Theoretical analysis of flow passing a single sphere moving in a micro-tube

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The flow passing a single sphere moving in a micro-tube is investigated theoretically. Two cases are studied. Case I is for a single sphere moving along the centreline, while case II is for a single sphere moving parallel to the centreline of a micro-tube. The range of Knudsen numbers to be studied lies between 0.01 and 0.1 so that the Navier–Stokes equations with slip boundary conditions are the appropriate system to be analysed. The stream-function method is used to obtain analytic solutions for case I, while the reflection method is used for case II. The effects due to the slip boundary conditions on the flow field, resultant force on the sphere, pressure drop over the sphere, terminal velocity of the sphere, etc., are calculated and discussed.

Keywords: micro-tube; Knudsen number; slip boundary;
stream-function method; reflection method

1. Introduction

In the past decade, modern microfabrication technologies have been able to produce micro-sized machines and apparatus. On account of their small length-scale, micro-sized machines usually possess properties different from those of the ordinary machines with which we are familiar (Duncan & Peterson 1994; Tien & Chen 1994; Ho & Tai 1998). Some of these properties in fluid mechanics are the invalidity of the no-slip boundary condition and the Navier–Stokes equations when the characteristic length is about the order of magnitude of the mean free path of a gas flow in reduced pressure conditions. Micro-channel flows are one of the typical examples which, when combined with micro-valves, are usually applied in biomedical and electronic cooling devices, etc.

The rarefaction of a gas flow is usually characterized by a dimensionless parameter known as the Knudsen number (Kn), which was first proposed by Knudsen in 1909. The definition of the Knudsen number is the ratio of the mean free path of the molecules (λ) to the characteristic gas-flow length (L), i.e. $Kn \equiv \lambda/L$.

Usually, when $Kn \geq 0.1$, the continuum model and, therefore, the Navier–Stokes equations become invalid. The Boltzmann equation has then to be applied to describe the molecular motion properly. When $Kn < 0.01$, the Navier–Stokes equation is a good approximation. However, when $0.01 \leq Kn \leq 0.1$, the slip boundary condition

has to be applied to provide a finite velocity and temperature jump at a solid boundary. It is the latter situation, i.e. a continuum flow field with slip boundary conditions, that is to be investigated in this study.

Flow passing through a tube with a solid sphere or droplet moving along or parallel to the centreline was investigated in the 1960s to simulate the motion of polymers, blood cells, etc., in a tube (Happel & Brenner 1957, 1983, pp. 96–157 and 286–321; Haberman & Sayre 1958; Goldsmith & Mason 1962). Pipe flow with non-Newtonian droplets was also studied (Hetsroni *et al.* 1970; Hyman & Skalak 1972; Wohl 1974).

However, when the channel size becomes extremely small or at a very low pressure condition, the Knudsen number of the flow may become larger than 0.01. In such situations, the validity of the no-slip boundary condition and even the Navier–Stokes equations becomes questionable. As a result, flow structure must be different from the observation of the conventional channel/pipe flow. Flow experiments with a micro-channel were not carried out until around a decade ago. Helium is mostly used in experiments for gas flow. The heat flux, mass flow rate, and/or the pressure distribution along the channel were measured to evaluate the size effect of the micro-channels (Pong *et al.* 1994; Harley *et al.* 1995; Shih *et al.* 1996; Ravigururajan *et al.* 1996; Arkilic 1997; Choquette *et al.* 1996; Flockhart & Dhariwal 1998). Wu *et al.* (1998) also measured the temperature distribution. The tangential momentum accommodation coefficient, which is an important constant in the slip boundary conditions for theoretical analysis, has also been derived from the experiments (Arkilic 1997). Liquid flow in a micro-channel became of interest to the researchers only in recent years (Koplik *et al.* 1989; Pfahler *et al.* 1990, 1991). The decrease in viscosity with decreasing channel size has been studied.

Three different approaches in theoretical analysis of micro-channel flows have been undertaken. One is derived from the kinetic theory, which simulates the molecule collisions directly. Although it is a general approach to describe the flow characteristics, the computing time is tremendously large due to its slow convergence (Ho & Tai 1998). Oh *et al.* (1997) have computed the helium flow in a micro-channel by direct simulation by the Monte Carlo method with monotonic Lagrangian grid (DSMC-MLG) for a Mach number greater than one and Knudsen numbers equal to 0.07, 0.14 and 0.19. Marviplis (1997) also computed, by DSMC, the heat transfer and velocity distribution between two parallel plates with a large aspect ratio. The velocity in his study ranges from very low to supersonic. Kandemir *et al.* (1998) employed the multicell molecular dynamics computational method to compute the one-dimensional heat-conduction problem and planar Couette flow with an aspect ratio of 1/3.5 and a Knudsen number ranging from 0.01 to 1.0.

Another approach is the numerical simulation of the Navier–Stokes equations and energy equation with slip boundary conditions. This method is valid when the magnitude of Kn is less than order one. Several works have been produced using this method (Beskok & Karniadakis 1994; Baker & Calvert 1996; Beskok *et al.* 1996; Shih *et al.* 1996; Kavehpour *et al.* 1997; Takamatsu *et al.* 1997; Papautsky *et al.* 1998). Liu *et al.* (1998) calculated flows past a micro-sphere in a pipe in continuum, slip and transitional flow regimes. However, only the situation with the sphere moving along the centreline of a pipe was computed. The off-centreline effects were not studied. Besides, the effect on the terminal velocity of the sphere due to the slip boundary condition was not analysed.

Figure 1. (a) Schematic of the physical model. (b) The associated cylindrical coordinate system. (c) The associated spherical coordinate system.

