AN ALGORITHM TO BUILD CONVEX HULLS FOR 3-D OBJECTS

Han-Ming Chen* and Tzung-Han Lin

ABSTRACT

In this paper, a new algorithm based on the Quickhull algorithm is proposed to find convex hulls for 3-D objects using neighbor trees. The neighbor tree is the data structure by which all visible facets to the selected furthest outer point can be found. The neighboring sequence of ridges on the outer boundary of all visible facets also can be found directly from the neighbor tree. This new algorithm is twice as efficient as Barber's algorithm.

Key Words: 3-D convex hull, quickhull algorithm, computational geometry.

I. INTRODUCTION

The convex hull of a set S of points is defined as the smallest convex set containing all the points in S . It is one of the most fundamental concepts in the field of computer graphics, computational geometry, pattern recognition, computer-aided design, image processing (Robert and Faugeras, 1995), and robotics. It also plays an important role in many applications such as shape determination (Kaasalainen and Torppa, 2001), collision detection (Sherali *et al.*, 2001), finding the minimum bounding box (O'Rourke, 1993), and constructing spherical Voronoi diagram (Sugihara, 2000).

Graham (1972) proposed an algorithm running in time complexity $O(n \log n)$ for finding the convex hull of a set of planar points. Anderson reevaluated Graham's algorithm for simplifying the computation to determine whether the second point in three consecutive points is convex or not (Anderson, 1978). Later, several quick methods based on Graham's algorithm were published to determine a 2-D convex hull for a set of planar points (Atwah *et al.*, 1995; Atwah and Baker, 2002; Koplowitz and Jouppi, 1978). Jarvis presented a convex hull identification method that includes point deletion checks and is especially effective when the number of points on the plane is relatively small (Jarvis, 1973). In 1996, an algorithm that is a parallel adaptation of Jarvis March and the

Quickhull algorithm was presented by Atwah *et al.* (1996).

In recent decades, at least four algorithms for finding the multi-dimensional convex hull have been proposed. They are gift wrapping, divide and conquer, incremental, and Quickhull algorithms (O'Rourke, 1993).

Chand and Kapur (1970) first presented the gift wrapping algorithm in arbitrary dimensions. In the gift wrapping algorithm, the convex hull of a point set is generated by continuing to find the hyperplane making the maximum interior angle with the adjacent facet until all facets found are closed. Assume that the number of points is n . The gift wrapping algorithm runs in time $O(n^2)$. Sugihara revised the conventional gift wrapping algorithm to increase numerical robustness and topological consistency and kept the same time complexity as the conventional algorithm (Sugihara, 1994).

The divide and conquer algorithm was first proposed by Preparata and Hong (1977) to apply a merge procedure for finding the convex hull of two non-intersecting convex hulls in two or three dimensions with $O(n \log n)$ operations. A more detailed implementation of the 3-D divide and conquer algorithm was presented by Day (1990).

O'Rourke (1993) presented a detailed implementation for the incremental algorithm. Initially, one tetrahedral hull is formed by randomly selecting four non-degeneracy points. Each time a new tetrahedron is formed with an outside point and a triangle on the old hull. Afterward this new tetrahedron is merged into the old hull and the concave parts of the merged hull are filled up. The incremental algorithm is sometimes called the beneath-beyond algorithm. Allison

*Corresponding author. (Tel: 886-2-23680375; Fax: 886-2-23631755; Email: hmchen@ntu.edu.tw)

The authors are with the Department of Mechanical Engineering, National Taiwan University, Taipei, Taiwan 106, R.O.C.

and Noga (1997) combined the divide and conquer algorithm with the incremental algorithm in computing 3-D convex hulls. In 3-D cases, the incremental algorithm requires $O(n \log n)$ expected time (Clarkson and Shor, 1988; Edelsbrunner, 1987; Kallay, 1984; Mulmuley, 1994; Preparata and Shamos, 1985; Seidel, 1990). In each iterative routine of the incremental algorithm, the merged point is randomly selected so that the vertices of the convex hull can't be found efficiently.

Generally speaking, the Quickhull algorithm is evolved from the incremental algorithm. In constructing 2-D convex hulls, Eddy (1977) presented a notion to successively partition a set of planar points into several regions with binary trees. With the "throw-away" principle, Akl and Toussaint (1978) presented an algorithm in which the furthest outside point is selected for the selected edge of the current convex hull in each iterative routine and then the selected edge is replaced with two new edges. Instead of a triangle used by Eddy and a quadrilateral used by Akl (1978) and Bykat (1978) uses a line segment to divide planar points into two regions in the first step. For general-dimension Quickhull algorithms, Barber *et al.*, (1996) proposed detailed procedures and empirical data. His algorithm is very efficient in constructing general-dimension convex hulls.

