Antiplane Deformations for Anisotropic Multilayered Media by Using the Coordinate Transform Method

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Green's functions for anisotropic elastic multilayered media subjected to antiplane shear deformation are presented in this study. The antiplane shear deformation due to a concentrated shear force and screw dislocation in an arbitrary layer was investigated in detail. A linear coordinate transformation is introduced in this study to simplify the problem. The linear coordinate transformation reduces the anisotropic multilayered problem to an equivalent isotropic problem without complicating the geometry of the problem. Explicit analytical solutions were derived using the Fourier transform and the series expansion technique. The complete solutions for the multilayered problem consist only of the simplest solutions obtained from an infinite homogeneous medium with concentrated loadings. Numerical results for the full-field stress distribution in multilayered media subjected to a point body force are presented. These numerical results were compared with the solutions obtained by considering the multilayered medium as one layer with effective elastic constants determined from the averaged material constants of the multilayered medium. It is found that the shear stress τ_{yz} of the homogeneous one layer solution is a very good approximation of the result for the multilayered medium; however, the shear stress τ_{xz} in these two solutions has a large discrepancy due to the fact that τ_{xz} is discontinuous at the interfaces of the multilayered medium. [S0021-8936(00)01703-7]

1 Introduction

Antiplane shear deformations are the simplest of two-dimensional deformations that arise in anisotropic or isotropic elastic bodies. For the antiplane shear deformation, the displacement is parallel to the axial coordinate that is normal to the plane and is dependent only on the coordinates in the plane. Such deformation field characterized by a single axial displacement can be regarded as complementary to that of plane-strain deformation (e.g., [1]). The antiplane problem plays a useful role as a pilot problem that reveals simpler aspects of elasticity solutions. The antiplane problem of two dissimilar anisotropic wedges has been considered by Ma [2] and Ma and Hour [3] using the Mellin transform method. They showed that the stress and displacement fields have reduced dependence on the elastic constants. Retting [4] did the antiplane stress analysis of a cracked beam made of orthotropic material. Wu and Chiu [5] discussed interface cracks in anisotropic bimetals subjected to antiplane shearing. Ting [6] also provided many basic discussions and investigated some fundamental problems for anisotropic antiplane deformations. In addition, considerable attention has been paid to the analysis of antiplane shear deformation in nonlinear elasticity theory for isotropic solids, cf. Jiang and Knowles [7], Polignone and Horgan [8] and the references therein. A comprehensive review of antiplane shear for both linear and nonlinear elasticity is given by Horgan [9].

Analysis of anisotropic elasticity problems is often tedious due to the presence of many elastic constants. It is desirable to reduce the dependence on elastic constants through theoretical consider-

ations in advance of the analysis of a given boundary value problem. For anisotropic elasticity, Lekhnitskii's formulation ([10]) and Stroh's formulation ([11]) are the two widely used methods. The general solutions obtained by these methods showed that the antiplane anisotropic problem can be converted to a corresponding isotropic problem by properly changing the geometry of the original configuration and the tractions on the boundary. In other words, the anisotropic antiplane problem can be simplified to an isotropic problem with the aid of a suitable coordinate transformation. In the isotropic case, the displacement equilibrium equation becomes the Laplace equation which can be solved easily. For the anisotropic problem, however, the governing equilibrium equation is a general second-order partial differential equation with constant coefficients. In particular, a variety of coordinate transformations could be used to convert the general second-order partial differential equation to the Laplace equation. The properties of the coordinate transformations for antiplane deformations have been investigated by Ma [12] and Horgan and Miller [1]. An orthotropic transformation concept was introduced by Yang and Ma [13] to analyze the much complicated and difficult in-plane deformations for planar anisotropic solids.

Because of the rapid expansion in the use of structural components made of laminated materials, predictions of the behavior of multilayered media subjected to arbitrary loads are needed. The stress states at the interfaces of multilayered media are of particular interest because delamination may occur if allowable levels are exceeded. For multilayered anisotropic media, the problem becomes more complicated than that of homogeneous isotropic counterparts. Based on the mixed formulation of elasticity, Bufler [14] and Bahar [15] independently constructed the transfer matrix approach to solve multilayered media problems. A local/global stiffness matrix approach which is similar to the transfer matrix approach was employed by Pindera [16] and Pindera and Lane [17,18] to solve round punch contact problems of arbitrarily lami-

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nated media. The local/global stiffness matrix method is based on the systematic construction of a global stiffness matrix in terms of local stiffness matrices of the individual layers.

According to the classical Saint-Venant principle, the precise decay factor of the stress with distance from the region of applied load depends on the geometry of the body, the boundary conditions, the applied load and material properties. For layered media, this issue has been examined by Baxter and Horgan [19,20]. They showed that material inhomogeneity significantly affects the practical application of Saint-Venant's principle to sandwich structures. Wu and Chiu [21] have solved the problem of a semi-infinite multilayered monoclinic strip loaded by shear stress at the end by using eigenfunction expansion in conjunction with Betti's reciprocity theorem.

In the current study, an anisotropic elastic multilayered medium with n layers subjected to antiplane loading within an arbitrary layer is investigated. The material properties and the thickness in each layer are different. One of the objectives of this study is to develop an effective analytical methodology to construct the full-field solutions for this complicated problem. A general linear coordinate transformation is introduced in this study to simplify the problem. This linear coordinate transformation will simplify the governing equilibrium equation without complicating the boundary and interface continuity conditions. Based on this transformation, the original anisotropic multilayered problem is converted to an equivalent isotropic multilayered problem. The analytical solutions for the stresses and displacement obtained in this study are exact and are expressed in an explicit closed form. For a numerical example, a multilayered medium with 12 layers is discussed in detail. The stress distribution for the multilayered solution is compared with the homogeneous single layer solution by averaging the material constants of the multilayered medium.

