Self-adjusted convex approximation method for structural optimization

Tien-Tung Chung^{*}, Chyou-Huey Chiou

Department of Mechanical Engineering, National Taiwan University, Taipei 10617, Taiwan, ROC

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Abstract

This study presents an approximation method called self-adjusted convex approximation for optimum design of structures. The proposed method is a generalization of the convex approximation method. The order of the approximation for each constraint is automatically adjusted in the optimization process. This self-adjusted capability makes the approximate constraint values conservative enough to maintain the optimum point of the approximate problem in the feasible region. This feature can prevent oscillation of the optimum design point and increase the convergence rate of the redesign process. Results obtained from three examples demonstrate the effectiveness of the proposed method. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The structural optimization problem considered herein consists of minimizing an objective function subject to constraints that ensure the feasibility of the structural design. The numerical optimization problem can be written in the following form:

$$\begin{aligned} &\text{minimize } F(\vec{x}) \\ &\text{subject to } h_q(\vec{x}) \leq 0, \quad q = 1, 2, \dots, n_c, \end{aligned} \quad (1)$$

where $\vec{x}$ denotes the design vector, x_i represents the i th component of the design vector, $F(\vec{x})$ is the objective function, $h_q(\vec{x})$ denotes the q th constraint and n_c represents the total number of constraints. In general, design constraints include behavior constraints and size constraints (e.g. move limits). Behavior constraints generally impose limitations on structural response quantities (e.g. placing upper bounds on stresses and displacements under static loading cases or lower bounds on natural frequencies).

As is well known, structural optimization is severely restricted by the high computational cost of solving large-scale problems. Many innovative approaches have been proposed in the recent decade to resolve this design optimization problem. Among these, the most widely used approach applies an approximation concept that replaces the original optimization problem with a sequence of explicit approximate sub-problems. This method generates the sub-problems through Taylor series expansion of the objective function and constraints in terms of intermediate design variables.

2. Approximation methods

2.1. Linear approximation

The most commonly used approximation is the first-order Taylor series expansion [1] in terms of the original design variable. It can be written in the following form:

$$h_d(\vec{x}) = h(\vec{x}_0) + \sum_{i=1}^n \left(\frac{\partial h}{\partial x_i} \right)_{\vec{x}_0} (x_i - x_{i0}), \quad (2)$$

^{*}Corresponding author.

where $h_d(\bar{x})$ denotes the direct linear approximation function of $h(\bar{x})$ and $(\partial h / \partial x_i)_{\bar{x}_0}$ represents the sensitivity of $h(\bar{x})$ with respect to the i th design variable at the original design $\bar{x}_0$. For some applications, the linear approximation is inaccurate even when the design point $\bar{x}$ is close to $\bar{x}_0$.

2.2. Reciprocal approximation

The reciprocal approximation is the first-order Taylor series expansion in terms of the reciprocals $1/x_i$ [2,3]. It can be expressed as

$$h_r(\bar{x}) = h(\bar{x}_0) + \sum_{i=1}^n \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} (x_i - x_{i0}) \left(\frac{x_{i0}}{x_i} \right), \quad (3)$$

where $h_r(\bar{x})$ is the reciprocal approximation of $h(\bar{x})$. Its scaling property makes this approximation more efficient for structures with stress and displacement constraints than the appropriate linear approximation does.

2.3. Convex approximation

The convex approximation, generally referred to as convex linearization (CONLIN) [4], is a hybrid of the linear and reciprocal approximations. It can be written in the following form:

$$h_c(\bar{x}) = h(\bar{x}_0) + \sum_i^{(+)} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} (x_i - x_{i0}) + \sum_i^{(-)} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} (x_i - x_{i0}) \left(\frac{x_{i0}}{x_i} \right), \quad (4)$$

where $h_c(\bar{x})$ denotes the convex approximation of $h(\bar{x})$ and $\sum_i^{(+)} (\sum_i^{(-)})$ represents summation over the variables whose derivatives are positive (negative). The convex approximation is attractive in that it generates the most conservative approximation among all the possible combinations of direct or reciprocal approximations.

If the constraints are underestimated, the design usually ends up in the infeasible region, and many iterations are required to bring it back to the feasible region. Therefore, the design process may either diverge or converge slowly with undesirably large oscillations between the feasible and infeasible regions. The fact that the convex approximation introduces some conservative curvatures in approximate functions accounts for why it may not require control parameters such as move limits. Another attractive feature is that the designer may stop the optimization process at any stage and still obtain a feasible design that is better than the initial design.

