to construct the weight functions for finite cracked bodies. Sha (1984) used the stiffness derivative technique coupled with singular crack tip elements to determine the weight functions, and he obtained the weight function for a single-edge crack with specified specimen width and length by means of the finite element method. Sha and Yang (1985) obtained the weight function for an oblique edge crack by means of the finite element method using the virtual crack extension technique as suggested by Parks (1974) and Hellen (1975). They have extended this method to non-symmetric mixed mode problems and used a special symmetric mesh in the vicinity of the crack tip such that the stress intensity factors for modes I and II could be determined independently. Recently, Tsai and Ma (1989), Ma et al. (1990) constructed the explicit form of the mixed mode crack face weight function for finite rectangular plates by using the finite element and curve-fitting techniques. These explicit weight functions are expressed in terms of a position coordinate, crack length, specimen width and length, which are certainly more useful in practical applications. In a study by Tsai and Ma (1992), the formulation of the thermal weight function is derived from the thermoelastic Betti reciprocal theorem. It was found that the rate of change of the mean stress with respect to the crack length of any arbitrary loading system is the thermal weight function.

In this paper, an efficient finite element method with singular elements for evaluation of the weight functions for hollow circular cylinder has been established by coupling the virtual crack extension technique. Explicit mechanical weight functions are expressed in terms of the nondimensional quantities of position coordinate x/a , crack length $a/(R_o - R_i)$, and geometrical configuration R_i/R_o , where R_i and R_o denote the inner and outer radius of the hollow cylinder, respectively. These explicit weight functions are then used to calculate the stress intensity factors and compare them with some known results. The thermal weight functions of a hollow circular cylinder for axial cracks at the internal or external wall are presented. The thermal weight function method presented in this paper is more convenient than the modified J -integral method (Wilson and Yu, 1979) or the superposition method (Oliveira and Wu, 1987) to evaluate the stress intensity factor of cracked bodies subjected to thermal loading. The formulation of the thermal weight function will be demonstrated first, and some examples will be studied to validate the accuracy and efficiency of this method. Particularly, stress intensity factors of hollow cylinders subjected to thermal shock are discussed in detail.

2 Weight Function Formulation

The weight function method as presented by Bueckner (1970) and Rice (1972) was used to compute mode I stress intensity factors for symmetric cracked bodies with symmetric loading.

Fig. 1 Configuration of a cracked body subjected to different loading sources

Following Bowie and Freese (1981) and Bortman and Banks-Sills (1983), Tsai and Ma (1992) extended Rice's displacement derivative definition of the weight function for mode I deformation to mixed fracture mode and mixed boundary conditions of applying mechanical and thermal loadings. Consider the geometrical configuration as shown in Fig. 1. There are different sorts of loading types, such as prescribed traction t^* at boundary S_t , prescribed displacement u^* at boundary S_u , body force f^* , and temperature θ^* at domain A . Stress intensity factors are expressed as a product of the applied load and the associated weight function as follows:

$$K_I = \int_{S_t} t^* \cdot h_I^t dS + \int_{S_u} u^* \cdot h_I^u dS + \int_A f^* \cdot h_I^f dA + \int_A \theta^* h_I^\theta dA \quad (1)$$

$$K_{II} = \int_{S_t} t^* \cdot h_{II}^t dS + \int_{S_u} u^* \cdot h_{II}^u dS + \int_A f^* \cdot h_{II}^f dA + \int_A \theta^* h_{II}^\theta dA \quad (2)$$

The different sorts of weight functions are denoted as

$$h_I^t = \frac{H}{2K} \left(K_{II}^{(2)} \frac{\partial u^{(1)}}{\partial a} - K_I^{(1)} \frac{\partial u^{(2)}}{\partial a} \right) \quad (3)$$

$$h_I^u = -\frac{H}{2K} \left(K_{II}^{(2)} \frac{\partial t^{(1)}}{\partial a} - K_I^{(1)} \frac{\partial t^{(2)}}{\partial a} \right) \quad (4)$$

$$h_I^f = \frac{H}{2K} \left(K_{II}^{(2)} \frac{\partial u^{(1)}}{\partial a} - K_I^{(1)} \frac{\partial u^{(2)}}{\partial a} \right) \quad (5)$$