Another approach to studying and understanding the micro-channel flows is to obtain the analytic solutions of the simplified Navier–Stokes equations with slip boundary conditions. The most common assumptions are low Reynolds number, low Mach number, and steady and isothermal conditions. With such assumptions, flow over a micro-sphere can be determined (Keh & Chen 1996; Ramkissoon 1997). Even with compressibility, analytic solutions can also be obtained for a micro-channel flow (Prud’homme *et al.* 1986; van den Berg *et al.* 1993; Choquette *et al.* 1996). However, the flow field and pressure distribution in a micro-tube with a sphere moving in it has not been solved. Such a flow situation has applications in biomedical engineering and polymer science.

2. Mathematical formulation

The problem to be investigated is an isothermal gas flow passing a single sphere moving along the centreline of a micro-tube with a constant speed, U . A schematic of the physical model and the associated coordinate system are shown in figure 1. The Knudsen number of the entire flow is between 0.01 and 0.1. Consequently, the flow field can be described properly by the Navier–Stokes equations with slip boundary conditions. The following assumptions are made to further simplify the problem.

- (i) The aspect ratio, defined as the ratio of the tube radius to tube length, is very small.
- (ii) The flow is steady and incompressible.
- (iii) The Reynolds number is small.
- (iv) The gravitational effect and other body forces are negligibly small.
- (v) The entrance and outlet effects of the micro-tube are neglected.

Under these assumptions, the governing equations are reduced to the following forms:

$$\nabla \cdot \mathbf{V} = 0, \quad (2.1)$$

$$\nabla P = \mu \nabla^2 \mathbf{V}. \quad (2.2)$$

In the above equations, $\mathbf{V}$ represents the velocity vector, P the pressure field, and μ the dynamic viscosity. The slip boundary conditions on the tube wall and the surface of the sphere and the upstream and downstream conditions will be given whenever the solutions are pursued.

3. Mathematical solutions of case I

In this section, the velocity field, pressure distribution, and the terminal velocity of a single sphere moving along the centreline of a micro-tube will be analysed. The stream-function method proposed by Haberman & Sayre (1958) is applied to obtain the analytic solutions. However, in §3*b*, the flow field through a micro-tube without a sphere is solved first, to provide the boundary conditions of the velocity distribution needed far upstream and downstream when a single sphere is present. Two coordinate systems, i.e. the cylindrical coordinate system ($\mathbf{e}_R, \mathbf{e}_\phi, \mathbf{e}_Z$) and the spherical coordinate system ($\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$) as shown in figure 1, will be used for analysis.

(a) Flow through a long micro-tube without a sphere

The governing equations are equations (2.1) and (2.2) above. The following slip boundary condition is applied on the inside of the tube:

$$\mathbf{V} = -\frac{2-\sigma}{\sigma} \lambda \frac{\partial \mathbf{V}}{\partial R} = -\alpha \frac{\partial \mathbf{V}}{\partial R} \quad \text{at } R = b. \quad (3.1)$$

Equation (3.1) is the first term of the Taylor series expansion extended by Beskok & Karniadakis (1993), with σ denoting the tangential momentum-accommodation coefficient, λ , the mean free path of molecules and $\alpha = (2-\sigma)\lambda/\sigma$.

By further assuming the flow is axi-symmetrical and fully developed, the velocity vector, $\mathbf{V} = (V_R, V_\phi, V_Z)$, in the cylindrical coordinate system can be easily solved as follows:

$$V_\phi = V_R = 0, \quad (3.2)$$

$$V_Z = \frac{1}{\mu} \frac{dP}{dZ} \left(\frac{1}{4} R^2 - \frac{1}{4} b^2 - \frac{1}{2} \alpha b \right). \quad (3.3)$$

In equation (3.3), R represents the radial coordinate, b is the radius of the micro-tube, and dP/dZ is a constant. When the mass flow rate, $\dot{m}$, is given, it can be shown that

$$\frac{dP}{dZ} = \frac{-8\mu\dot{m}}{\pi\rho(b^4 + 4\alpha b^3)} = \frac{-8\nu\dot{m}}{\pi(b^4 + 4\alpha b^3)}, \quad (3.4)$$

where ν is the kinematic viscosity of the gas.

(b) *The stream-function method*

The stream-function method proposed by Haberman & Sayre (1958) will be employed to solve the flow over a single sphere moving along the centreline of a micro-tube. In this section, the governing equation satisfied by the stream function will be given first. Then, the solution in either the cylindrical or spherical coordinate system is obtained without temporarily satisfying the slip boundary conditions. Finally, the slip boundary condition on the inside surface of the micro-tube is made to be satisfied by the solution in the cylindrical coordinate system, while that on the surface of the sphere is satisfied by the solution in the spherical coordinate system. These two particular solutions are then forced to equal to each other by series expansion and matching the corresponding coefficients. The final solution in terms of an infinite series, either in the cylindrical or in the spherical coordinate system, satisfies not only the governing equation but also the slip boundary conditions at both the tube wall and the surface of the sphere. A flow chart illustrating the stream-function method is given in figure 2.

As derived in detail by Brenner (1962), the governing equation satisfied by the stream function, ψ , is

$$E^4\psi = E^2(E^2\psi) = 0, \quad (3.5)$$

where

$$E^2 = R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial}{\partial R} \right) + \frac{\partial^2}{\partial Z^2}$$

in the cylindrical coordinate system and

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right)$$

in the spherical coordinate system.

Since the problem to be considered is the flow passing a single sphere moving with a constant speed U along the centreline of an infinitely long micro-tube, the origin of the coordinate system is located, for convenience, at the centre of the moving sphere. The boundary conditions are now prescribed as follows.