2 Antiplane Shear Deformations of an Anisotropic Elastic Solid

In the absence of body forces, the equilibrium equations for the elastostatic problem are

$$\sigma_{ij,j} = 0, \quad (1)$$

where the repeated indices imply summation and a comma stands for differentiation. The generalized Hooke's law for an anisotropic, homogeneous, and linearly elastic solid is given by

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \quad (2)$$

where $\varepsilon_{kl} = 1/2(u_{k,l} + u_{l,k})$ denotes the infinitesimal strain tensor, and C_{ijkl} are the elastic stiffnesses satisfying the usual symmetry conditions. Due to the symmetry of C_{ijkl} , Eq. (2) can be rewritten as

$$\sigma_{ij} = C_{ijkl} u_{k,l}. \quad (3)$$

For a general anisotropic elastic material, an antiplane deformation or a plane deformation in general does not exist. For some special anisotropic materials possessing elastic stiffnesses C_{ijkl} which are written in a contracted notation in the form

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & \text{sym.} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}, \quad (4)$$

the plane and antiplane deformations will be uncoupled and such anisotropic elastic materials are capable of an antiplane deformation (see, e.g., Horgan and Miller [1], Ting [6]).

For two-dimensional problems, the Cartesian coordinate system is chosen such that the antiplane deformation is in the z -direction.

Let u , v , and w , respectively, represent the displacement components in the x , y , and z -direction of the Cartesian coordinate system. For antiplane shear deformations,

$$u = v = 0, \quad w = w(x, y), \quad (5)$$

and the relevant shear stresses are denoted by τ_{yz} and τ_{xz} . If the material constants have the form represented in Eq. (4), then the equilibrium equations in the x and y -directions are automatically satisfied, and the equilibrium equation in the z -direction can be written in terms of the displacement w as

$$C_{55} \frac{\partial^2 w}{\partial x^2} + 2C_{45} \frac{\partial^2 w}{\partial x \partial y} + C_{44} \frac{\partial^2 w}{\partial y^2} = 0. \quad (6)$$

Equation (6) is the governing equation for an anisotropic antiplane deformation problem, and is a homogeneous second-order partial differential equation for displacement w . The nonzero stresses are related to the displacement as follows:

$$\begin{aligned} \tau_{yz} &= C_{45} \frac{\partial w}{\partial x} + C_{44} \frac{\partial w}{\partial y}, \\ \tau_{xz} &= C_{55} \frac{\partial w}{\partial x} + C_{45} \frac{\partial w}{\partial y}, \\ \sigma_{zz} &= C_{35} \frac{\partial w}{\partial x} + C_{34} \frac{\partial w}{\partial y}. \end{aligned} \quad (7)$$

For monoclinic materials with the plane of symmetry $z=0$, $C_{34} = C_{35} = 0$ so that $\sigma_{zz} = 0$.

3 Linear Coordinate Transformation

The governing equation expressed in Eq. (6) is a general homogeneous second-order partial differential equation with constant coefficients. Such a linear partial differential equation can be transformed into the Laplace equation by a linear coordinate transformation (see, e.g., Horgan and Miller [1]). A special linear coordinate transformation is introduced as

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & \beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (8)$$

where $\alpha = -C_{45}/C_{44}$, $\beta = \mu^e/C_{44}$ and $\mu^e = \sqrt{C_{44}C_{55} - C_{45}^2}$. Assume that C_{44} and C_{55} as well as $\sqrt{C_{44}C_{55} - C_{45}^2}$ are all positive. After the coordinate transformation, Eq. (6) can be rewritten as the standard Laplace equation in the (X, Y) coordinate system

$$\mu^e \left(\frac{\partial^2 w}{\partial X^2} + \frac{\partial^2 w}{\partial Y^2} \right) = 0. \quad (9)$$

It is interesting to note that the mixed derivative disappears from Eq. (6). The relationships between the shear stresses in the two coordinate systems are given by

$$\begin{aligned} \tau_{yz}(x, y) &= \mu^e \frac{\partial w(X, Y)}{\partial Y} = \tau_{YZ}(X, Y), \\ \tau_{xz}(x, y) &= \beta \mu^e \frac{\partial w(X, Y)}{\partial X} - \alpha \mu^e \frac{\partial w(X, Y)}{\partial Y} \\ &= \beta \tau_{XZ}(X, Y) - \alpha \tau_{YZ}(X, Y). \end{aligned} \quad (10)$$

In a mathematical sense, Eqs. (6) and (7) are transformed to Eqs. (9) and (10) by the linear coordinate transformation expressed in Eq. (8), or in a physical sense, the governing Eq. (6) and the stress displacement relation (7) of an anisotropic antiplane problem are converted into an equivalent isotropic problem by properly changing the geometry of the body using the linear coordinate transformation, Eq. (8). The coordinate transformation in Eq. (8) has the following characteristics: (a) it is linear and continuous, (b) an anisotropic problem is converted to an isotropic problem after the transformation, and (c) there is no stretching and rotation in the

Fig. 1 Configuration and coordinates system of an anisotropic multilayered medium

horizontal direction. These important features offer advantages in dealing with straight boundaries and interfaces in the multilayered system discussed in the present study. The most interesting feature is that a straight line $(x_1, y_0), (x_2, y_0)$ that is parallel to the x -axis will remain a straight line $(X_1, Y_0), (X_2, Y_0)$ parallel to the X -axis after the transformation, and the length of the line will not change, i.e., $X_2 - X_1 = x_2 - x_1$. The relationship between the anisotropic problem and the corresponding isotropic problem in the polar coordinate system and more detailed discussions of this linear coordinate transformation were presented by Ma [12].