The convex approximation is more conservative than either the linear or the reciprocal approximations.

However, this method does not ensure that the approximate constraint is conservative with respect to the real constraint. Therefore, although the convex approximation achieves stable numerical results, the optimum process may either converge slowly if approximate functions are too conservative or continue to oscillate if they are not conservative enough.

2.4. Two-point exponential approximation and multi-point approximation

Fadel et al. [5] presented a two-point exponential approximation, which is a linear approximation in terms of intermediate variables $x_i^{p_i}$. The exponents p_i ($i = 1, 2, \dots, n$) are determined by matching function derivatives at the previous data point. It can be expressed in the following form:

$$h_t(\bar{x}) = h(\bar{x}_0) + \sum_{i=1}^n \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left[\left(\frac{x_{i1}}{x_{i0}} \right)^{p_i} - 1 \right] \left(\frac{x_{i0}}{p_i} \right), \quad (5)$$

$$p_i = 1 + \frac{\ln \left[\left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_1} / \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \right]}{\ln(x_{i1}/x_{i0})}, \quad (6)$$

where $h_t(\bar{x})$ denotes the two-point exponential approximation of $h(\bar{x})$.

Wang and Grandhi [6] developed an improved two-point function approximation, which not only matches the previous point gradients but also matches its function value. This can be expressed as

$$h_{ta}(\bar{x}) = h(\bar{x}_0) + \sum_{i=1}^n \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left[\left(\frac{x_{i1}}{x_{i0}} \right)^{p_i} - 1 \right] \left(\frac{x_{i0}}{p_i} \right) + \frac{1}{2} \varepsilon_2 \sum_{i=1}^n (x_i^{p_i} - x_{i0}^{p_i})^2, \quad (7)$$

where $h_{ta}(\bar{x})$ represents the two-point adaptive nonlinear approximation of $h(\bar{x})$ and ε_2 is a parameter to be defined.

Differentiating Eq. (7), n equations are obtained by matching the derivatives at the previous point. Another equation is obtained by matching the function value at the previous point. The $n + 1$ equations are solved by the numerical iteration method to determine the p_i and ε_2 . However, this approach is limited by the difficulty in obtaining the exponents p_i . In a related work, Wang and Grandhi [7] developed a multi-point approximation which used Hermite interpolation concepts and the multi-point information generated at the optimization iterations.

Zhang and Fleury [8] presented a modified convex approximation in the following form:

$$h_{mc}(\bar{x}) = h_c(\bar{x}) + \frac{1}{2} \alpha \left\| \bar{x} - \bar{x}_0 \right\|^2, \quad (8)$$

where $h_{mc}(\bar{x})$ denotes the modified convex approximation of $h(\bar{x})$ and α represents the convexity. The value of α can be determined by matching the function value at the previous point. If α is negative, a small positive value will be imposed.

2.5. Higher-order convex approximation

When x_i^2 is the intermediate variable if the sensitivity $(\partial h / \partial x_i)_{\bar{x}_0} > 0$ and $1/x_i^2$ is the intermediate variable if $(\partial h / \partial x_i)_{\bar{x}_0} < 0$, then the convex approximation of order 2 can be written in the following form:

$$h_2(\bar{x}) = h(\bar{x}_0) + \sum_i \frac{1}{2} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left(\frac{x_i^2}{x_{i0}} - x_{i0} \right) + \sum_i \frac{1}{2} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left(\frac{x_i^2}{x_{i0}} - x_{i0} \right) \left(\frac{x_{i0}}{x_i} \right)^2, \quad (9)$$

where $h_2(\bar{x})$ denotes the convex approximation of order 2.

Similarly, if x_i^n and $1/x_i^n$ are selected as intermediate design variables, the convex approximation of order n can be formulated as the following form:

$$h_n(\bar{x}) = h(\bar{x}_0) + \sum_i \frac{1}{n} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left[\left(\frac{x_i}{x_{i0}} \right)^{n-1} x_i - x_{i0} \right] + \sum_i \frac{1}{n} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left[\left(\frac{x_i}{x_{i0}} \right)^{n-1} x_i - x_{i0} \right] \left(\frac{x_{i0}}{x_i} \right)^n, \quad (10)$$

where $h_n(\bar{x})$ represents the convex approximation of order n , n is an integer and $n \geq 2$.