$$h_I^\theta = \frac{\alpha H}{2K} \left(K_{II}^{(2)} \frac{\partial \sigma_{kk}^{(1)}}{\partial a} - K_I^{(1)} \frac{\partial \sigma_{kk}^{(2)}}{\partial a} \right) \quad (6)$$

for mode I, and

$$h_{II}^t = \frac{H}{2K} \left(K_I^{(1)} \frac{\partial u^{(2)}}{\partial a} - K_{II}^{(2)} \frac{\partial u^{(1)}}{\partial a} \right) \quad (7)$$

$$h_{II}^u = -\frac{H}{2K} \left(K_I^{(1)} \frac{\partial t^{(2)}}{\partial a} - K_{II}^{(2)} \frac{\partial t^{(1)}}{\partial a} \right) \quad (8)$$

$$h_{II}^f = \frac{H}{2K} \left(K_I^{(1)} \frac{\partial u^{(2)}}{\partial a} - K_{II}^{(2)} \frac{\partial u^{(1)}}{\partial a} \right) \quad (9)$$

$$h_{II}^\theta = \frac{\alpha H}{2K} \left(K_I^{(1)} \frac{\partial \sigma_{kk}^{(2)}}{\partial a} - K_{II}^{(2)} \frac{\partial \sigma_{kk}^{(1)}}{\partial a} \right) \quad (10)$$

for mode II, in which

$$K = K_I^{(1)} K_{II}^{(2)} - K_I^{(2)} K_{II}^{(1)} \quad (11)$$

where $H=E$ (Young's modulus) for generalized plane stress and $H=E/(1-\nu^2)$ for plane strain, ν being Poisson's ratio. While α is the coefficient of thermal expansion, $\sigma_{kk} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ is the sum of normal stresses. The first three terms ((3)-(5)) are known as the traction, displacement, and body force mode I weight function, respectively; the last term (Eq. (6)) is known as the thermal mode I weight function. Configurations (1) and (2) are geometrically equivalent to the original problem and are subjected to arbitrary loadings. The corresponding stress intensity factors and displacement vectors will be denoted by $K_I^{(1)}$, $\mathbf{u}^{(1)}$ and $K_I^{(2)}$, $\mathbf{u}^{(2)}$, respectively. Once the weight functions ((3)-(8)) are determined from any particular load system, the stress intensity factor induced by any other load system can be obtained from Eqs. (1) and (2). In this paper, only the mechanical loading (traction boundary condition) and thermal loading are considered for pure mode I deformation. If only mechanical loading is considered and is restricted to mode I loading, the mode I stress intensity factor can be written as

$$K_I = \int_{S_I} t^* h' dS \quad (12)$$

where

$$h' = \frac{H}{2K_I^{(1)}} \frac{\partial u_y^{(1)}}{\partial a} \quad (13)$$

The weight functions are defined for the boundary S_I of the cracked body as indicated in (12). However, by employing the linear superposition principle, only the weight functions along the prospective crack-face are of primary interest when evaluating the stress intensity factor. If thermal loading is considered and only induced mode I deformation, then (1) becomes

$$K_I = \int_A \Theta^* h^\theta dA \quad (14)$$

where

$$h^\theta = \frac{\alpha H}{2K_I^{(1)}} \frac{\partial \sigma_{kk}^{(1)}}{\partial a} \quad (15)$$

Since the mean stress rate ($\partial \sigma_{kk}/\partial a$) is difficult to evaluate by the finite element method, we reformulate (14) and (15) by Green's theorem to an alternative form which can be obtained more easily and accurately by the finite element method, which yields

$$K_I = \frac{\alpha H'}{2K_I^{(1)}} \left[\int \Theta^* \left(\frac{\partial u_x^{(1)}}{\partial a} dy - \frac{\partial u_y^{(1)}}{\partial a} dx \right) - \iint \left(\frac{\partial \Theta^*}{\partial x} \frac{\partial u_x^{(1)}}{\partial a} + \frac{\partial \Theta^*}{\partial y} \frac{\partial u_y^{(1)}}{\partial a} \right) dx dy \right] \quad (16)$$

where $H' = E^2/(1-\nu)$ for plane stress and $H' = E^2/[1-2\nu(1-\nu^2)]$ for plane strain.