The linear coordinate transformation described by Eq. (8) can be used to solve the anisotropic antiplane problem for only a single material. However, for a multilayered anisotropic medium with straight interfaces shown in Fig. 1, a modification of the linear coordinate transformation will be introduced in the following section to transform the multilayered anisotropic problem to an equivalent multilayered isotropic problem.

4 Formulations for Multilayered Media Subjected to Interior Loadings

In the following sections, the Green's function for an anisotropic n -layered medium will be constructed. Consider an anisotropic n -layered medium with n layers subjected to a line of constant force f_z and a screw dislocation of magnitude b_z along the z -axis located in the m th layer. The displacement and shear stresses are independent of the z -axis and so we can consider this problem as a two-dimensional antiplane problem. In other words, the original problem can be simplified as a two-dimensional multilayered medium subjected to a point shear force and a screw dislocation shown in Fig. 1. The displacement equilibrium equation in each layer is expressed as

$$C_{55}^j \frac{\partial^2 w^j}{\partial x^2} + 2C_{45}^j \frac{\partial^2 w^j}{\partial x \partial y} + C_{44}^j \frac{\partial^2 w^j}{\partial y^2} = 0, \quad j = 1, 2, \dots, n. \quad (11)$$

The boundary conditions on the top and bottom surfaces of the layered medium are

$$\tau_{yz}^1(x, 0) = 0, \quad \tau_{yz}^n(x, h_n) = 0. \quad (12)$$

The jump conditions for the shear stress and displacement across the point loads within the m th layer are

$$\begin{aligned} \tau_{yz}^{m+}(x, h^+) - \tau_{yz}^{m-}(x, h^-) &= -f_z \delta(x), \\ w^{m+}(x, h^+) - w^{m-}(x, h^-) &= -b_z U(x), \end{aligned} \quad (13)$$

Fig. 2 Configuration and coordinates system for the multilayered medium after the linear coordinate transformation

where $\delta()$ is the delta function and $U()$ is the heaviside function. In Eq. (13), τ_{yz}^{m+} and τ_{yz}^{m-} indicate the shear stress above and below the plane of applied loadings in the m th layer; h^+ and h^- denote the position just above and below the applied loadings at $y = h$ (see Fig. 1). Application of the traction and displacement continuity conditions at the interface between the j th and $j+1$ th layer, yields

$$\begin{aligned} \tau_{yz}^j(x, h_j) &= \tau_{yz}^{j+1}(x, h_j), \\ w^j(x, h_j) &= w^{j+1}(x, h_j), \end{aligned} \quad j = 1, 2, \dots, n-1. \quad (14)$$

In order to maintain the geometry of the layered configuration, the linear coordinate transformation described in Eq. (8) is modified for each layer as follows:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & \alpha_j \\ 0 & \beta_j \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \sum_{k=1}^{j-1} h_k \begin{bmatrix} \alpha_k - \alpha_{k+1} \\ \beta_k - \beta_{k+1} \end{bmatrix}, \quad j = 1, 2, \dots, n. \quad (15)$$

Comparing with Eq. (8), the first term on the right-hand side of Eq. (15) retains exactly the same form while the second summation term becomes the modified term. The new coordinate transformation possesses the following characteristics: (a) no gaps or overlaps are generated along the interface and (b) no sliding and no mismatches occur along the interface. The geometric configuration in the transformed (X, Y) coordinate is shown in Fig. 2. Note that while the thickness of each layer is changed, the interfaces are parallel to the X -axis. Thus, the new geometric configuration is similar to the original problem.

The equilibrium equations in the transformed coordinate are governed by the standard Laplace equation expressed by

$$\mu_j^e \frac{\partial^2 w^j}{\partial X^2} + \mu_j^e \frac{\partial^2 w^j}{\partial Y^2} = 0. \quad (16)$$

The displacement w and the shear stress τ_{YZ} are continuous along the interfaces in the transformed coordinates,

$$\begin{aligned} w^j(X, H_j) &= w^{j+1}(X, H_j), \quad \tau_{YZ}^j(X, H_j) = \tau_{YZ}^{j+1}(X, H_j), \\ j &= 1, 2, \dots, n-1, \end{aligned} \quad (17)$$

where

$$H_j = \beta_j h_j + \sum_{k=1}^{j-1} (\beta_k - \beta_{k+1}) h_k.$$

The top and bottom surfaces are traction free and can be expressed as

$$\tau_{YZ}^1(X, 0) = 0, \quad \tau_{YZ}^n(X, H_n) = 0. \quad (18)$$

Fig. 3 The jump conditions for applied loadings at the m th layer

The jumps within the m th layer caused by the applied antiplane body force and screw dislocation are shown in Fig. 3, and the jump conditions are

$$\begin{aligned}\tau_{YZ}^{m+}(X, H^+) - \tau_{YZ}^{m-}(X, H^-) &= -f_z \delta(X - H'), \\ w^{m+}(X, H^+) - w^{m-}(X, H^-) &= -b_z U(X - H'),\end{aligned}\quad (19)$$

where

$$\begin{aligned}H' &= \alpha_m h + \sum_{k=1}^{m-1} (\alpha_k - \alpha_{k+1}) h_k, \\ H &= \beta_m h + \sum_{k=1}^{m-1} (\beta_k - \beta_{k+1}) h_k.\end{aligned}$$

Here τ_{YZ}^{m+} and τ_{YZ}^{m-} indicate the shear stress above and below the applied loadings, respectively, in the m th layer (see Fig. 3). The location of the applied loadings is shifted by an amount of H' in the horizontal direction. The stress displacement relations expressed in the (X, Y) coordinates within each layer become

$$\begin{aligned}\tau_{XZ}^j(X, Y) &= \mu_j^e \frac{\partial w^j(X, Y)}{\partial X}, \\ \tau_{YZ}^j(X, Y) &= \mu_j^e \frac{\partial w^j(X, Y)}{\partial Y}.\end{aligned}\quad (20)$$

The boundary value problem described by Eqs. (16)–(20) is similar to the multilayered problem for an isotropic material. Hence the linear coordinate transformation presented in Eq. (15) changes the original anisotropic multilayered problem to the corresponding isotropic multilayered problem with a similar geometric configuration and boundary conditions. In the next section, the boundary value problem described by Eqs. (16)–(20) will be solved, and the relationships between the shear stresses in the (x, y) and (X, Y) coordinates established in Eq. (10) will be used to obtain the solutions for the original anisotropic multilayered problem.