The higher-order, convex approximation is attractive in that it yields more conservative results with the higher convex approximation order. Showing that $h_n(\bar{x}) < h_m(\bar{x})$ for $n < m$ proves this remarkable property. For example, subtracting Eq. (4) from Eq. (9) yields

$$h_2(\bar{x}) - h_c(\bar{x}) = \sum_i \frac{x_{i0}}{2} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left(\frac{x_i}{x_{i0}} - 1 \right)^2 - \sum_i \frac{x_{i0}}{2} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left(\frac{x_i}{x_{i0}} - 1 \right)^2 \times \left(\frac{x_{i0}}{x_i} \right)^2 \geq 0. \quad (11)$$

For the more general form of higher-order, convex approximation, the real order r_i for each x_i can be used instead of the integer order n . It can be written as

$$h_h(\bar{x}, \bar{r}) = h(\bar{x}_0) + \sum_i \frac{1}{r_i} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left[\left(\frac{x_i}{x_{i0}} \right)^{r_i-1} x_i - x_{i0} \right] + \sum_i \frac{1}{r_i} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left[\left(\frac{x_i}{x_{i0}} \right)^{r_i-1} x_i - x_{i0} \right] \left(\frac{x_{i0}}{x_i} \right)^{r_i}, \quad (12)$$

where $h_h(\bar{x}, \bar{r})$ denotes the higher-order, convex approximation of $h(\bar{x})$, r_i is a real number and representing the order or degree of convexity of the constraint $h(\bar{x})$ with respect to the design variable x_i . The value of r_i ($i = 1, 2, \dots, n$) varies for each constraint at different iterations. It can be proven that $h_h(\bar{x}, \bar{r}) > h_h(\bar{x}, \bar{s})$ if $r_i > s_i$ ($i = 1, 2, \dots, n$). The proof is given in Appendix A.

Many investigations use the three-bar truss, shown in Fig. 1, as an example. This study also uses it to examine the conservative tendency of the proposed method.

The truss is subjected to a concentrated load, and the cross-sectional areas of members 1 and 3 are the same. Two design variables, x_1 and x_2 are the cross-sectional areas of members 1 and 2, respectively.

Here, this work examines the approximations of the constraint on the stress in member 1, which stipulates that the stress is less than 20. The theoretical stress of member 1 can be calculated as

$$\sigma_1 = 20 \frac{x_2 + \sqrt{2}x_1}{2x_1x_2 + \sqrt{2}x_1^2} \leq 20. \quad (13)$$

Therefore, the actual constraint can be written as

$$h(\bar{x}) = \frac{x_2 + \sqrt{2}x_1}{2x_1x_2 + \sqrt{2}x_1^2} - 1 \leq 0. \quad (14)$$

Fig. 1. Three-bar truss.

Assume that $h(\bar{x})$ is expanded at the point (1.0,1.0). Different kinds of approximations can be constructed as follows:

$$\begin{aligned} h_d(\bar{x}) &= 8.28 - 11.714x_1 - 2.426x_2, \\ h_r(\bar{x}) &= 11.714/x_1 + 2.426/x_2 - 20, \\ h_c(\bar{x}) &= 11.714/x_1 + 2.426/x_2 - 20, \\ h_2(\bar{x}) &= 5.857/x_1^2 + 1.213/x_2^2 - 12.93, \\ h_h(\bar{x}, (3.5, 3.5)) &= 3.347/x_1^{3.5} + 0.693/x_2^{3.5} - 9.90. \end{aligned} \quad (15)$$

Fig. 2 depicts the different approximations. Notably, the higher the chosen order the more conservative the constraint value. This conservative tendency suggests that if the constraint is not sufficiently conservative, design variable values may oscillate during the optimization steps. In such a case, the approximation order r_i in the next iteration should be increased to achieve numerical stability.

2.6. The self-adjusted convex approximation method

This study presents an approximation method called self-adjusted convex approximation (SACA). In this method, the order r_i for each design variable is adjusted to prevent design variable values oscillating between two consecutive iteration steps. For a lower-order value, the approximate constraint values are probably underesti-

mated. Restated, the approximate constraint values are less conservative than the actual constraint values. Under such a condition, the design variable values of the solution of the approximate problem usually oscillates between consecutive iteration steps. Increasing the order is the way to resolve this situation.