In this paper, a new Quickhull algorithm for building 3-D convex hulls is presented. We propose some efficient routines to find a tetrahedron as the initial convex hull and then build successively the new convex hull for each new selected vertex. In finding the boundary of all visible facets for a new selected vertex, a neighbor tree is used in this paper. Because Barber's algorithm is one of the latest and efficient Quickhull algorithms, the efficiency of our algorithm is compared with that of Barber's algorithm.

II. METHODS

In some mathematical models, the vertex or facet enumeration is used for representing the convex hull. The vertex enumeration only contains all the vertices of the convex hull in random order. The facet enumeration lists the inside halfspace for each facet of the convex hull. In this paper, each facet and its three neighboring facets of the current convex hull are recorded in a linked list. Compared with the data structures of previous Quickhull algorithms, a much simpler data structure is proposed in this paper to raise the efficiency of our algorithm. The procedure of our algorithm is described as follows and its pseudo-code is shown in Fig. 1 starting from the main routine CONVEX_HULL().

(1) Record all the points of input point set in a linked list L_s . Remove duplicated points from L_s with an

octree. Substitute the coordinate of each point in L_s into four formulas $x + y + z$, $x - y - z$, $-x + y - z$, and $-x - y + z$ separately. Find out a point A in L_s with the maximum value for the formula $x + y + z$ and then take out point A from L_s . Repeat the same process for points B , C , and D and remove them from L_s one by one. Points B , C , and D have the maximum values for formulas $x - y - z$, $-x + y - z$, and $-x - y + z$ respectively. Hence, points A , B , C , and D are definitely different. The initial tetrahedron is built with points A , B , C , and D if any three points of them are not collinear. The normal vector of each facet of the initial tetrahedron must point outward. For each facet of the initial tetrahedron, its three neighboring facets are stored in a linked list. This part of the work has been done by the function FindInitialHull() in Fig. 1.

- (2) All the points in L_s are partitioned into five sets by four facets of the initial tetrahedron in line (a) of Fig. 1. Four sets are outside the tetrahedron and the fifth set is inside the tetrahedron. For each facet T_n , if there are points in front of T_n , put T_n in the tail of a linked list L_p . Otherwise, put T_n in the tail of another linked list L_e . After these four facets have been put into L_p or L_e , connect L_e after L_p forming a new linked list L_h . Assume that the head of L_h is the facet T_j .
- (3) The work in line (b) of Fig. 1 is described as follows. If the facet T_j is NULL, the whole procedure has been finished and all facets in L_h form the final convex hull. If T_j is inside the current convex hull, remove T_j from the linked list L_h , set the next facet in L_h as T_j , and then go back to step (3). If there is no point in front of T_j , set the next facet in L_h as T_j and then go back to step (3).
- (4) Among all points in front of T_j , find the point P furthest from T_j . In line (c) of Fig. 1, set the facet T_j as the root of a neighbor tree. Set three neighboring facets of T_j as its three branches. Assume that the node T_k is one of these three branches.
- (5) For the node T_k , assume that the facet T_a is the upper node of T_k and T_a , T_b , and T_c are three neighboring facets of T_k in the neighbor tree. If T_k is visible to point P , put T_b and T_c as two branches of T_k into the neighbor tree in the same surrounding order as T_a , T_b , and T_c are around T_k on the convex hull. If T_k has already appeared in this neighbor tree, end the work of branch T_k . If T_k is not visible to point P , set T_k as a boundary facet B_n and end the work of branch T_k . If there is any branch T_k whose work has not been ended, go back to step (5).
- (6) Search the neighbor tree and take out the ridge shared by each boundary facet B_n and the upper node facet of B_n with the depth-first search. Connect each ridge with point P to form a facet for

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typedef struct {
    FACET neighboringFacet*[3], *nextFacet;
    bool isInsideFacet = False;
    int branchState[3]; /*record visible, repeated, or boundary for each neighboringFacet*/
    VERTEX *frontVertex; /*the furthest point stored in the head of this linked list*/
} FACET;

NeighborTree(FACET *currentNode, VERTEX *furthest, VERTEX *outsideVertex, FACET *facetList)
{
    currentNode->isInsideFacet = True;