Far upstream ($Z \rightarrow +\infty$) and far downstream ($Z \rightarrow -\infty$) of the sphere, the velocity distribution is assumed to be the solution obtained previously for a micro-tube without the presence of a sphere, that is, as $Z \rightarrow +\infty$,

$$V_Z = -U + V \left(1 - \frac{R^2}{b^2} \right) + W, \quad (3.6a)$$

$$V_R = 0, \quad (3.6b)$$

with

$$V = -\frac{b^2}{4\mu} \frac{dP}{dZ}, \quad (3.7a)$$

$$W = -\frac{\alpha b}{2\mu} \frac{dP}{dZ}, \quad (3.7b)$$

Figure 2. The flow chart of the stream-function method.

where U is the velocity of the sphere. The boundary conditions at the inner surface of the micro-tube, i.e. $R = b$, are

$$V_Z = -U - \alpha \frac{\partial V_Z}{\partial R} = -U - \frac{Wb}{2V} \frac{\partial V_Z}{\partial R}, \quad (3.8a)$$

$$V_R = 0, \quad (3.8b)$$

or

$$\psi = \frac{1}{2}(U - W)b^2 - \frac{1}{4}Vb^2. \quad (3.9)$$

The boundary conditions at the surface of the sphere, i.e. $r = c$, are

$$V_\theta = \alpha \frac{\partial V_\theta}{\partial r} = \frac{Wc}{2V} \frac{\partial V_\theta}{\partial r}, \quad (3.10 a)$$

$$V_r = 0. \quad (3.10 b)$$

Since the boundary conditions and the governing equations of the stream function are symmetrical about the (x, y) -plane, the stream function must be an even function of $\cos \theta$ in the spherical coordinates and an even function of Z in the cylindrical coordinates. Therefore, the general solution of the stream function in the spherical coordinates is

$$\psi(r, \theta) = \sum_{2, \text{ even}}^{\infty} C_n^{-1/2} (\cos \theta) (A_n r^n + B_n r^{-n+1} + C_n r^{n+2} + D_n r^{-n+3}), \quad (3.11)$$

where $C_n^{-1/2}$ is the first-kind Gegenbauer function of order n with degree $-1/2$; A_n , B_n , C_n and D_n are undetermined constants. To satisfy the boundary condition at infinity and the Dirichlet condition of the Fourier integral, the general solution of the stream function in cylindrical coordinates is

$$\begin{aligned} \psi(R, z) = \int_0^\infty [RK_1 f_1(a) + R^2 K_0 F_1(a) + RI_1 g_1(a) + R^2 I_0 G_1(a)] \cos(aZ) da \\ + \frac{1}{2}(U - V - W)R^2 + \frac{V}{4b^2}R^4. \end{aligned} \quad (3.12)$$

In the above expression, (I_0, I_1) and (K_0, K_1) are the modified first- and second-kind Bessel functions of zeroth and first order, respectively; $f_1(a)$, $g_1(a)$, $F_1(a)$ and $G_1(a)$ are undetermined functions of the dummy variable a .

In order to transform the solution in cylindrical coordinates to that in spherical coordinates, let

$$f_1(a) = \sum_0^\infty a_n a^n, \quad (3.13 a)$$

$$F_1(a) = \sum_0^\infty b_n a^n. \quad (3.13 b)$$

It can be shown (Haberman & Sayre 1958) that the odd terms of a_n and even terms of b_n are even functions of r , so only the odd a_n and even b_n are chosen in the above summations, which results in the following expression:

$$\begin{aligned} \psi(R, Z) = \int_0^\infty RK_1(aR)(a_1 a + a_3 a^3 + a_5 a^5 + \dots) \cos(aZ) da \\ + \int_0^\infty R^2 K_0(aR)(b_0 + b_2 a^2 + b_4 a^4 + \dots) \cos(aZ) da \\ + \int_0^\infty [RI_1(aR)g_1(a) + R^2 I_0(aR)G_1(a) \dots] \cos(aZ) da \\ + \frac{1}{2}(U - V - W)R^2 + \frac{V}{4b^2}R^4. \end{aligned} \quad (3.14)$$

Expanding the last integral by Taylor series about the origin, it becomes

$$\begin{aligned}
 \psi(R, Z) = & a_1 \frac{\pi}{r} C_2^{-1/2} - a_3 \frac{12\pi}{r^3} C_4^{-1/2} + a_5 \frac{360\pi}{r^5} C_6^{-1/2} - \dots \\
 & + b_0 \pi r C_2^{-1/2} + b_2 \frac{2\pi}{5r} C_2^{-1/2} - b_2 \frac{12\pi}{5r} C_4^{-1/2} \\
 & - b_4 \frac{16\pi}{r^3} C_4^{-1/2} + b_4 \frac{40\pi}{r^3} C_6^{-1/2} + \dots \\
 & + \frac{1}{2} R^2 \left[\int_0^\infty (S_1 a + 2S_3)(a_1 a + a_3 a^3 + a_5 a^5 + \dots) da \right. \\
 & \quad \left. + \int_0^\infty (S_2 a + 2S_4)(b_0 + b_2 a^2 + b_4 a^4 + \dots) da + U - V - W \right] \\
 & - \frac{1}{4} R^2 Z^2 \left[\int_0^\infty (S_1 a + 2S_3)(a_1 a + a_3 a^3 + a_5 a^5 + \dots) da \right. \\
 & \quad \left. + \int_0^\infty (S_2 a + 2S_4)(b_0 + b_2 a^2 + b_4 a^4 + \dots) da \right] \\
 & + \frac{1}{24} R^4 \left[\int_0^\infty (\frac{3}{2} S_1 a^3 + 6S_3 a^2)(a_1 a + a_3 a^3 + a_5 a^5 + \dots) da \right. \\
 & \quad \left. + \int_0^\infty (\frac{3}{2} S_2 a^3 + 2S_4 a^2)(b_0 + b_2 a^2 + b_4 a^4 + \dots) da + \frac{6V}{b^2} \right] \\
 & - \frac{1}{48} R^4 Z^2 \left[\int_0^\infty (\frac{3}{2} S_1 a^3 + 6S_3 a^2)(a_1 a + a_3 a^3 + a_5 a^5 + \dots) da \right. \\
 & \quad \left. + \int_0^\infty (\frac{3}{2} S_2 a^3 + 2S_4 a^2)(b_0 + b_2 a^2 + b_4 a^4 + \dots) da \right] \\
 & + \dots .
 \end{aligned} \tag{3.15}$$