5 Fundamental Solutions in the Transformed Domain

The boundary value problem of the previous section can be solved by the integral transform technique. The expressions for the field variables will be found by applying a Fourier transform over the spatial coordinate X with parameter ω . Take the Fourier transform pairs defined as

$$\begin{aligned}\tilde{g}(\omega, Y) &= \int_{-\infty}^{\infty} g(X, Y) e^{-i\omega X} dX, \\ g(X, Y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega, Y) e^{i\omega X} d\omega,\end{aligned}$$

and apply to the governing Eq. (16). Then, Eq. (16) becomes an ordinary differential equation with the following general solution in the Fourier transformed domain:

$$\begin{bmatrix} \tilde{w}^j \\ \tilde{\tau}_{YZ}^j \end{bmatrix} = \begin{bmatrix} e^{\omega Y} & e^{-\omega Y} \\ \mu_j^e \omega e^{\omega Y} & -\mu_j^e \omega e^{-\omega Y} \end{bmatrix} \begin{bmatrix} c_j \\ d_j \end{bmatrix}. \quad (21)$$

Here c_j and d_j are undetermined coefficients for each layer and can be obtained by applying the boundary, continuity and jump conditions. Substituting the general solution into the continuity conditions at the interfaces, the recurrence relation for the coefficients for each layer can be expressed as

$$\begin{bmatrix} c_{j+1} \\ d_{j+1} \end{bmatrix} = \frac{1}{s_j'} \mathbf{G}^+(j) \begin{bmatrix} c_j \\ d_j \end{bmatrix}, \quad (22)$$

or

$$\begin{bmatrix} c_j \\ d_j \end{bmatrix} = \frac{1}{s_j''} \mathbf{G}^-(j) \begin{bmatrix} c_{j+1} \\ d_{j+1} \end{bmatrix}, \quad (23)$$

where

$$\begin{aligned}\mathbf{G}^+(j) &= \begin{bmatrix} 1 & t_j e^{-2\omega H_j} \\ t_j e^{2\omega H_j} & 1 \end{bmatrix}, \\ \mathbf{G}^-(j) &= \begin{bmatrix} 1 & -t_j e^{-2\omega H_j} \\ -t_j e^{2\omega H_j} & 1 \end{bmatrix},\end{aligned}$$

$$t_j = \frac{\mu_{j+1}^e - \mu_j^e}{\mu_{j+1}^e + \mu_j^e}, \quad s_j' = \frac{2\mu_{j+1}^e}{\mu_j^e + \mu_{j+1}^e}, \quad s_j'' = \frac{2\mu_j^e}{\mu_j^e + \mu_{j+1}^e}.$$

Here t_j is called the reflection coefficient and s_j', s_j'' are the refraction coefficients. By applying the boundary condition for the top layer $\tilde{\tau}_{YZ}^1(\omega, 0) = 0$, $c_1 = d_1$ is obtained, and along with Eq. (22), the coefficients for the m th layer can be related to those of the first layer as follows:

$$\begin{bmatrix} c_m^+ \\ d_m^+ \end{bmatrix} = \prod_{k=1}^{m-1} \frac{1}{s_k'} \mathbf{G}^+(m-k) \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}, \quad (24)$$

where

$$\prod_{k=1}^n a_k = a_1 \cdot a_2 \cdots a_n.$$

Similarly, by applying the boundary condition for the bottom layer $\tilde{\tau}_{YZ}^n(\omega, H_n) = 0$, $c_n = d_n e^{-2\omega H_n}$ is obtained, and along with Eq. (23), the coefficients for the m th layer can be represented by

$$\begin{bmatrix} c_m^- \\ d_m^- \end{bmatrix} = \prod_{k=m}^{n-1} \frac{1}{s_k''} \mathbf{G}^-(k) \begin{bmatrix} d_n e^{-2\omega H_n} \\ d_n \end{bmatrix}. \quad (25)$$

Finally, the jump conditions in the m th layer in the transformed domain are employed

$$c_m^+ e^{\omega H} + d_m^+ e^{-\omega H} - c_m^- e^{\omega H} - d_m^- e^{-\omega H} = \frac{i b_z}{\omega} e^{-i\omega H'}, \quad (26)$$

$$c_m^+ e^{\omega H} - d_m^+ e^{-\omega H} - c_m^- e^{\omega H} + d_m^- e^{-\omega H} = \frac{-f_z}{\mu_m^e \omega} e^{-i\omega H'},$$

where $i = \sqrt{-1}$ and Eq. (26) can be written in a matrix form as

$$\begin{bmatrix} e^{\omega H} & e^{-\omega H} \\ e^{\omega H} & -e^{-\omega H} \end{bmatrix} \left\{ \begin{bmatrix} c_m^+ \\ d_m^+ \end{bmatrix} - \begin{bmatrix} c_m^- \\ d_m^- \end{bmatrix} \right\} = \begin{bmatrix} \frac{i b_z}{\omega} \\ \frac{-f_z}{\mu_m^e \omega} \end{bmatrix} e^{-i\omega H'}. \quad (27)$$