    AddVertexToList(currentNode->frontVertex, outsideVertex); /*collect all points outside visible facets*/

    for (each branch FACET Tk of currentNode) {
        if (Tk has not appeared in the tree) {
            if (furthestVertex is in front of Tk)
                SetbranchState(visible); /*Tk is a visible facet*/
            else
                SetbranchState(boundary); /*Tk is a boundary facet*/
        }
        else
            SetbranchState(repeated); /*Tk is a repeated facet*/
    }

    for (each branch FACET Tj of currentNode)
        switch(branchState of Tj) {
            case "visible": /*Tj is a visible facet*/
                NeighborTree(Tj, furthestVertex, outsideVertex, facetList);
                break;
            case "boundary": /*Tj is a boundary facet*/
                newFacet = BuildOneFacet(); /*furthestVertex and two vertices shared by currentNode and Tj*/
                RecordNewNeighbor(); /*replace currentNode with newFacet as one neighboring facet of Tj*/
                AddToHead(newFacet, facetList);
                break;
            case "repeated": /*Tj is a repeated facet*/
                break;
            default: break;
        }
    }
}

PARTITION(VERTEX *vertexList, FACET *facetList, FACET *convexFacetList)
{
    for (each FACET Tn in facetList) {
        for (each VERTEX Pj left in vertexList)
            if (VERTEX Pj in front of FACET Tn)
                VertexAddToFacet(Pj, Tn); /*add Pj into the frontVertex of Tn*/
        if (Tn->frontVertex != NULL) /*some points are in front of Tn*/
            AddToList(Tn, convexFacetList);
            /*in convexFacetList, add Tn after the last facet that has outside points*/
        else /*no point is in front of Tn*/
            AddToListTail(Tn, convexFacetList); /*add Tn into the tail of convexFacetList*/
    }
}

CONVEX_HULL(VERTEX *vertexList)
{
    FACET *facetList, *convexFacetList = NULL;
    VERTEX *outsideVertex;

    facetList = FindInitialHull(vertexList); /*build the initial tetrahedron*/
    PARTITION(vertexList, facetList, convexFacetList); /*line (a)*/

    for (each FACET Tj in convexFacetList) { /*line (b)*/
        if (Tj->isInsideFacet) /*Tj is inside the convex hull*/
            DeleteFacet(Tj);
        else {
            if (Tj->frontVertex != NULL) {
                Tj->FindBoundary(Tj, frontVertex, outsideVertex, facetList); /*line (c)*/
                RecordNeighboringFacet(facetList); /*record neighboring facets for each new facet*/
                Rearrange(facetList); /*avoid two successive facets neighboring on convex hull*/
                PARTITION(outsideVertex, facetList, convexFacetList); /*line (d)*/
                Tj = convexFacetList; /*go back to the head of convexFacetList after each partitioning*/
            }
        }
    }
}

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Fig. 1 The pseudo-code of our algorithm.

Fig. 2 The procedures of our algorithm

the new convex hull and put this facet into the tail of L_h . Collect all points outside the facets of the new convex hull.

- (7) In line (d) of Fig. 1, partition all outside points with each new facet formed in step (6) using the same process as step (2). Set $j = j + 1$ and go to step (3).

The procedures whole procedures of our algorithm whose starting point and terminal point are [Start] and [End] respectively is shown in Fig. 2.

For 3-D cases mentioned above, four vertices A , B , C , and D are required to build the initial tetrahedron. In 2-D cases, only three points P_1 , P_3 , and P_5 shown in Fig. 3 are needed to form the initial triangle. To demonstrate the whole procedure of our algorithm in a simpler way, a 2-D example is used. In order to describe one important process in our algorithm as shown in Fig. 3(b), the highest point in Fig. 3(a) is not chosen as one vertex, i.e., the point P_1 , for the initial triangle.

In Fig. 3(a), all points are partitioned by facets in sequence of T_1 , T_2 , and T_3 . There are points in front of T_2 and T_3 but there is no point in front of T_1 .

Facet T_3 is after facet T_2 in the partitioning sequence so T_3 is put after T_2 in the linked list of Fig. 4(a) and T_1 is put in the tail of Fig. 4(a). Facet T_2 is the first element with outside points in Fig. 3(a) so T_2 is set as the root of the neighbor tree in Fig. 5(a). Among all points in front of T_2 in Fig. 3(a), find the furthest point P_2 from T_2 . Facets T_1 and T_3 are the neighboring facets of T_2 and not visible to P_2 so T_1 and T_3 are the boundary facets of this tree and set as the branches of the root in Fig. 5(a). The ridge shared by T_1 and T_2 is point P_3 and the ridge shared by T_2 and T_3 is point P_1 . Connect P_2 to P_3 and P_1 separately to get two new facets T_4 and T_5 . All points which are in front of T_2 are partitioned by T_4 and T_5 to get Fig. 3(b) and Fig. 4(b). Facet T_2 becomes an inside facet in the current convex hull and is marked with a cross in Fig. 4(b) because T_2 will not appear on the surface of the final convex hull.