Usually, the formulation (14) is more suitable for analytical analysis of the thermal weight function, and (16) is especially convenient for the finite element calculation of finite cracked bodies. The formulation shown in (16) reveals a significant physical meaning, that the stress intensity factor is influenced by the temperature at boundary and heat flux in the entire domain. This expression can also be obtained from the body force analogy method which transfer the thermal loading to mechanical traction and body force.

The values of $u_x^{(1)}$ and $u_y^{(2)}$ in (13) and (16) are the displacement components along the x and y axes, respectively. As indicated in (13) and (16), the determination of the weight functions for the cracked body of interest requires both the stress intensity

factor and the displacement derivatives, established for any arbitrary loading system applied to the same cracked body. Unfortunately, the exact solution of the displacement field is available only for very few ideal crack problems. An efficient finite element method for determining the weight functions for cracked bodies of interest has been achieved in this study by combining the singular crack tip elements with the virtual crack extension technique.

3 Weight Functions for Mechanical Loadings

By the principle of superposition, the stresses in a cracked body subjected to external loading can be separated into two parts: (a) the stresses that would arise if the cracks were not present, and (b) the stresses that nullify the unwanted tractions on the prospective crack faces in (a). Since the stresses in (a) are finite, only the stresses in (b) give rise to the stress intensity factors, which are amplitudes of the $r^{-1/2}$ singularity at the crack tips. In other words, the stress intensity factor under crack face loading is equivalent to loading the cracked body with remote loading that produces the same pressure loading on the prospective crack face in the absence of the crack. Hence, only the weight functions on the crack faces are of primary interest for evaluating the stress intensity factors. The weight function of interest in the present work for axial cracks in hollow cylinder is restricted to mode I loadings. Once the weight functions are determined for a given geometry, then the stress intensity factor for any loading system applied to the same cracked body can be calculated very efficiently by

$$K_I = \int_0^a t^* h'(x, a) dx \quad (17)$$

where the crack face weight function $h'(x, a)$ is represented as

$$h'(x, a) = \frac{H}{K_I^{(1)}} \frac{\partial u_y^{(1)}}{\partial a} \quad (18)$$

Once the stress intensity factor $K_I^{(1)}$ and the crack opening displacement $u_y^{(1)}$ are established for a simple reference problem, the stress intensity factor for any other loading of the cracked body can be easily calculated by integration. Although the evaluation of Eq. (17) is a straightforward integration, this formula requires preliminary work to establish the appropriate values of the derivatives $\partial u_y^{(1)}/\partial a$. Unfortunately, the exact solution of the crack face displacement is available only for very few crack problems. Gregory (1977, 1979, 1989) gave exact solutions of the single-edge cracked circular disk for several types of loadings by using the Wiener-Hopf technique. To evaluate the weight functions for important engineering problems of cracked body with finite dimensions, the main difficulty arises in the determination of the crack opening displacement field corresponding to a given loading. Conceivably, the crack face weight functions, which depend on geometry alone, can be accurately obtained by means of finite element analysis from a relatively simple loading condition.

The existing weight functions were usually derived for particular geometry or simple geometry and loading. In this paper, we will construct the explicit form of the mode I crack face weight function for axial cracks in a hollow cylinder by using the finite element and curve-fitting technique. These explicit weight functions are expressed in terms of the nondimensional quantities of position coordinate x/a , crack length $a/(R_O - R_I)$ and hollow cylinder geometry R_I/R_O , where R_I and R_O are radii for the inner and outer walls, respectively. The crack geometries considered are single-axial crack at the inner or outer wall; the specimen configurations are shown in Fig. 2. These explicit weight functions are then used to calculate the stress intensity factors and compare them with some known results.