For convenience, set $c_m^+ = A_1 c_1$, $d_m^+ = A_2 c_1$, $c_m^- = B_1 d_n$ and $d_m^- = B_2 d_n$, where A_1 , A_2 , B_1 and B_2 are constants which can be expressed in matrix form as

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \prod_{k=1}^{m-1} \frac{1}{s_k'} \mathbf{G}^+(m-k) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \prod_{k=m}^{n-1} \frac{1}{s_k''} \mathbf{G}^-(k) \begin{bmatrix} e^{-2\omega H_n} \\ 1 \end{bmatrix}. \quad (28)$$

Therefore, Eq. (27) can be reformulated and coefficients c_1 and d_n are found as follows:

$$\begin{bmatrix} c_1 \\ d_n \end{bmatrix} = \frac{1/2}{A_1 B_2 - A_2 B_1} \begin{bmatrix} B_2 & -B_1 \\ A_2 & -A_1 \end{bmatrix} \begin{bmatrix} e^{-\omega H} & e^{-\omega H} \\ e^{\omega H} & -e^{\omega H} \end{bmatrix} \times \begin{bmatrix} \frac{i b_z}{\omega} \\ -f_z \\ \frac{\mu_m^e \omega}{\mu_m^e \omega} \end{bmatrix} e^{-i\omega H'} \quad (29)$$

The undetermined constants c_j and d_j for each layer are obtained with the aid of the recurrence relations given in Eqs. (22) and (23). After substituting the coefficients c_j and d_j into Eq. (21), the full-field solutions for each layer are completely determined in the transformed domain. Omitting the lengthy algebraic derivation, the general solutions for each layer are finally expressed as

$$\begin{bmatrix} \tilde{w}^j \\ \tilde{\tau}_{YZ}^j \end{bmatrix} = \frac{-e^{-i\omega H'}}{2(A_2 B_1 - A_1 B_2) \omega \mu_m^e} (B_2 e^{-\omega H} (f_z - i \mu_m^e b_z) + B_1 e^{\omega H} (f_z + i \mu_m^e b_z)) \cdot \begin{bmatrix} e^{\omega Y} & e^{-\omega Y} \\ \mu_j^e \omega e^{\omega Y} & -\mu_j^e \omega e^{-\omega Y} \end{bmatrix} \times \left\{ \prod_{k=1}^{j-1} \frac{1}{s_k'} \begin{bmatrix} 1 & t_{j-k} e^{-2\omega H_{j-k}} \\ t_{j-k} e^{2\omega H_{j-k}} & 1 \end{bmatrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (30)$$

for $1 \leq j \leq m$. And for $m \leq j \leq n$,

$$\begin{bmatrix} \tilde{w}^j \\ \tilde{\tau}_{YZ}^j \end{bmatrix} = \frac{-e^{-i\omega H'}}{2(A_2 B_1 - A_1 B_2) \omega \mu_m^e} (A_2 e^{-\omega H} (f_z - i \mu_m^e b_z) + A_1 e^{\omega H} (f_z + i \mu_m^e b_z)) \cdot \begin{bmatrix} e^{\omega Y} & e^{-\omega Y} \\ \mu_j^e \omega e^{\omega Y} & -\mu_j^e \omega e^{-\omega Y} \end{bmatrix} \times \left\{ \prod_{k=j}^{n-1} \frac{1}{s_k''} \begin{bmatrix} 1 & -t_k e^{-2\omega H_k} \\ -t_k e^{2\omega H_k} & 1 \end{bmatrix} \right\} \begin{bmatrix} e^{-2\omega H_n} \\ 1 \end{bmatrix}. \quad (31)$$

6 Full-Field Solutions and Their Physical Meaning

The complete solutions in the Fourier transformed domain for multilayered media have been presented in Eqs. (30) and (31) in the previous section. The solutions are so complicated that it is not easy to obtain the inverse Fourier transform. In order to construct the explicit analytical full-field solution of the n -layered medium, the Taylor series expansion is used in this study.

Because of the denominators in Eqs. (29)–(31), it is impossible to invert the Fourier transform directly. By examining the structure of the denominator of Eq. (29), both the numerator and denominator are multiplied by a constant $S = (\prod_{k=1}^{m-1} s_k') (\prod_{k=m}^{n-1} s_k'')$.

Then it becomes

$$\begin{bmatrix} c_1 \\ d_n \end{bmatrix} = \frac{S/2}{S(A_1 B_2 - A_2 B_1) \mu_m^e \omega} \begin{bmatrix} B_2 & -B_1 \\ A_2 & -A_1 \end{bmatrix} \begin{bmatrix} e^{-\omega H} & e^{-\omega H} \\ e^{\omega H} & -e^{\omega H} \end{bmatrix} \times \begin{bmatrix} i \mu_m^e b_z \\ -f_z \end{bmatrix} e^{-i\omega H'}. \quad (32)$$

The denominator in Eq. (32), $S(A_1 B_2 - A_2 B_1)$, can be decomposed into the form of $(1-p)$ where $p = 1 - S(A_1 B_2 - A_2 B_1)$. It can be shown that $p < 1$ for $\omega > 0$. By a series expansion, we obtain $1/(1-p) = \sum_{l=0}^{\infty} p^l$ so that Eq. (32) can be rewritten as

$$\begin{bmatrix} c_1 \\ d_n \end{bmatrix} = \frac{S}{2\omega} \begin{bmatrix} B_2 & -B_1 \\ A_2 & -A_1 \end{bmatrix} \begin{bmatrix} e^{-\omega H} & e^{-\omega H} \\ e^{\omega H} & -e^{\omega H} \end{bmatrix} \begin{bmatrix} i b_z \\ -f_z \end{bmatrix} \frac{e^{-i\omega H'}}{\mu_m^e} \cdot \sum_{l=0}^{\infty} p^l. \quad (33)$$

Now, the solutions for c_1 and d_n are linear combinations of exponential functions, i.e., $M_i e^{\pm f(\omega, H_i)}$, and so are c_j and d_j . Note that M_i are constants which denote the magnitude of the exponential function, and $f(\omega, H_i)$ are functions of ω and H_i .