After T_2 has been deleted in Fig. 3(b), T_3 becomes the first element with outside points in the linked list. Hence, set T_3 as the root of the neighbor tree and T_1 and T_5 as two branches of T_3 in Fig. 5(b). Among all points which are in front of T_3 , P_4 is the

Fig. 3 A 2-D example and the processes of forming its final convex hull

Fig. 4 The linked lists for the 2-D example in Fig. 2

furthest point from T_3 in Fig. 3(b). Facet T_1 is not visible to P_4 so T_1 is set as a boundary facet in Fig 5 (b). Facet T_5 is visible to P_4 so T_5 is set as a visible facet. Aside from T_3 , T_4 is another neighboring facet of T_5 and not visible to P_4 so T_4 is set as a boundary facet. Facets T_1 and T_4 are two boundary facets to P_4 . The ridge between T_1 and its upper node, i.e., T_3 , is P_5 . Point P_2 is the ridge between T_4 and its upper

Fig. 5 Four neighbor trees for the 2-D example in Fig. 2

Fig. 6 (a) Light gray facets are visible to the black point and dark gray facets are not visible to the black point. (b) Merge new facets into the old convex hull to form a new convex hull

node, i.e., T_5 . Connect P_4 to P_5 and P_2 to form T_6 and T_7 respectively. All points outside visible facets T_3 and T_5 are partitioned by T_6 and T_7 . Facets T_3 and T_5 become inside facets in the current convex hull so they are marked with crosses in Fig 4(c). There are points outside T_6 and T_7 so T_6 and T_7 are inserted successively into the linked list after T_3 in Fig. 4(c).

Points P_6 and P_7 are the furthest points from T_6 and T_7 respectively in Fig. 3(c) and 3(d). After all inside facets in Fig. 3(e) have been deleted, all the facets of the final convex hull are composed of T_1 , T_4 , T_8 , T_9 , T_{10} , and T_{11} as shown in Fig. 3(f) and 4(f). Only the vertices of the initial triangle or the furthest points can be the vertices of the final convex hull in 2-D examples.

In Fig. 6(a) that is a 3-D example, light gray facets are visible and dark gray facets are not visible to the black point respectively. All visible facets and boundary facets are found with a neighbor tree. All ridges are located on the boundary between visible facets and invisible facets. Connect the black point to the two end points of each ridge to form a new facet. All visible facets will be inside the new convex hull so they will be deleted.

In the 3-D example shown in Fig. 7(a), assume

Fig. 7 (a) A 3-D example. (b) The neighbor tree of the example in (a)

that facet T_1 is set as the root of the neighbor tree in Fig. 7(b). Among all points that are in front of T_1 , assume that P is the furthest point from T_1 . On the convex hull, facets T_1 through T_6 are visible to P . Facets T_2 , T_3 , and T_4 are three neighboring facets of T_1 so set these three facets as the branches of T_1 .

Facets T_1 , B_1 , and B_2 are three neighboring facets of T_2 and their surrounding order around T_2 in the neighbor tree of Fig. 7(b) must be accorded with that around T_2 on the convex hull of Fig. 7(a). Facet T_1 is the upper node of T_2 so set B_1 , and B_2 as two branches of T_2 . Facets B_1 and B_2 are not visible to P so they are boundary facets. Facets T_1 , B_3 , and T_5 are three neighboring facets of T_3 and T_1 is the upper node of T_3 so set B_3 and T_5 as two branches of T_3 . In the similar way, B_4 and T_6 are set as two branches of T_5 and B_5 and T_4 are set as two branches of T_6 . Facet T_4 has already appeared in the neighbor tree so T_4 is a repeated facet. Facets B_3 , B_4 , and B_5 are not visible to P so they are boundary facets. Other parts of the neighbor tree are built in the same way. The boundary of all visible facets is found when every leaf node in a neighbor tree is either a boundary facet or a repeated facet.

Fig. 8 Figures (a), (c), and (e) are three 3-D models. Figures (b), (d), and (f) are the convex hulls of models (a), (c), and (e) respectively

III. IMPLEMENTATION AND COMPARISON

Assume that the plane equation of the facet A is $f_A(x, y, z) = 0$ and the coordinate of the point P is (x_P, y_P, z_P) . When building a neighbor tree in our algorithm, if $f_A(x_P, y_P, z_P) > -\epsilon$, facet A is defined as a visible facet to point P . When doing the partitioning process in our algorithm, if $f_A(x_P, y_P, z_P) > \epsilon$, P is defined as a point in front of facet A . In this paper, the value of ϵ is set to 16 times of the machine epsilon ($2.22\text{E-}16$). Under these two conditions, degenerate cases will not cause any vague situation in our algorithm.