The solution of the stream function in the spherical coordinates can also be transformed ($R = r \cdot \sin \theta$ and $Z = r \cdot \cos \theta$) to

$$\begin{aligned}
 \psi = & C_2^{-1/2} \left(B_2 \frac{1}{r} + D_2 r \right) + \frac{1}{2} R^2 A_2 + R^2 Z^2 \left(\frac{1}{2} A_4 + \frac{1}{2} C_2 \right) + R^4 \left(-\frac{1}{8} A_4 + \frac{1}{2} C_2 \right) \\
 & + C_4^{-1/2} \left(B_4 \frac{1}{r^3} + D_4 \frac{1}{r} \right) + R^2 Z^4 \left(\frac{1}{2} A_6 + \frac{1}{2} C_4 \right) + R^4 Z^2 \left(-\frac{3}{4} A_6 + \frac{3}{8} C_4 \right) \\
 & + R^6 \left(\frac{1}{16} A_6 - \frac{1}{8} C_4 \right) + C_6^{-1/2} \left(B_6 \frac{1}{r^5} + D_6 \frac{1}{r^3} \right) + R^2 Z^6 \left(\frac{1}{2} A_8 + \frac{1}{2} C_6 \right) \\
 & + R^4 Z^4 \left(-\frac{15}{8} A_8 - \frac{1}{4} C_6 \right) + R^6 Z^2 \left(\frac{15}{16} A_8 - \frac{11}{16} C_6 \right) + R^8 \left(-\frac{5}{128} A_8 + \frac{1}{16} A_6 \right) \\
 & + \dots .
 \end{aligned} \tag{3.16}$$

Equating the above two equations, the following relationships between A_n , B_n , C_n , D_n and a_n , b_n have to be satisfied:

$$B_n = (-1)^{n/2+1} \pi \left[\frac{n!}{2} a_{n-1} + \frac{n(n-1)n!}{2(2n+1)} b_n \right], \tag{3.17}$$

$$D_n = (-1)^{n/2+1} \pi \frac{n!}{2(2n-3)} b_{n-2}, \tag{3.18}$$

$$A_2 = \int_0^\infty (S_1 a + 2S_3)(a_1 a + a_3 a^3 + a_5 a^5 + \dots) da + \int_0^\infty (S_2 a + 2S_4)(b_0 + b_2 a^2 + b_4 a^4 + \dots) da + U - V - W, \quad (3.19 a)$$

$$A_4 = -\frac{1}{2} \left[\frac{a_1}{b^5} \int_0^\infty S_1(ab)^4 d(ab) + \frac{a_3}{b^7} \int_0^\infty S_1(ab)^6 d(ab) + \dots \right. \\ \left. + \frac{b_0}{b^3} \int_0^\infty (S_2 a)(ab)^2 d(ab) + \frac{b_2}{b^5} \int_0^\infty (S_2 a)(ab)^4 d(ab) + \dots \right] \\ - \frac{6}{5} \left[\frac{a_1}{b^5} \int_0^\infty \left(\frac{S_3}{a} \right) (ab)^4 d(ab) + \frac{a_3}{b^7} \int_0^\infty \left(\frac{S_3}{a} \right) (ab)^6 d(ab) + \dots \right. \\ \left. + \frac{b_0}{b^3} \int_0^\infty S_4(ab)^2 d(ab) + \frac{b_2}{b^5} \int_0^\infty S_4(ab)^4 d(ab) + \dots \right] - \frac{2}{5} \frac{V}{b^2}, \quad (3.19 b)$$

$$A_n = \frac{-1}{(n-2)!} \left[\frac{a_1}{b^{n+1}} \int_0^\infty S_1(ab)^n d(ab) \right. \\ \left. + \frac{a_3}{b^{n+3}} \int_0^\infty S_1(ab)^{n+2} d(ab) + \frac{a_5}{b^{n+5}} \int_0^\infty S_1(ab)^{n+4} d(ab) \right] \\ + \frac{n-n^2}{(n-2)!(2n-3)} \left[\frac{a_1}{b^{n+1}} \int_0^\infty \left(\frac{S_3}{a} \right) (ab)^n d(ab) \right. \\ \left. + \frac{a_3}{b^{n+3}} \int_0^\infty \left(\frac{S_3}{a} \right) (ab)^{n+2} d(ab) + \dots \right. \\ \left. + \frac{b_0}{b^{n-1}} \int_0^\infty S_4(ab)^{n-2} d(ab) \right. \\ \left. + \frac{b_2}{b^{n+1}} \int_0^\infty S_4(ab)^n d(ab) + \dots \right], \quad (3.19 c)$$

$$C_2 = \frac{1}{5} \left[\frac{a_1}{b^5} \int_0^\infty \left(\frac{S_3}{a} \right) (ab)^4 d(ab) + \frac{a_3}{b^7} \int_0^\infty \left(\frac{S_3}{a} \right) (ab)^6 d(ab) + \dots \right. \\ \left. + \frac{b_0}{b^3} \int_0^\infty \left(\frac{S_4}{a^2} \right) (ab)^4 d(ab) + \frac{b_2}{b^5} \int_0^\infty \left(\frac{S_4}{a^2} \right) (ab)^6 d(ab) + \dots \right] + \frac{2}{5} \frac{V}{b^2}, \quad (3.20 a)$$