Fig. 2 Geometry and coordinate definitions for hollow cylinder with axial cracks in (a) outer wall, (b) inner wall

In order to determine the weight function, a reference stress intensity factor for any stress system together with the corresponding crack opening displacement field must be known. The main difficulty with the method is the determination of the crack opening displacement field. Exact solutions are available for only a very few crack problems. The only exact solution is given by Gregory (1977) for the single-edge cracked circular disk subjected to a constant internal pressure; exact expressions were obtained for the stress intensity factor, total strain energy, and the opening at the mouth of the crack. This exact solution was used as the reference solution for deriving the weight function by Wu (1991). For most cases that are of practical interest, i.e., cracks in finite bodies, exact solutions hardly exist at all. Apparently, good approximations to the displacement field for such cases are very desirable from the engineering point of view. Therefore, Tsai and Ma (1989) have proposed an approximation for the crack opening displacement field for an edge crack in plates with finite width and length. Following a similar consideration provided by Tsai and Ma (1989), we also assume that the crack opening displacement of an axial crack in a hollow cylinder has the following form:

$$u_y(a, x) = u_0(a/(R_0 - R_1), R_1/R_0) \sqrt{a} \sqrt{a-x} + u_1(a/(R_0 - R_1), R_1/R_0) \frac{(a-x)^{3/2}}{\sqrt{a}} \quad (19)$$

Table 1 The interpolation coefficients C_{mn} of external crack face weight functions for a hollow cylinder

$C_{mn}(A_1)$	1	2	3	4	5	6
1	.4519940E+0	.0000000E+0	.0000000E+0	.0000000E+0	.0000000E+0	.0000000E+0
2	.9585456E+2	-.9952838E+3	.3977954E+4	-.7699939E+4	.7222599E+4	-.2632472E+4
3	-.9032765E+3	.9491210E+4	-.3803427E+5	.7377713E+5	-.6934246E+5	.2532853E+5
4	.2245826E+4	-.2351598E+5	.9425496E+5	-.1828611E+6	.1720286E+6	-.6291329E+5
5	-.1638583E+4	.1715131E+5	-.6871587E+5	.1333148E+6	-.1255140E+6	.4596778E+5

$C_{mn}(A_2)$	1	2	3	4	5	6
1	.2410970E+0	.0000000E+0	.0000000E+0	.0000000E+0	.0000000E+0	.0000000E+0
2	-.1108925E+3	.1167713E+4	-.4635839E+4	.8886143E+4	-.8247202E+4	.2974238E+4
3	.1066751E+4	-.1108958E+5	.4405219E+5	-.8455044E+5	.7858280E+5	-.2838133E+5
4	-.2576936E+4	.2669594E+5	-.1059510E+6	.2032524E+6	-.1887895E+6	.6812816E+5
5	.1816732E+4	-.1875821E+5	.7431760E+5	-.1423464E+6	.1319997E+6	-.4752997E+5

Table 2 The interpolation coefficients C_{mn} of internal crack face weight functions for a hollow cylinder

$C_{mn}(A_1)$	1	2	3	4	5	6
1	.4519940E+0	.0000000E+0	.0000000E+0	.0000000E+0	.0000000E+0	.0000000E+0
2	.5179675E+2	-.6243824E+3	.2638403E+4	-.5276902E+4	.5086739E+4	-.1903685E+4
3	-.7135086E+3	.7769903E+4	-.3192930E+5	.6331715E+5	-.6080709E+5	.2270333E+5
4	.2969651E+4	-.3142925E+5	.1279758E+6	-.2524830E+6	.2419222E+6	-.9024775E+5
5	-.4838370E+4	.5064577E+5	-.2053626E+6	.4038616E+6	-.3861562E+6	.1439091E+6
6	.2698282E+4	-.2803446E+5	.1134037E+6	-.2224781E+6	.2122450E+6	-.7895822E+5

$C_{mn}(A_2)$	1	2	3	4	5	6
1	.2410970E+0	.0000000E+0	.0000000E+0	.0000000E+0	.0000000E+0	.0000000E+0
2	.1517968E+2	-.1754828E+3	.7478605E+3	-.1501785E+4	.1449548E+4	-.5422346E+3
3	-.1531806E+3	.1758374E+4	-.7438711E+4	.1498062E+5	-.1455010E+5	.5491137E+4
4	.3958849E+3	-.4795574E+4	.2056246E+5	-.4165209E+5	.4065716E+5	-.1546638E+5
5	-.3876289E+3	.5064261E+4	-.2201072E+5	.4473825E+5	-.4363964E+5	.1665775E+5
6	.1127650E+3	-.1749772E+4	.7705382E+4	-.1560334E+5	.1502570E+5	-.5675550E+4