In this study, the orders r_i for all design variables x_i are considered to be the same, i.e.

$$r_i = r, \quad i = 1, 2, \dots, n.$$

Therefore, this simplified form of SACA can be written as

$$\begin{aligned} h_a(\bar{x}, r) &= h(\bar{x}_0) + \sum_i^{(+)} \frac{1}{r} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left[\left(\frac{x_i}{x_{i0}} \right)^{r-1} x_i - x_{i0} \right] \\ &\quad + \sum_i^{(-)} \frac{1}{r} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_0} \left[\left(\frac{x_i}{x_{i0}} \right)^{r-1} x_i - x_{i0} \right] \left(\frac{x_{i0}}{x_i} \right)^r, \end{aligned} \quad (16)$$

where $h_a(\bar{x}, r)$ represents the self-adjusted convex approximation of $h(\bar{x})$.

In the $(k-1)$ th iteration, $r^{(k-1)}$ is the order of convex approximation and an approximate problem with respect to $\bar{x}_{k-1}$ can be generated. The optimum solution of this $(k-1)$ th approximate problem is denoted as $\bar{x}_k$. With $\bar{x}_k$ obtained, the value of $h_a(\bar{x}, r^{(k-1)})$ evaluated at $\bar{x}_k$ can be obtained and denoted as $h_a(\bar{x}_k, r^{(k-1)})$.

Fig. 2. Comparison of higher-order, convex approximations.

In the k th iteration, $\bar{x}_k$ is the current referent point and the actual constraint value $h(\bar{x}_k)$ can be obtained from the problem analysis procedure. This allows the actual constraint value $h(\bar{x}_k)$ to be compared with the estimated constraint value $h_a(\bar{x}_k, r^{(k-1)})$. If $h_a(\bar{x}_k, r^{(k-1)}) < h(\bar{x}_k)$, it implies that the approximate function of the $(k-1)$ th iteration is not conservative enough. Restated, it underestimates the actual constraint. Therefore, the order $r^{(k)}$ must be increased to avoid oscillation of design variables and prevent an infeasible design from being obtained. In contrast, if $h_a(\bar{x}_k, r^{(k-1)}) > h(\bar{x}_k)$, it implies that the order is sufficiently conservative; although, it is probably too large and will slow down the convergence. To accelerate the convergence in this case, the order $r^{(k)}$ should be decreased.

The criterion for adjusting the order $r^{(k)}$ is that the approximate function does not underestimate or overestimate the actual constraint value at $\bar{x}_k$. That is, $r^{(k)}$ is obtained from solving the equation $h_a(\bar{x}_k, r) = h(\bar{x}_k)$. Substituting this equation into Eq. (16) yields

$$h(\bar{x}_{k-1}) + \sum_i \frac{1}{r} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_{k-1}} \left[\left(\frac{x_{ik}}{x_{i(k-1)}} \right)^{r-1} x_{ik} - x_{i(k-1)} \right] + \sum_i \frac{1}{r} \left(\frac{\partial h}{\partial x_i} \right)_{\bar{x}_{k-1}} \left[\left(\frac{x_{ik}}{x_{i(k-1)}} \right)^{r-1} x_{ik} - x_{i(k-1)} \right] \times \left(\frac{x_{i(k-1)}}{x_{ik}} \right)^r = h(\bar{x}_k). \quad (17)$$

The order r is the only unknown in Eq. (17). Because $h_a(\bar{x}_k, r)$ is an increasing function with respect to r , as the proof in Appendix A, Eq. (17) can be easily solved by the following steps:

(1) For the increasing function $h_a(\bar{x}_k, r)$, if the upper bound r_U and lower bound r_L of the order r can be determined, the golden section method or other interpolation methods can be used to obtain the proper order r within any given accuracy ε .

(2) r_U and r_L of the order r can be obtained as follows:

(a) If $h_a(\bar{x}_k, r^{(k-1)}) < h(\bar{x}_k)$, the order of the previous iteration $r^{(k-1)}$ is too small to match the actual constraint value at $\bar{x}_k$. Therefore, $r^{(k-1)}$ can be set as the lower bound r_L of the order r . In this case, the upper bound r_U can be determined by sequentially adding Δr to r_L and checking if $h_a(\bar{x}_k, r) > h(\bar{x}_k)$, where $r = r_L + \Delta r$.