$$C_n = \frac{1}{(n-2)!(2n+1)} \left[\frac{a_1}{b^{n+3}} \int_0^\infty \left(\frac{S_3}{a} \right) (ab)^{n+2} d(ab) \right. \\ \left. + \frac{a_3}{b^{n+5}} \int_0^\infty \left(\frac{S_3}{a} \right) (ab)^{n+4} d(ab) + \dots \right. \\ \left. + \frac{b_0}{b^{n+1}} \int_0^\infty \left(\frac{S_4}{a^2 b^2} \right) (ab)^{n+2} d(ab) \right. \\ \left. + \frac{b_2}{b^{n+3}} \int_0^\infty \left(\frac{S_4}{a^2 b^2} \right) (ab)^{n+4} d(ab) + \dots \right]. \quad (3.20 b)$$

With the boundary conditions

$$V_r = 0 \quad \text{and} \quad V_\theta = \alpha \frac{\partial V_\theta}{\partial r} = \frac{Wa}{2V} \frac{\partial V_\theta}{\partial r}$$

on the surface of the sphere, the following two equations for A_n , B_n , C_n and D_n can be obtained.

From $V_r = 0$,

$$A_n c^{n-2} + B_n c^{-n-1} + C_n c^n + D_n c^{-n+1} = 0. \quad (3.21)$$

From $V_\theta = \alpha \partial V_\theta / \partial r$,

$$\begin{aligned} [nc^{n-2} - \alpha n(n-2)c^{n-3}]A_n - [\alpha(n-1)(n+1)c^{-n-2} + (n-1)c^{-n-1}]B_n \\ + [(n+2)c^n - \alpha n(n+2)c^{n-1}]C_n - [(n-3)c^{-n+1} + \alpha(n-1)(n-3)c^{-n}]D_n = 0. \end{aligned} \quad (3.22)$$

Then

$$A_n = \frac{(2n+1)c^{1-2n}(c-\alpha)}{4\alpha n - 2c} B_n + \frac{(2n-1)c^{3-2n}(c-3\alpha)}{4\alpha n - 2c} D_n, \quad (3.23)$$

$$C_n = \frac{(2n-1)c^{-2n-1}(c+\alpha)}{2c - 4\alpha n} B_n + \frac{(2n-3)c^{1-2n}(\alpha-c)}{4\alpha n - 2c} D_n. \quad (3.24)$$

Substituting equations (3.27)–(3.30) into the above two equations results in $2n$ equations of $2n$ unknowns, i.e. a_n and b_n , as shown in the following.

From A_2 ,

$$\begin{aligned} \int_0^\infty (S_1 a + 2S_3)(a_1 a + a_3 a^3 + a_5 a^5 + \dots)^4 da \\ + \int_0^\infty (S_2 a + 2S_4)(b_0 + b_2 a^2 + b_4 a^4 + \dots) da \\ - \frac{5c^{-3}(c-\alpha)}{8\alpha - 2c} \pi [a_1 + \frac{2}{5}b_2] - \frac{3c^{-1}(c-3\alpha)}{8\alpha - 2c} \pi b_0 = -U + V + W. \end{aligned} \quad (3.25)$$

From A_4 ,

$$\begin{aligned} -\frac{1}{2} \left[\frac{a_1}{b^5} \int_0^\infty S_1(ab)^4 d(ab) + \frac{a_3}{b^7} \int_0^\infty S_1(ab)^6 d(ab) + \dots \right. \\ \left. + \frac{b_0}{b^3} \int_0^\infty (S_2 a)(ab)^2 d(ab) + \frac{b_2}{b^5} \int_0^\infty (S_2 a)(ab)^4 d(ab) + \dots \right] \\ - \frac{6}{5} \left[\frac{a_1}{b^5} \int_0^\infty \left(\frac{S_3}{a} \right) (ab)^4 d(ab) + \frac{a_3}{b^7} \int_0^\infty \left(\frac{S_3}{a} \right) (ab)^6 d(ab) + \dots \right. \\ \left. + \frac{b_0}{b^3} \int_0^\infty S_4(ab)^2 d(ab) + \frac{b_2}{b^5} \int_0^\infty S_4(ab)^4 d(ab) + \dots \right] \\ + \frac{9c^{-7}(c-\alpha)}{16\alpha - 2c} \pi [12a_3 + \frac{2}{3}b_4] + \frac{7c^{-5}(c-3\alpha)}{16\alpha - 2c} \pi (\frac{12}{5})b_2 = \frac{2}{5} \frac{V}{b^2}. \end{aligned} \quad (3.26)$$

From A_n ($n \neq 2, 4$),

$$\begin{aligned}
 & \frac{-1}{(n-2)!} \sum_{m=1,3,\dots}^{\infty} \frac{a_m}{b^{n+m}} \int_0^{\infty} S_1(ab)^{n+m-1} d(ab) \\
 & - \frac{1}{(n-2)!} \sum_{m=0,2,\dots}^{\infty} \frac{b_m}{b^{n+m-1}} \int_0^{\infty} S_2(ab)^{n+m-2} d(ab) \\
 & + \frac{(n-n^2)}{(n-2)!(2n-3)} \left[\sum_{m=1,3,\dots}^{\infty} \int_0^{\infty} \frac{a_m}{b^{n+m}} \left(\frac{S_3}{a} \right) (ab)^{n+m-1} d(ab) \right. \\
 & \quad \left. + \sum_{m=0,2,\dots}^{\infty} \frac{b_m}{b^{n+m-1}} \int_0^{\infty} S_4(ab)^{n+m-2} d(ab) \right] \\
 & - \frac{(2n+1)c^{1-2n}(c-\alpha)}{4\alpha n-2c} (-1)^{n/2+1} \pi \left[\frac{n!}{2} a_{n-1} + \frac{n(n-1)n!}{2(2n+1)} b_n \right] \\
 & - \frac{(2n-1)c^{3-2n}(c-3\alpha)}{4\alpha n-2c} (-1)^{n/2+1} \pi \frac{n!}{2(2n-3)} b_{n-2} = 0. \tag{3.27}
 \end{aligned}$$