The solutions for the displacement and shear stress in Eqs. (30) and (31) can also be expressed in similar forms. By a complicated algebraic derivation, the explicit expressions for displacement and shear stress in the Fourier transformed domain are obtained as follows:

$$\tilde{w}^j = \sum_{l=0}^{\infty} \sum_{k=1}^N \frac{M_k}{\mu_m^e \omega} \begin{bmatrix} -(f_z - i \mu_m^e b_z) e^{\omega(Y-H+F_k^{c'})} - (f_z + i \mu_m^e b_z) e^{\omega(Y+H+F_k^{c''})} \\ -(f_z - i \mu_m^e b_z) e^{-\omega(Y+H-F_k^{d'})} - (f_z + i \mu_m^e b_z) e^{-\omega(Y-H-F_k^{d''})} \end{bmatrix} e^{-i\omega H'}, \quad (34)$$

$$\tilde{\tau}_{YZ}^j = \sum_{l=0}^{\infty} \sum_{k=1}^N \frac{\mu_j^e M_k}{\mu_m^e} \begin{bmatrix} -(f_z - i \mu_m^e b_z) e^{\omega(Y-H+F_k^{c'})} - (f_z + i \mu_m^e b_z) e^{\omega(Y+H+F_k^{c''})} \\ +(f_z - i \mu_m^e b_z) e^{-\omega(Y+H-F_k^{d'})} + (f_z + i \mu_m^e b_z) e^{-\omega(Y-H-F_k^{d''})} \end{bmatrix} e^{-i\omega H'},$$

where

$$\begin{cases} N = 2^{n+j-m-1} \cdot (2^n - 1)^l, & 1 \leq j \leq m, \\ N = 2^{n+m-j-1} \cdot (2^n - 1)^l, & m \leq j \leq n. \end{cases}$$

Here n is the number of layers, m denotes the layer that is subjected to the applied loading, and j is the j th layer where the solution is required. The terms M_k , $F_k^{c'}$, $F_k^{c''}$, $F_k^{d'}$, and $F_k^{d''}$ in Eq. (34) are very complicated and difficult to present. The following functions are first defined as

$$\begin{cases} a_1 = 1, f_1^A = 0, \\ a_{i+2k-1} = a_i t_k, & k = 1, 2, \dots, m-1, \\ f_{i+2k-1}^A = -(f_i^A + 2H_k), & i = 1, 2, \dots, 2^{k-1}, \end{cases} \quad (35a)$$

$$\begin{cases} b_1 = 1, f_1^{B1} = -2H_n, f_1^{B2} = 0, \\ b_{i+2k-1} = -b_i t_{n-k}, & k = 1, 2, \dots, n-m, \\ f_{i+2k-1}^{B1} = -(f_i^{B1} + 2H_n + 2H_{n-k}), & i = 1, 2, \dots, 2^{k-1}, \\ f_{i+2k-1}^{B2} = -(f_i^{B2} + 2H_n - 2H_{n-k}), & \end{cases} \quad (35b)$$

$$\begin{cases} r_{(i-1) \cdot 2^{n-m}+k}^p = -a_i b_k, & r_{2^{n-1}+(i-1) \cdot 2^{m-1}+k}^p = a_i b_k, & i=1,2,\dots,2^{m-1}, \\ g_{(i-1) \cdot 2^{n-m}+k}^p = f_i^A + f_k^{B_2}, & g_{2^{n-1}+(i-1) \cdot 2^{m-1}+k}^p = f_k^{B_1} - f_i^A, & k=1,2,\dots,2^{n-m}, \end{cases} \quad (35c)$$

$$\begin{cases} r_k^l = \prod_{i=1}^l r_{i_0}^p, & i_1, i_2, i_3, \dots, i_l = 2, 3, \dots, 2^n, \\ g_k^l = \sum_{i=1}^l g_{i_0}^p, & k = \sum_{i=1}^{l-1} (i_0 - 2)(2^n - 1) + (i_l - 1), \end{cases} \quad (35d)$$

for $l=0$, $r_{i_0}^0 = 1$ and $g_{i_0}^0 = 0$.

For the case of $1 \leq j \leq m$, the following functions are defined:

$$\begin{cases} f_{(i-1) \cdot 2^{n-m}+k}^{c'} = f_k^{B_2} + f_i^A, \\ f_{(i-1) \cdot 2^{n-m}+k}^{c''} = f_k^{B_1} + f_i^A, & i=1,2,\dots,2^{j-1}, \\ f_{(i-1) \cdot 2^{n-m}+k}^{d'} = f_k^{B_2} - f_i^A, & k=1,2,\dots,2^{n-m}. \\ f_{(i-1) \cdot 2^{n-m}+k}^{d''} = f_k^{B_1} - f_i^A, \end{cases} \quad (35e)$$

Finally, the terms M_k , $F_k^{c'}$, $F_k^{c''}$, $F_k^{d'}$, and $F_k^{d''}$ indicated in Eq. (34) can be expressed explicitly as

$$\begin{cases} M_{(k-1) \cdot 2^{n+j-m-1}+i} = \frac{-1}{2} \left(\prod_{j=0}^{m-1} S_0' \right) r_k^l r_i^p, \\ F_{(k-1) \cdot 2^{n+j-m-1}+i}^{c'} = g_k^l + f_i^{c'}, & k=1,2,\dots,(2^n-1)^l, \\ F_{(k-1) \cdot 2^{n+j-m-1}+i}^{c''} = g_k^l + f_i^{c''}, & i=1,2,\dots,2^{n+j-m-1}. \\ F_{(k-1) \cdot 2^{n+j-m-1}+i}^{d'} = g_k^l + f_i^{d'}, \\ F_{(k-1) \cdot 2^{n+j-m-1}+i}^{d''} = g_k^l + f_i^{d''}, \end{cases} \quad (35f)$$