(b) If $h_a(\bar{x}_k, r^{(k-1)}) > h(\bar{x}_k)$, the order of the previous iteration $r^{(k-1)}$ can be set as the upper bound r_U of the order r . Similarly, the lower bound r_L can be determined by sequentially subtracting Δr from r_U and checking if $h_a(\bar{x}_k, r) < h(\bar{x}_k)$, where $r = r_U - \Delta r$.

(3) By the above means, the proper order r of the constraint can be determined. The order r is adopted as

the order $r^{(k)}$ of the convex approximation in the k th iteration.

At each iteration, the order for each individual constraint should be solved, respectively. In the following three examples, the time required for a Pentium MMX-200 CPU to determine the order for each constraint is less than 0.01 s. After the approximate constraints are constructed, the approximate structural design problem can be solved by general optimization methods. This study uses the interior penalty method. The total CPU time of one iteration for SACA is about the same as for CONLIN and linear approximation.

The SACA method has three advantages over the two-point approximation methods. First, two-point exponential approximation may not produce an exact p_i value due to its negative value in the logarithmic function. However, SACA can always obtain the proper order r because the approximation function is increasing with respect to the order r . Second, SACA can easily determine the order r in Eq. (17). In other words, unlike certain two-point approximation methods SACA does not need to utilize the least square method or even to perform another optimization algorithm. Third, the order of SACA is more flexible than the convexity of Eq. (8). The order can be less than 1 or even negative, in other words, the SACA method can be less conservative than CONLIN.

3. Numerical examples

3.1. A three-bar truss

Consider the three-bar truss in Fig. 1. Table 1 summarizes the design data.

The weight of the truss is considered to be the objective function. The exact solution of this problem can be obtained as [9] $x_1 = 0.788$ in., $x_2 = 0.410$ in., minimum weight = 263.9 lb.

Table 2 compares CONLIN and SACA with respect to the structural weight iteration history. Table 3 lists the orders for SACA in the optimization process. It indicates that all the orders are less than 1. This finding implies that CONLIN is too conservative for this problem. CONLIN takes three more iterations than SACA to converge. Furthermore, comparing the two methods' computation times shows that CONLIN takes 10.11 s to converge while SACA takes only 5.6 s.

Table 1
Design data for the three-bar truss problem

Modulus of elasticity = 1 psi
$-15 \leq \sigma_i \leq 20$ psi ($i = 1, 2, 3$)
$\bar{x}_i > 0$ ($i = 1, 2$)
Initial design $x_{i0} = 1$ in. ² ($i = 1, 2$)

Table 2
Iteration history of the weight for the three-bar truss problem

Iteration no.	CONLIN	SACA
1	267.5	267.5
2	265.1	265.0
3	264.4	263.9
4	264.1	263.9
5	264.0	
6	263.9	
7	263.9	

Table 3
The orders for the three-bar truss problem

Iteration no.	Constraint 1	Constraint 2	Constraint 3
1	0.90	0.82	0.05
2	−0.18	−0.30	0.38
3	−0.28	−0.44	0.54

3.2. A cantilever beam with a frequency constraint

This problem involves a cantilever beam with ten cross-sectional area variables x_i ($i = 1, 2, \dots, 10$), as shown in Fig. 3, and with the first axial natural frequency constrained to be greater than 410 Hz. Table 4 presents the beam design data and dimensions.

Table 5 compares the iteration history of structural weight for SACA and CONLIN. Some of CONLIN iteration results fall into the infeasible region, in other words, the first natural frequency violates the design constraint. This observation implies that CONLIN is not conservative enough for this example. To illustrate the underestimation of the constraint using convex ap-

Fig. 3. Cantilever beam with a frequency constraint.

Table 4
Design data for the cantilever beam problem

Modulus of elasticity = 10.3×10^6 psi
Material density 0.1 lb in.^{-3}
$\nu = 0.3$
$Mo = 38.64 \text{ lb}$
Initial design $x_{i0} = 9.2 \text{ in.}^2$ ($i = 1, 2, \dots, 10$)
First frequency $\omega_1 > 410 \text{ Hz}$

Table 5
Iteration history of the structural weight (lb) for the cantilever beam problem

Iteration no.	CONLIN	SACA
1	71.6	71.6
2	80.2	88.4
3	70.5	80.1
4	77.6	80.7
5	70.6	80.7
6	77.6	
7	70.6	
8	77.6	
9	70.6	
10	77.6

Fig. 4. Natural frequencies in the optimization process.

proximation, Fig. 4 also shows the natural frequency in the iteration history. Table 6 lists the orders of SACA at each iteration step. This table reveals that most orders are greater than 1, which means CONLIN is not conservative enough.