From C_2 ,

$$\begin{aligned}
 & \frac{1}{5} \left[\frac{a_1}{b^5} \int_0^{\infty} \left(\frac{S_3}{a} \right) (ab)^4 d(ab) + \frac{a_3}{b^7} \int_0^{\infty} \left(\frac{S_3}{a} \right) (ab)^6 d(ab) + \dots \right. \\
 & \quad \left. + \frac{b_0}{b^3} \int_0^{\infty} \left(\frac{S_4}{a^2 b^2} \right) (ab)^4 d(ab) + \frac{b_2}{b^5} \int_0^{\infty} \left(\frac{S_4}{a^2 b^2} \right) (ab)^6 d(ab) + \dots \right] \\
 & - \frac{3c^{-5}(c+\alpha)}{2c-8\alpha} \pi [a_1 + \frac{2}{5}b_2] - \frac{c^{-3}(\alpha-c)}{8\alpha-2c} \pi b_0 = -\frac{2}{5} \frac{V}{b^2}. \tag{3.28}
 \end{aligned}$$

From C_n ($n \neq 2$)

$$\begin{aligned}
 & \frac{1}{(n-2)!(2n+1)} \left[\sum_{m=1,3,\dots}^{\infty} \frac{a_m}{b^{n+m+2}} \int_0^{\infty} \left(\frac{S_3}{a} \right) (ab)^{n+m+1} d(ab) \right. \\
 & \quad \left. + \sum_{m=0,2,\dots}^{\infty} \frac{b_m}{b^{n+m+1}} \int_0^{\infty} \left(\frac{S_4}{a^2 b^2} \right) (ab)^{n+m+2} d(ab) \right] \\
 & - \frac{(2n-1)c^{-2n-1}(c+\alpha)}{2c-4\alpha n} (-1)^{n/2+1} \pi \left[\frac{n!}{2} a_{n-1} + \frac{n(n-1)n!}{2(2n+1)} b_n \right] \\
 & - \frac{(2n-3)c^{1-2n}(\alpha-c)}{4\alpha n-2c} (-1)^{n/2+1} \pi \frac{n!}{2(2n-3)} b_{n-2} = 0. \tag{3.29}
 \end{aligned}$$

These form a system of $2n$ linear equations which can be expressed as $A \cdot X = B$, with A being a $2n \times 2n$ coefficient matrix, X a $2n \times 1$ column matrix of the unknowns, and B a $2n \times 1$ column matrix of the driving terms of the flow field. Their expressions

are as follows:

$$A = \left[\begin{array}{cccc} -\frac{3c^{-1}(c-3\alpha)}{8\alpha-2c}\pi + \int_0^\infty (S_2a + 2S_4) da & \dots & \dots & \dots \\ \vdots & & \ddots & \\ \vdots & & & \ddots \\ \vdots & & & \ddots \end{array} \right],$$

$$X = \begin{bmatrix} b_0 \\ a_1 \\ b_2 \\ a_3 \\ \vdots \end{bmatrix}, \quad B = \begin{bmatrix} -U + V + W \\ -\frac{2}{5} \frac{V}{b^2} \\ \frac{2}{5} \frac{V}{b^2} \\ 0 \\ \vdots \end{bmatrix}.$$

(3.30)

The unknown b_0 can be decomposed into two parts, b_{01} and b_{02} , as

$$b_0 = b_{01} - b_{02}, \tag{3.31}$$

with b_{01} and b_{02} being determined from the following two systems of algebraic equations, respectively:

system 1: $A \cdot X = B_1,$

$B_1 = \begin{bmatrix} -U + W \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix};$

system 2: $A \cdot X = B_2,$

$B_2 = \begin{bmatrix} -V \\ \frac{2}{5} \frac{V}{b^2} \\ -\frac{2}{5} \frac{V}{b^2} \\ 0 \\ \vdots \end{bmatrix}.$

As proved by Haberman & Sayre (1958), the resultant force experienced by the sphere can be expressed as

$$F_{\text{result}} = -4\pi^2\mu b_0 = -4\pi^2\mu(b_{01} - b_{02}), \tag{3.32}$$

where F_{result} is taken to be positive in the negative z -direction. Two wall-correction factors, K_1 and K_2 , are now defined as follows:

$$K_1 \text{ (first wall-correction factor)} = -\frac{4\pi^2\mu b_{01}}{6\pi\mu(U-W)c} = -\frac{2\pi}{3} \frac{b_{01}}{(U-W)c}, \tag{3.33}$$

Figure 3. (a) The physical model for a sphere moving along a line parallel to the axis. (b) The associated coordinate systems.

$$K_2 \text{ (second wall-correction factor)} = -\frac{4\pi^2\mu b_{02}}{6\pi\mu Vc} = -\frac{2\pi}{3}\frac{b_{02}}{Vc}. \quad (3.34)$$

Hence, F_{result} can be re-expressed as

$$F_{\text{result}} = 6\pi\mu c[(U - W)K_1 - VK_2]. \quad (3.35)$$

Since when the sphere reaches terminal velocity the resultant force must vanish, i.e. drag equals pressure difference,

$$U_{\text{terminal}} = V\frac{K_2}{K_1} + W. \quad (3.36)$$

4. Mathematical solutions of case II

The schematic of the physical model and the associated coordinate system are shown in figure 3. In this section, a sphere translating in a micro-tube along a line parallel to the axis is considered. The stream-function method becomes invalid because of the asymmetry. To solve the problem, the reflection method as proposed by Happel & Brenner (1957) is employed.