Fig. 3 Comparison between the calculated and curve-fitting crack face weight function: (a) $a/(R_0 - R_1) = 0.5$ and different R_1/R_0 ratio, (b) $R_1/R_0 = 0.5$ and different $a/(R_0 - R_1)$ ratio for external axial cracks, (c) $a/(R_0 - R_1) = 0.5$ for different R_1/R_0 ratio, (d) $R_1/R_0 = 0.5$ and different $a/(R_0 - R_1)$ ratio for internal axial cracks

where x is measured from the crack mouth and $x=a$ is the crack tip. The first term is associated with the near-tip singular field and the second term is the first-order correction for larger distances from the crack tip. The crack face weight function for a single-edge crack as a function of distance can be shown to take the following form:

$$\sqrt{ah'}(x/a) = \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{1-x/a}} + A_1(a/(R_0 - R_1), R_1/R_0) \times \sqrt{1-x/a} + A_2(a/(R_0 - R_1), R_1/R_0)(1-x/a)^{3/2} \quad (20)$$

Coefficients $A_1(a/(R_0 - R_1), R_1/R_0)$ and $A_2(a/(R_0 - R_1), R_1/R_0)$ which are functions of crack length a , hollow cylinder thickness $R_0 - R_1$, and ratio of radius R_1/R_0 , can be obtained by finite element analysis and least-squares fitting procedures. It is noted that the general form of the weight function in Eq. (20) is valid for inner and outer axial cracks.

The numerical results of the weight function are calculated by using the virtual crack extension technique. Several different crack geometries are considered in the numerical calculation. The numerical results form the basic data for curve-fitting. By using least-squares procedures, the discretized values of the crack face weight functions are approximated with Eq. (20). The coefficients A_1 and A_2 in Eq. (20) are represented by the following polynomial form as:

$$A_i(a/(R_0 - R_1), R_1/R_0) = \sum_{m=1}^{6.5} \sum_{n=1}^6 C_{mn} \left(\frac{a}{R_0 - R_1} \right)^{(n-1)} \left(\frac{R_1}{R_0} \right)^{(m-1)}, \quad i = 1 \text{ or } 2 \quad (21)$$

The tabulation of the coefficients C_{mn} for different crack

Fig. 4 Comparison of normalized stress intensity factors calculated from the weight function method and Andrasic and Park (1984) for applying uniform loading at crack faces of (a) external crack, (b) internal crack

geometries are given in Tables 1 and 2. The weight functions of four crack geometries for $a/(R_0 - R_1) = 0.5$ and $R_1/R_0 = 0.5$ are plotted in Figs. 3(a)–(d). In these figures, the actual nodal weight functions obtained from finite element analysis are represented by discrete points, and those calculated from the fitted nodal weight functions are plotted as solid lines. Close agreement between these two results shows that the fitted weight functions of Eq. (20) are good approximations to the actual FEM results. At the crack tip (i.e., $x/a = 1$), the singular behavior of the weight function along the crack face is shown clearly in the figures.

Any error in calculating $K_I^{(1)}$ and $\partial u_y^{(1)}/\partial a$ by means of the finite element and curve-fitting methods will contribute to the inaccuracy of the weight function evaluations. The error from the finite element method calculations is related to the mesh arrangement and the value of the virtual crack extension da of the crack length. A da/a that is equal to 10^{-5} is regarded as good for the numerical results and is used in this paper. The stress intensity factors with less than 0.3 percent error are obtained and the error that results from the process of curve-fitting by means of the least-squares method is less than 1 percent in general.