For the case of $m \leq j \leq n$, the expressions are

$$\begin{cases} f_{(i-1) \cdot 2^{m-1}+k}^{c'} = f_i^{B_1} + f_k^A, \\ f_{(i-1) \cdot 2^{m-1}+k}^{c''} = f_i^{B_1} + f_k^A, & i=1,2,\dots,2^{n-j}, \\ f_{(i-1) \cdot 2^{m-1}+k}^{d'} = f_i^{B_2} - f_k^A, & k=1,2,\dots,2^{m-1}. \\ f_{(i-1) \cdot 2^{m-1}+k}^{d''} = f_i^{B_2} + f_k^A, \end{cases} \quad (35g)$$

The terms M_k , $F_k^{c'}$, $F_k^{c''}$, $F_k^{d'}$, and $F_k^{d''}$ are presented by

$$\begin{cases} M_{(k-1) \cdot 2^{n-j+m-1}+i} = \frac{1}{2} \left(\prod_{m=0}^{j-1} S_0'' \right) r_k^l r_{2^{n-1}+i}^p, \\ F_{(k-1) \cdot 2^{n-j+m-1}+i}^{c'} = g_k^l + f_i^{c'}, & k=1,2,\dots,(2^n-1)^l, \\ F_{(k-1) \cdot 2^{n-j+m-1}+i}^{c''} = g_k^l + f_i^{c''}, & i=1,2,\dots,2^{n+j-m-1}. \\ F_{(k-1) \cdot 2^{n-j+m-1}+i}^{d'} = g_k^l + f_i^{d'}, \\ F_{(k-1) \cdot 2^{n-j+m-1}+i}^{d''} = g_k^l + f_i^{d''}, \end{cases} \quad (35h)$$

The structures of the complete solutions given in Eq. (34) have some interesting characteristics. The solutions are composed of infinite terms, and it is interesting to note that each term represents the solution in the transformed domain for a concentrated loading in an infinite homogeneous medium. The term with $(Y \pm H \pm F_k)$ in the exponential functions indicates the location of the loading. However, $M_k b_z$ and $\mu_j^e / \mu_m^e M_k f_z$ represent the magnitudes of the concentrated dislocation and force, respectively. F_k is dependent on the locations of the interfaces, i.e., H_j , and M_k depends only on the reflection and refraction coefficients, i.e., t_j , s_j' , and s_j'' .

Only one term in the infinite series of Eq. (34) represents the

applied concentrated force f_z and dislocation b_z in an infinite medium at $Y=H$ and $X=H'$, all the remaining terms are image forces and dislocations that are induced to satisfy the boundary and interface conditions. This method is referred to as the method of images. The advantage of this method is that the solutions of problems with complicated geometric configurations can be constructed by superposing the solution in an infinite medium. The mathematical derivation in this section provides an automatic determination for the locations and magnitudes of all the image forces.

Since the solutions in the transformed domain expressed in Eq.

(34) are exponential functions of ω , only two different inverse Fourier transformations for exponential functions are required, which are

$$\int_{-\infty}^{\infty} e^{-\omega(Y \pm H \pm F_k)} \cdot e^{-i\omega H'} \cdot e^{i\omega X} d\omega = \frac{1}{\pi} \frac{Y \pm H \pm F_k}{(X - H')^2 + (Y \pm H \pm F_k)^2},$$

$$\int_{-\infty}^{\infty} \frac{1}{\omega} e^{-\omega(Y \pm H \pm F_k)} \cdot e^{-i\omega H'} \cdot e^{i\omega X} d\omega = \frac{-1}{2\pi} \ln[(X - H')^2 + (Y \pm H \pm F_k)^2]. \quad (36)$$

Therefore, the inverse Fourier transformation for Eq. (34) can be easily derived and the results are

$$w^j = \sum_{l=0}^{\infty} \sum_{k=1}^N \frac{M_k}{\pi} \left\{ b_z \left[-\tan^{-1} \frac{Y - H + F_k^{c'}}{X - H'} + \tan^{-1} \frac{Y + H + F_k^{c''}}{X - H'} \right] + \tan^{-1} \frac{Y + H + F_k^{d'}}{X - H'} - \tan^{-1} \frac{Y - H - F_k^{d''}}{X - H'} \right\} + \frac{f_z}{2\mu_m^e} \left[\ln((X - H')^2 + (Y - H + F_k^{c'})^2) + \ln((X - H')^2 + (Y + H + F_k^{c''})^2) + \ln((X - H')^2 + (Y + H - F_k^{d'})^2) + \ln((X - H')^2 + (Y - H - F_k^{d''})^2) \right] \quad (37)$$

$$\tau_{yz}^j = \sum_{l=0}^{\infty} \sum_{k=1}^N \frac{\mu_j^e M_k}{\pi} \left\{ b_z \left[\frac{-(X - H')}{(X - H')^2 + (Y - H + F_k^{c'})^2} + \frac{(X - H')}{(X - H')^2 + (Y + H + F_k^{c''})^2} \right] + \frac{(X - H')}{(X - H')^2 + (Y + H - F_k^{d'})^2} - \frac{(X - H')}{(X - H')^2 + (Y - H - F_k^{d''})^2} \right] + \frac{f_z}{\mu_m^e} \left[\frac{Y - H + F_k^{c'}}{(X - H')^2 + (Y - H + F_k^{c'})^2} + \frac{Y + H + F_k^{c''}}{(X - H')^2 + (Y + H + F_k^{c''})^2} + \frac{Y + H - F_k^{d'}}{(X - H')^2 + (Y + H - F_k^{d'})^2} + \frac{Y - H - F_k^{d''}}{(X - H')^2 + (Y - H - F_k^{d''})^2} \right] \right\}.$$

Finally, by substituting X and Y defined in Eq. (15) into Eq. (37), and using the displacement and stress relations in Eq. (10), the complete solutions for the original problem of the anisotropic multilayered medium can be obtained. Equation (37) is the explicit expression of the Green's function for the multilayered medium subjected to the antiplane shear deformation.