(a) The reflection method

The reflection method basically is to solve a problem by a series of reflections. At each reflection, a solution satisfying the governing equations is obtained so that its combination with the solution obtained by the previous reflection successively satisfies one and only one of the boundary conditions. If the reflection method converges,

the summation of the solutions obtained by a series of reflections will approximate the exact solution reasonably. Due to the linearity of the governing equations and slip boundary conditions, $\mathbf{V}$ and P can be expressed as

$$\mathbf{V} = \mathbf{V}^{(0)} + \mathbf{V}^{(1)} + \mathbf{V}^{(2)} + \mathbf{V}^{(3)} + \dots, \quad (4.1)$$

$$P = P^{(0)} + P^{(1)} + P^{(2)} + P^{(3)} + \dots. \quad (4.2)$$

$\mathbf{V}^{(i)}$ and $P^{(i)}$ satisfy the corresponding equations (2.1), (2.2) and the following boundary conditions, respectively:

$$\mathbf{V}^{(0)} \Rightarrow \left\{ \begin{array}{l} V_R^{(0)} = 0, \quad V_\phi^{(0)} = -\alpha \frac{\partial V_\phi^{(0)}}{\partial R}, \\ V_Z^{(0)} = -U - \alpha \frac{\partial V_Z^{(0)}}{\partial R}, \end{array} \right\} \quad \text{at } R = b, \quad (4.3)$$

$$\mathbf{V}^{(1)} \Rightarrow \left\{ \begin{array}{l} V_r^{(1)} = -V_r^{(0)}, \quad V_\theta^{(1)} + V_\theta^{(0)} = \alpha \frac{\partial (V_\theta^{(1)} + V_\theta^{(0)})}{\partial r}, \\ V_\omega^{(1)} + V_\omega^{(0)} = \alpha \frac{\partial (V_\omega^{(1)} + V_\omega^{(0)})}{\partial r}, \\ \mathbf{V}^{(1)} = \mathbf{0}, \quad \text{at } r, z \rightarrow \pm\infty, \end{array} \right\} \quad \text{at } r = c, \quad (4.4)$$

$$\mathbf{V}^{(2)} \Rightarrow \left\{ \begin{array}{l} V_R^{(2)} = -V_R^{(1)}, \quad V_\phi^{(2)} + V_\phi^{(1)} = -\alpha \frac{\partial (V_\phi^{(2)} + V_\phi^{(1)})}{\partial R}, \\ V_Z^{(2)} + V_Z^{(1)} = -\alpha \frac{\partial (V_Z^{(2)} + V_Z^{(1)})}{\partial R}, \\ \mathbf{V}^{(2)} = \mathbf{0} \quad \text{at } Z \rightarrow \pm\infty, \end{array} \right\} \quad \text{at } R = b, \quad (4.5)$$

$$\mathbf{V}^{(3)} \Rightarrow \left\{ \begin{array}{l} V_r^{(2)} = -V_r^{(3)}, \quad V_\theta^{(2)} + V_\theta^{(3)} = \alpha \frac{\partial (V_\theta^{(2)} + V_\theta^{(3)})}{\partial r}, \\ V_\omega^{(2)} + V_\omega^{(3)} = \alpha \frac{\partial (V_\omega^{(2)} + V_\omega^{(3)})}{\partial r}, \\ \mathbf{V}^{(3)} = \mathbf{0} \quad \text{at } r, z \rightarrow \pm\infty, \end{array} \right\} \quad \text{at } r = c, \quad (4.6)$$

$$\text{etc.} \quad (4.7)$$

The initial field $\mathbf{V}^{(0)}$ is simply the same as the slip flow through a micro-tube without a sphere. The solutions of $\mathbf{V}^{(0)}$ are

$$\mathbf{V}^{(0)} = \left[-U + V \left(1 - \frac{R^2}{b^2} \right) + W \right] \mathbf{e}_Z, \quad (4.8a)$$

$$V = -\frac{b^2}{4\mu} \frac{dP}{dz}, \quad (4.8b)$$

$$W = -\frac{\alpha b}{2\mu} \frac{dP}{dz}. \quad (4.8c)$$

Once the velocity field $\mathbf{V}^{(0)}$ has been obtained, $\mathbf{V}^{(1)}$, corresponding to the flow field around a sphere immersed in an infinite domain with specified boundary conditions

Figure 4. Flow chart of the reflection method.

as described in equation (4.4), is purposely included so that the combination of $\mathbf{V}^{(0)} + \mathbf{V}^{(1)}$ satisfied the slip boundary conditions on the sphere surface. However, the slip boundary conditions at the tube wall are now sacrificed. To take care of this deficiency, $\mathbf{V}^{(2)}$ is sought so that $\mathbf{V}^{(1)} + \mathbf{V}^{(2)}$ (or $\mathbf{V}^{(0)} + \mathbf{V}^{(1)} + \mathbf{V}^{(2)}$) satisfies the slip boundary conditions on the tube wall. But the slip boundary conditions on the sphere surface are sacrificed this time. The flow field is hence approximated by such a recursive procedure of reflections with a certain accuracy, if it converges. A flow chart illustrating the procedure of the reflection method is given in figure 4.