3.3. A 10-bar truss with stress constraints

Fig. 5 shows the minimum weight design of a 10-bar truss [10]. The allowable stress limits are set to ± 25 ksi for all members except member 9. For member 9, the allowable stress limit is ± 75 ksi. The minimum cross-sectional area is 0.1 in.^2 , and Table 7 summarizes the design data.

Table 6
The order for the cantilever beam problem

Iteration no.	1	2	3	4	5
Order	1.00	3.90	0.32	2.44	2.61

Fig. 5. 10-bar truss with stress constraints.

Table 7
Design data for the 10-bar truss problem

Modulus of elasticity = 1×10^7 psi
Density of the truss material = 0.1 lb/in.^3
$l = 360''$, $P = 100 \text{ Kips}$
Initial design $x_{i0} = 5 \text{ in.}^2$ ($i = 1, 2, \dots, 10$)

Table 8
Iteration history of the structural weight (lb) for the 10-bar truss problem

Iteration no.	Ref. [10]				This study, SACA
	Linear	Quadratic	CONLIN	Linear force	
1	1845	2002	2361	1891	1882
2	1637	1741	1960	1688	1653
3	1601	1650	1722	1589	1550
4	1558	1586	1641	1549	1553
5	1531	1547	1587	1526	1448
6	1514	1525	1566	1511	1491
7	1507	1514	1555	1504	1498
8	1502	1507	1546	1501	1498
9	1500	1503	1540	1500	
10	1500	1501	1538	1499	
11	1500	1500	1535	1499	
12	1499	1499	1532		

Table 8 compares the SACA results with the results in Ref. [10]. In Ref. [10], linear, reciprocal, conservative (CONLIN), quadratic, reciprocal quadratic and linear force approximation was used. The best result was obtained with linear force approximation. Table 8 sum-

marizes the iteration history of the structural weight for this problem with Ref. [10] and SACA. This table indicates that SACA obtains the optimum solution with fewer design iterations. Notably, the minimum weight obtained is 1498 lb instead of the 1499 lb obtained in Ref. [10]. This may be attributed to using different finite element numerical algorithms.

4. Conclusions

This study presents a method called SACA. The proposed method is a generalization of the conventional convex approximation method. Our results demonstrate that SACA has two merits. First, the order of approximation for each constraint can be easily determined because SACA yields a higher constraint value with a higher order. Second, SACA can adjust the convex order at each redesign iteration to prevent the underestimation or overestimation of design constraints. This improves accuracy and calculation speed by reducing oscillations and speeding up the convergence rate. This study also applies the SACA method to the optimum design processes of three common structural examples, confirming that it yields quite satisfactory results.

Appendix A. The conservative tendency of the higher-order convex approximation

To prove the conservative tendency of the higher-order convex approximation method, two different set real numbers $\vec{r}$ and $\vec{s}$ are chosen as the order of Eq. (12) and $r_i > s_i$ ($i = 1, 2, \dots, n$). Then, $h_h(\vec{x}, \vec{r})$, $h_h(\vec{x}, \vec{s})$ can be expressed as

$$h_h(\vec{x}, \vec{r}) = h(\vec{x}_0) + \sum_i^{(+)} \frac{1}{r_i} \left(\frac{\partial h}{\partial x_i} \right)_{\vec{x}_0} \left[\left(\frac{x_i}{x_{i0}} \right)^{r_i-1} x_i - x_{i0} \right] + \sum_i^{(-)} \frac{1}{r_i} \left(\frac{\partial h}{\partial x_i} \right)_{\vec{x}_0} \left[\left(\frac{x_i}{x_{i0}} \right)^{r_i-1} x_i - x_{i0} \right] \left(\frac{x_{i0}}{x_i} \right)^{r_i}, \quad (\text{A.1})$$

$$h_h(\vec{x}, \vec{s}) = h(\vec{x}_0) + \sum_i^{(+)} \frac{1}{s_i} \left(\frac{\partial h}{\partial x_i} \right)_{\vec{x}_0} \left[\left(\frac{x_i}{x_{i0}} \right)^{s_i-1} x_i - x_{i0} \right] + \sum_i^{(-)} \frac{1}{s_i} \left(\frac{\partial h}{\partial x_i} \right)_{\vec{x}_0} \left[\left(\frac{x_i}{x_{i0}} \right)^{s_i-1} x_i - x_{i0} \right] \left(\frac{x_{i0}}{x_i} \right)^{s_i}. \quad (\text{A.2})$$