By combining the uncracked stress fields with the explicit weight functions obtained, accurate stress intensity factors for different loading systems can be economically estimated according to Eq. (17). In order to demonstrate the accuracy of the weight functions obtained in determining the stress intensity factors, several crack problems have been considered. The stress intensity factors calculated for these problems by means of the present weight functions are shown in Fig. 4. The maximum difference in the stress intensity factors obtained by the weight function technique are less than 1 percent when compared with the results of Andrasic and Parker (1984). We have also obtained the stress intensity factors for applying uniform pressure P_0 in inner and outer walls and the results are shown in Fig. 5. It is noted that the explicit form of the mechanical weight function obtained in this section is valid for the range $0 < a/(R_0 - R_1) \leq 0.8$ and $0.3 \leq R_1/R_0 \leq 0.8$, which covers a large range of hollow cylindrical geometry for practical use. The similar problem was also studied by Tracy (1979) and Delale and Erdogan (1982). Tracy (1979) obtained the solution for one to four radial cracks emanating from either the inner or outer surface of a circular ring by using the modified mapping collocation method. Delale and Erdogan (1982) gave the results for a hollow cylinder or a disk containing a radial crack by using the basic dislocation solutions as Green's functions; the crack may be an external edge, an internal edge crack, or an embedded crack.

Fig. 5 Normalized stress intensity factors calculated from the weight function method for applying uniform pressure at (a) inner wall, (b) outer wall

Fig. 6 Temperature distribution of a hollow cylinder with an external axial crack subjected to prescribed temperature at boundary and crack faces

4 Weight Functions for Thermal Loadings

In the analysis of a finite cracked body, the numerical method of finite elements is used to construct the thermal weight function, and the formulation expressed in (16) is used. We consider first a hollow cylinder with an external axial crack, subjected to uniform temperature distribution Θ_i in the inner wall and Θ_o in the outer wall, as shown in Fig. 2.. The temperature in the crack faces is assumed to be the same as that on the external wall. The temperature distribution is solved by using the finite element method. Since the singular behavior $r^{-1/2}$ for the temperature gradient in the vicinity of the crack tip is predominant in this problem, the quarter-point quadratic isoparametric elements are used in the crack tip region. The remaining parts are surrounded by standard eight-noded quadratic isoparametric elements. The temperature distribution of the cracked hollow cylinder is displayed in Fig. 6. The numerical results for thermal weight functions are shown in Fig. 7. Then from (16), the stress intensity factor can be evaluated directly by numerical integration. The solutions of the stress intensity factors obtained by the thermal weight function for different crack length are shown in Table 3, which are all negative values. The stress intensity factors are normalized with respect to $E\alpha\Delta\Theta\sqrt{W/(1-\nu)}$, where $W=R_o-R_i$ is the wall thickness of the cylinder. The next problem to be considered is an internal crack in a hollow cylinder. The temperature distribution is displayed in Fig. 8 and the thermal weight functions are shown in Fig. 9. The stress intensity factors for different crack lengths, shown in Table 4, are all positive values. From Tables 3 and 4, it can be concluded that if the temperature in the outer wall is higher than that in the inner wall, inner cracks will open

Fig. 7 Numerical results of (a) $(E(K_I^2/(1-\nu^2)))\partial u_r/\partial a$, (b) $(E(K_I^2/(1-\nu^2)))\partial u_y/\partial a$ for a hollow cylinder with an external axial crack

Table 3 Stress intensity factors of an external crack for different crack lengths subjected to prescribed temperature at boundary and crack faces

a/W	$K_I(1-\nu)/(E\alpha\Delta\Theta\sqrt{W})$
0.1	-.2314
0.2	-.3269
0.3	-.4040
0.4	-.4759
0.5	-.5402

and might propagate so that inner cracks are much more dangerous than outer cracks.

Finally, we would like to investigate the axial cracks in a hollow cylinder subject to thermal shock. The cracking of a brittle material due to thermal shock has been recognized as an important fracture phenomenon in cases such as turbines, combustion chambers, or pressure vessels subjected to abnormal operation or workpieces under the quenching process. Sometimes the transient, thermally induced stress singular behavior is strong enough to fracture the structure. The existing results related to cracks subjected to thermal shock are rather limited. This is probably due to the fact that the calculation must employ the finite element method, which can be time-consuming and expensive for the analysis. The thermal weight function method will be used in this study, which will provide a more efficient methodology in the analyses of thermal shock problems.