Models in Figs. 8(a) and 8(c) were created by the software. The model in Fig. 8(e) was made with clay and then scanned with the laser scanning system to get image data. Figs. 8(b), 8(d), and 8(f) are the convex hulls of models Figs. 8(a), 8(c), and 8(e) respectively. The last six cases in Table 1 were created by random distribution of points in a unit cube or on the surface of a unit sphere respectively.

Our algorithm has been successfully implemented in C language with an Intel Pentium-4 1.6G personal computer. The source code of Barber's algorithm is available via network address <http://www.qhull.org/download>. In order to compare the net efficiency between Barber's algorithm and our algorithm, the time

Table 1 The efficiency comparison between Barber's algorithm and our algorithm on nine models

Models	Number of points	Number of point on the convex hull	Number of faces on the convex hull	Barber's algorithm (ms)	Our algorithm (ms)	Efficiency ratio
Fig. 8(a)	131,329	31,871	63,738	1,425	498	2.86
Fig. 8(c)	288,000	25,867	51,730	2,140	793	2.70
Fig. 8(e)	634,712	3,497	6,990	1,320	541	2.44
Points in a cube	30,000	171	338	54	24	2.25
	300,000	240	476	550	216	2.55
	3,000,000	372	740	6,265	3,120	2.01
Points on a sphere	10,000	10,000	19,996	272	114	2.39
	100,000	100,000	199,996	2,971	1,306	2.27
	1,000,000	1,000,000	1,999,996	45,530	18,215	2.50

for reading data files is subtracted from the whole running time. Each datum in Table 1 is the average time for running the corresponding algorithm 30 times on the corresponding model.

For storing facets, double linked lists are used in Barber's algorithm, but single linked lists are used in our algorithm. As a visible facet is deleted from the double linked list in Barber's algorithm, the previous element and the next element of the deleted facet will be connected. Hence, any element in the double linked list must have two pointers to the previous element and the next element separately. In our algorithm, visible facets are found and marked in traversing the neighbor tree, and deleted in the next traverse of the linked list. Hence, the time for double linked lists is longer than that for linked lists in doing the similar work. This is the first point where our algorithm is better than Barber's algorithm.

In Barber's algorithm, the memory cell for each ridge is allocated first and then two neighboring facets and two endpoints are stored into this cell. Ridges found by Barber on the outer boundary of all visible facets are randomly stored and then these ridges form new facets with the furthest point. All new facets must be searched with a hash table to find neighboring relations among them. This extra work is not necessary in our algorithm because the neighboring sequence of ridges appears in the neighbor tree. This is the second point where our algorithm is better than Barber's algorithm.

All outer points to visible facets must be partitioned by new facets which are formed with the outer boundary of these visible facets. In Barber's algorithm, any two successively selected partitioning facets may be neighboring on the new convex hull. In our algorithm, two successively selected partitioning facets are arranged to avoid being neighboring facets on the new convex hull. Therefore, the total number of operations of substituting an outer point into the plane

equation of a partitioning facet in our algorithm is less than that in Barber's algorithm. This is the third point where our algorithm is better than Barber's algorithm.

The time complexity of our algorithm is $O(n \log r)$, where n is the number of input points and r is the number of processed points. Although the time complexity order of our algorithm is the same as that of Barber's algorithm, our algorithm is more efficient than Barber's algorithm due to above three points in qualitative comparisons. In quantitative comparisons, the efficiency of our algorithm is over twice that of Barber's algorithm for those nine models in Table 1.

IV. CONCLUSIONS

In this paper, the neighbor tree is proposed to find visible facets and ridges on the outer boundary of these visible facets. The data structure of our algorithm is simpler than that of Barber's algorithm. New facets found in our algorithm appear in neighboring sequence but these new facets must be arranged again in an extra process in Barber's algorithm. The order of partitioning facets to partition outer points in our algorithm is more efficient than that in Barber's algorithm. In both qualitative and quantitative comparisons, our algorithm is more efficient than Barber's algorithm.

ACKNOWLEDGMENTS

We gratefully acknowledge the support of the National Science Council of the Republic of China for this research under the grant NSC 93-2212-E-002-061, and specially thank Ms. Yue-Jun Chen for making the clay model of Fig. 8(e) and Mr. Ming-Hui Lin for getting the image data of this model with the laser scanning system.

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Manuscript Received: Jul. 08, 2005

Revision Received: Dec. 15, 2005

and Accepted: Feb. 07, 2006