Subtracting Eq. (A.2) from Eq. (A.1) yields

$$h_h(\vec{x}, \vec{r}) - h_h(\vec{x}, \vec{s}) = \sum_i^{(+)} \alpha_i \kappa_i(\xi_i, r_i, s_i) + \sum_i^{(-)} \beta_i \lambda_i(\xi_i, r_i, s_i), \quad (\text{A.3})$$

where

$$\xi_i = \frac{x_i}{x_{i0}},$$

$$\alpha_i = x_{i0} \left(\frac{\partial h}{\partial x_i} \right)_{\mathbf{x}_0} > 0 \quad \text{if } x_{i0} > 0,$$

$$\beta_i = x_{i0} \left(\frac{\partial h}{\partial x_i} \right)_{\mathbf{x}_0} < 0 \quad \text{if } x_{i0} > 0,$$

$$\kappa_i(\xi_i, r_i, s_i) = \frac{\xi_i^{r_i}}{r_i} - \frac{\xi_i^{s_i}}{s_i} + \frac{1}{s_i} - \frac{1}{r_i}, \quad (\text{A.4})$$

$$\lambda_i(\xi_i, r_i, s_i) = \frac{\xi_i^{-s_i}}{s_i} - \frac{\xi_i^{-r_i}}{r_i} + \frac{1}{r_i} - \frac{1}{s_i}. \quad (\text{A.5})$$

Next, we are going to prove that $\kappa_i(\xi_i, r_i, s_i) > 0$ with respect to the variables ξ_i for any two given parameters $r_i, s_i \in \mathbb{R}$ and $r_i > s_i$. The first and second order derivatives of $\kappa_i(\xi_i, r_i, s_i)$ with respect to ξ_i are

$$\kappa_i'(\xi_i, r_i, s_i) = \frac{\partial \kappa_i(\xi_i, r_i, s_i)}{\partial \xi_i} = \xi_i^{r_i-1} - \xi_i^{s_i-1}, \quad (\text{A.6})$$

$$\begin{aligned} \kappa_i''(\xi_i, r_i, s_i) &= \frac{\partial^2 \kappa_i(\xi_i, r_i, s_i)}{\partial \xi_i^2} \\ &= (r_i - 1)\xi_i^{r_i-2} - (s_i - 1)\xi_i^{s_i-2}. \end{aligned} \quad (\text{A.7})$$

The stationary points of $\kappa_i(\xi_i, r_i, s_i)$ can be obtained by solving $\kappa_i'(\xi_i, r_i, s_i) = 0$,

$$\xi_i^{r_i-1} = \xi_i^{s_i-1}, \quad (\text{A.8})$$

$$\xi_i^* = 1. \quad (\text{A.9})$$

Substituting $\xi_i^* = 1$ into $\kappa_i''(\xi_i, r_i, s_i)$ and $\kappa_i(\xi_i, r_i, s_i)$ gives

$$\kappa_i''(\xi_i^*, r_i, s_i) = r_i - s_i > 0, \quad (\text{A.10})$$

$$\kappa_i(\xi_i^*, r_i, s_i) = 0. \quad (\text{A.11})$$

It states that the function $\kappa_i(\xi_i, r_i, s_i)$ has only one minimum point at $\xi_i^* = 1$ and the function value at $\xi_i^* = 1$ is 0. That is,

$$\kappa_i(\xi_i, r_i, s_i) > 0. \quad (\text{A.12})$$

Similarly, it can be proved that

$$\lambda_i(\xi_i, r_i, s_i) < 0. \quad (\text{A.13})$$

From Eqs. (A.12) and (A.13), Eq. (A.3) can be rewritten as

$$h_h(\vec{x}, \vec{r}) - h_h(\vec{x}, \vec{s}) = \sum_i^{(+)} \alpha_i \kappa_i(\xi_i, r_i, s_i) + \sum_i^{(-)} \beta_i \lambda_i(\xi_i, r_i, s_i) > 0. \quad (\text{A.14})$$

That is,

$$h_h(\vec{x}, \vec{r}) \geq h_h(\vec{x}, \vec{s}) \quad \text{for } r_i > s_i \quad (i = 1, 2, \dots, n).$$

Therefore, higher-order approximation results in a higher-approximate function value, proving its conservative tendency.

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