7 Numerical Examples and Discussions

The full-field analysis of shear stresses in a layered medium consisting of 12 layers subjected to interior forces is presented in this section. In the analysis of a nonhomogeneous multilayered medium, it is sometimes the practice to treat the multilayered medium as a single homogeneous layer with effective or homogenized stiffness constants. The differences in the stress distributions between layered medium with 12 layers and one homogeneous layer with the effective material constants obtained from the averaged material properties of 12 layers will be discussed in detail. The effective stiffness constants of one homogeneous layer, $\tilde{C}_{ij}$, are obtained from the homogenized weighted properties of the 12 layers using the formula

$$\tilde{C}_{ij} = \frac{\sum_{k=2}^n C_{ij}^k (h_k - h_{k-1}) + C_{ij}^1 h_1}{h_n}. \quad (38)$$

As indicated in Eq. (38), the effective stiffness constants are obtained by averaging the stiffness constants over the thickness of each layer. The complete explicit expressions for the displacement and shear stress given in Eqs. (35) and (37) are used to construct the program for numerical calculations. For the 12-layered me-

dium, the thickness of each layer is the same and equal to 1, and the elastic stiffness constants for each layer are listed as follows:

$$[C_{44}^1, C_{45}^1, C_{55}^1; C_{44}^2, C_{45}^2, C_{55}^2; \dots; C_{44}^{12}, C_{45}^{12}, C_{55}^{12}] = [3960, 0, 7170; 7170, 1220, 3960; 3960, -1220, 7170; 3585, 610, 3960; 3960, -610, 3585; 4430, 0, 0, 4430; 1980; 0, 1980; 7170, 1220, 3585; 5158, -610, 3585; 3585, 0, 5185; 3960, 0, 3960; 6160, -2440, 3330] \text{ GPa}.$$

The effective elastic constants for one homogeneous layer obtained from Eq. (38) are

$$[\tilde{C}_{44}, \tilde{C}_{45}, \tilde{C}_{55}] = [4347, -169, 4131] \text{ GPa}.$$

An antiplane body force f_z is applied within the seventh layer at $x=0$ and $y=6.5$. The distributions of the shear stress τ_{yz} along the y -axis at $x=0$ and $x=2$ are shown in Figs. 4 and 5, respectively. In addition, the τ_{xz} stress distributions along the y -axis at $x=0$ and $x=2$ are displayed in Figs. 6 and 7, respectively. To generate the shear stress distributions, the series solution is truncated if the numerical calculation is within the accuracy of 0.1 percent.

As shown in Figs. 4 and 5, the multilayered solution for τ_{yz} is continuous at the interfaces and approaches zero at the top and bottom boundaries. The results of τ_{yz} for multilayered solutions and one homogeneous layer solution are closely related at $x=0$, and only a small difference appears at $x=2$. This implies that the

Fig. 4 Distribution of τ_{yz} at $x=0$ along y -axis for the 12-layered medium and one homogeneous layer

Fig. 5 Distribution of τ_{yz} at $x=2$ along y -axis for the 12-layered medium and one homogeneous layer

Fig. 6 Distribution of τ_{xz} at $x=0$ along y -axis for the 12-layered medium and one homogeneous layer

Fig. 7 Distribution of τ_{xz} at $x=2$ along y -axis for the 12-layered medium and one homogeneous layer

shear stress τ_{yz} of the multilayered problem can be obtained with good approximation from the result of a much simpler one homogeneous layer solution. As indicated in Figs. 6 and 7, since τ_{xz} is discontinuous at the interfaces for the multilayered solution, the results for multilayered solution and one homogeneous layer solution differ substantially. This indicates that it is not suitable to use the single homogeneous layer solution to simulate the multilayered problem for the shear stress τ_{xz} .

8 Concluding Remarks

An analytical investigation of multilayered anisotropic media subjected to concentrated forces and screw dislocations has been provided in this study. A general linear coordinate transformation for multilayered media was introduced to simplify the governing equilibrium equation without complicating the boundary and interface continuity conditions. With this linear coordinate transformation, the original anisotropic multilayered problem can be reduced to an equivalent isotropic multilayered problem. The linear coordinate transformation introduced in this study substantially reduces the dependence on elastic constants in the antiplane anisotropic multilayered media. By using the Fourier transform technique and a series expansion, analytical solutions for displacement and stresses are presented in an explicit form. The complete solutions for this complicated problem consist only of very simple solutions obtained from an infinite homogeneous medium with concentrated loadings. Except for the original applied loading, the remaining terms in the infinite series are image forces which are induced to satisfy the boundary and interface conditions. The mathematical approach introduced in this study provides an automatic determination for the locations and magnitudes of all the image forces. A computational program for numerical calculation of the full-field analysis is easily constructed by using the explicit formulation of the solutions. In order to simplify the problem, the nonhomogeneous multilayered medium is sometimes treated as a single homogeneous layer with effective material constants. Hence the numerical results for the multilayered problem were compared with the solutions obtained by considering the multilayered medium as a single homogeneous layer with effective material constants.

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