We consider a hollow cylinder which is initially at a uniform temperature Θ_o . At time $t=0$, the temperature at the inner wall is suddenly changed to Θ_i while the temperature at the outer wall is kept at Θ_o . We assume that the presence of the crack does not affect the temperature distribution for simplicity. This implies that the transient temperature distribution is independent of crack geometry and is the same for all crack

Fig. 11 Transient stress intensity factors due to thermal shock for external crack

In this study, the efficient weight function method is used to calculate the stress intensity factors for a hollow cylinder with an axial crack subjected to mechanical or thermal loadings. Explicit weight functions for applying mechanical loadings are presented in terms of the nondimensional quantities of geometric parameters, which are very useful in practical applications. The weight function for mechanical loading is now a common method to determine the stress intensity factor. However, the thermal weight function concept is not widely used yet and it is proved that the method is more convenient and efficient than other methods used in the literature. If the temperature distribution is known, then the stress intensity factor of a cracked body can be obtained by a simple integration over the predetermined thermal weight function. In the previous study of thermal transient problem, the maximum stress intensity factor may occur during the transient period, so that the analysis of the transient problem is necessary. By the concept of the thermal weight function, the time consumed for the numerical calculation of the transient problem can be reduced at every transient time step.

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References

- Andrasic, C. P., and Parker, A. P., 1984, "Dimensionless Stress Intensity Factors for Cracked Thick Cylinders under Polynomial Crack Face Loadings," *Engineering Fracture Mechanics*, Vol. 19, pp. 187-193.
- Bahr, H.-A., and Balke, H., 1987, "Fracture Analysis of a Single Edge Cracked Strip under Thermal Shock," *Theoretical and Applied Fracture Mechanics*, Vol. 8, pp. 33-39.
- Bortman, Y., and Banks-Sills, L., 1983, "An Extended Weight Function Method for Mixed-Mode Elastic Crack Analysis," *ASME Journal of Applied Mechanics*, Vol. 50, pp. 907-909.
- Bowie, O. L., and Freese, C. E., 1981, "Cracked-Rectangular Sheet with Linearly Varying End Displacements," *Engineering Fracture Mechanics*, Vol. 14, pp. 519-526.
- Bueckner, H. F., 1970, "A Novel Principle for the Computation of Stress Intensity Factors," *ZAMM*, Vol. 50, pp. 529-546.
- Carlsaw, H. S., and Jaeger, J. C., 1950, *Conduction of Heat in Solids*, Oxford University Press, U.K.
- Delale, F., and Erdogan, F., 1982, "Stress Intensity Factors in a Hollow Cylinder Containing a Radial Crack," *International Journal of Fracture*, Vol. 20, pp. 251-265.
- Emery, A. F., Walker, G. E., and Williams, J. A., 1969, "A Green Function for the Stress Intensity Factors of Edge Cracks and Its Application to Thermal Stress," *ASME Journal of Applied Mechanics*, Vol. 36, pp. 618-624.
- Florence, A. L., and Goodier, J. N., 1960, "Thermal Stresses Due to Disturbance of Uniform Heat Flow by an Insulated Ovaloid Hole," *ASME Journal of Applied Mechanics*, Vol. 27, pp. 635-639.
- Florence, A. L., and Goodier, J. N., 1963, "The Linearly Thermoelastic Problem of Uniform Heat Flow Disturbed by a Penny-Shaped Crack," *International Journal of Engineering Science*, Vol. 1, pp. 533-540.
- Gregory, R. D., 1977, "A Circular Disc Containing a Radial Edge Crack Opened by a Constant Internal Pressure," *Mathematical Proceedings of Cambridge Philosophical Society*, Vol. 81, pp. 497-521.
- Gregory, R. D., 1979, "The Edge-Cracked Circular Disc Under Symmetric Pin-Loading," *Mathematical Proceedings of Cambridge Philosophical Society*, Vol. 85, pp. 523-538.
- Gregory, R. D., 1989, "The Spinning Circular Disc With a Radial Edge Crack; An Exact Solution," *International Journal of Fracture*, Vol. 41, pp. 39-50.
- Hellen, T. K., 1975, "On the Method of Virtual Crack Extensions," *International Journal of Numerical Methods in Engineering*, Vol. 9, pp. 187-207.
- Hellen, T. K., and Cesari, F., 1979, "On the Solution of the Center Cracked Plate with a Quadratic Thermal Gradient," *Engineering Fracture Mechanics*, Vol. 12, pp. 469-478.
- Kassir, M. K., and Sih, G. C., 1968, "Thermal Stress in a Solid Weakened by an External Circular Crack," *International Journal of Solids and Structures*, Vol. 5, pp. 351-367.
- Kassir, M. K., and Bregman, A. M., 1971, "Thermal Stress in a Solid Containing Parallel Circular Crack," *Applied Scientific Research*, Vol. 25, pp. 262-280.
- Ma, C. C., Chang, Z., and Tsai, C. H., 1990, "Weight Functions of Oblique Edge and Center Cracks in Finite Bodies," *Engineering Fracture Mechanics*, Vol. 36, pp. 267-285.
- Nied, H. F., 1983, "Thermal Shock Fracture in an Edge-Cracked Plate," *Journal of Thermal Stress*, Vol. 6, pp. 217-229.
- Olesiak, Z., and Sneddon, I. N., 1959, "The Distribution of Thermal Stress in an Infinite Elastic Solid Containing a Penny-Shaped Crack," *Archives of Rational Mechanics Analysis*, Vol. 4, pp. 238-254.
- Oliveira, R., and Wu, X. R., 1987, "Stress Intensity Factor for Axial Cracks in Hollow Cylinders Subjected to Thermal Shock," *Engineering Fracture Mechanics*, Vol. 27, pp. 185-197.
- Parks, D. M., 1974, "A Stiffness Derivative Finite Element Technique for Determination of Crack Tip Stress Intensity Factors," *International Journal of Fracture*, Vol. 10, pp. 487-501.
- Rice, J. R., 1972, "Some Remarks on Elastic Crack-Tip Stress Field," *International Journal of Solids and Structures*, Vol. 8, pp. 751-758.
- Rooke, D. P., and Cartwright, D. J., 1976, *Compendium of Stress Intensity Factors*, Her Majesty's Stationery Office, London, U.K.
- Sha, G. T., 1984, "Stiffness Derivative Finite Element Technique to Determine Nodal Weight Functions with Singularity Elements," *Engineering Fracture Mechanics*, Vol. 19, pp. 685-699.
- Sha, G. T., and Yang, C. T., 1985, "Weight Function Calculations for Mixed-Mode Fracture Problems with the Virtual Crack Extension Technique," *Engineering Fracture Mechanics*, Vol. 21, pp. 1119-1149.
- Sih, G. C., 1962, "On the Singular Character of Thermal Stress Near a Crack Tip," *ASME Journal of Applied Mechanics*, Vol. 29, pp. 587-589.
- Sih, G. C., 1973, *Handbook of Stress Intensity Factors*, Institute of Fracture and Solid Mechanics, Lehigh University.
- Tada, H., Paris, P. C., and Irwin, G. R., 1973, *The Stress Analysis of Cracks Handbook*, Del Research Corporation, Hellertown, PA.
- Tracy, P. G., 1979, "Elastic Analysis of Radial Cracks Emanating from the Outer and Inner Surfaces of a Circular Ring," *Engineering Fracture Mechanics*, Vol. 11, pp. 291-300.
- Tsai, C. H., and Ma, C. C., 1989, "Weight Functions for Cracks in Finite Rectangular Plates," *International Journal of Fracture*, Vol. 40, pp. 43-63.
- Tsai, C. H., and Ma, C. C., 1992, "Thermal Weight Function of Cracked Bodies Subjected to Thermal Loading," *Engineering Fracture Mechanics*, Vol. 41, pp. 27-40.
- Wilson, W. K., and Yu, I. W., 1979, "The Use of the *J*-Integral in Thermal Stress Crack Problems," *International Journal of Fracture*, Vol. 15, pp. 377-387.
- Wu, X. R., 1991, "The Arbitrarily Loaded Single-Edge Cracked Circular Disc Accurate Weight Function Solutions," *International Journal of Fracture*, Vol. 49, pp. 239-256.