When the velocity field has been determined by the reflection method, the drag force and the torque experienced by the sphere can then be evaluated by summing the corresponding contributions due to each reflected flow field. Since there exists no singularity inside the sphere for each reflected field $\mathbf{V}^{(j)}$, with j an even number, these fields produce neither drag nor torque on the sphere. Therefore, the drag force

and torque exerted on the sphere can be expressed as

$$\mathbf{F} = \mathbf{F}^{(1)} + \mathbf{F}^{(3)} + \mathbf{F}^{(5)} + \dots, \quad (4.9)$$

$$\mathbf{T} = \mathbf{T}^{(1)} + \mathbf{T}^{(3)} + \mathbf{T}^{(5)} + \dots. \quad (4.10)$$

The rate of convergence of the reflection method is of an order of c/b , which was proved by Brenner (1962).

(b) *The first reflection*

The boundary conditions on the sphere surface for the first reflection field $V^{(1)}$, as described in equation (4.4), are

$$\begin{aligned} V_r^{(1)} &= -V_r^{(0)}, \\ V_\theta^{(1)} + V_\theta^{(0)} &= \alpha \frac{\partial(V_\theta^{(1)} + V_\theta^{(0)})}{\partial r}, \\ V_\omega^{(1)} + V_\omega^{(0)} &= \alpha \frac{\partial(V_\omega^{(1)} + V_\omega^{(0)})}{\partial r}. \end{aligned}$$

The general solutions of $\mathbf{V}^{(1)}$ and $\partial\mathbf{V}^{(1)}/\partial r$ are

$$\begin{aligned} \mathbf{V}^{(1)} = \sum_{-\infty}^{\infty} \left[\nabla\phi_n + \frac{(n+3)}{2\mu(n+1)(2n+3)} r^2 \nabla P_n \right. \\ \left. - \frac{n}{\mu(n+1)(2n+3)} \mathbf{r} P_n + \nabla \times (\mathbf{r} \chi_n) \right], \end{aligned} \quad (4.11)$$

$$\begin{aligned} \frac{\partial\mathbf{V}^{(1)}}{\partial r} = \frac{1}{r} \sum_{-\infty}^{\infty} \left[n \nabla \times (\mathbf{r} \chi_n) + (n-1) \nabla\phi_n \right. \\ \left. + \frac{(n+3)}{2\mu(2n+3)} r^2 \nabla P_n - \frac{n}{\mu(2n+3)} \mathbf{r} P_n \right], \end{aligned} \quad (4.12)$$

with ϕ_n , P_n and χ_n being the spherical harmonic functions.

To fit the form of the general solutions in spherical coordinates, the boundary conditions on the spherical surface have to be converted to algebraic equations in terms of ϕ_n , χ_n and P_n , which yield, after a tedious mathematical manipulation (Wen 1999) and the application of the Euler theorem on spherical harmonic functions,

$$\begin{aligned} \mathbf{V}^{(1)} = \sum_{n=1}^{\infty} \left[\nabla\phi_{-(n+1)} - \frac{n-2}{\mu 2n(2n-1)} r^2 \nabla P_{-(n+1)} \right. \\ \left. + \frac{n+1}{\mu n(2n-1)} \mathbf{r} P_{-(n+1)} + \nabla \times (\mathbf{r} \chi_{-(n+1)}) \right], \end{aligned} \quad (4.13)$$

$$\begin{aligned} \frac{\partial\mathbf{V}^{(1)}}{\partial r} = \frac{1}{r} \sum_{n=1}^{\infty} \left[- (n+1) \nabla \times (\mathbf{r} \chi_{-(n+1)}) - (n+2) \nabla\phi_{-(n+1)} \right. \\ \left. + \frac{n-2}{2\mu(2n-1)} r^2 \nabla P_n - \frac{n+1}{\mu(2n-1)} \mathbf{r} P_{-(n+1)} \right]. \end{aligned} \quad (4.14)$$

The associated boundary conditions used for the first reflected field $\mathbf{V}^{(1)}$ become as follows:

$$[V_r^{(1)}]_{r=c} = -[V_r^{(0)}]_{r=c}, \quad (4.15 a)$$

$$\left[\mathbf{r} \cdot \nabla \times \left(\mathbf{V}^{(1)} - \alpha \frac{\partial \mathbf{V}^{(1)}}{\partial r} \right) \right]_{r=c} = - \left[\mathbf{r} \cdot \nabla \times \left(\mathbf{V}^{(0)} - \alpha \frac{\partial \mathbf{V}^{(0)}}{\partial r} \right) \right]_{r=c}, \quad (4.15 b)$$

$$\left[\mathbf{r} \cdot \nabla \times \mathbf{r} \times \left(\mathbf{V}^{(1)} - \alpha \frac{\partial \mathbf{V}^{(1)}}{\partial r} \right) \right]_{r=c} = - \left[\mathbf{r} \cdot \nabla \times \mathbf{r} \times \left(\mathbf{V}^{(0)} - \alpha \frac{\partial \mathbf{V}^{(0)}}{\partial r} \right) \right]_{r=c}. \quad (4.15 c)$$

Since any function in the spherical angular coordinates (θ, ω) can be uniquely expanded into a series of surface spherical harmonics, we then have the following expressions for the velocity field $\mathbf{V}^{(1)}$ and the corresponding pressure distribution $P^{(1)}$:

$$\begin{aligned} \mathbf{V}^{(1)} = & \nabla \times (\mathbf{r}\chi_{-2}) + \nabla \left(\phi_{-2} + \phi_{-3} + \phi_{-4} + \frac{1}{2}r^2 \frac{P_{-2}}{\mu} - \frac{1}{30}r^2 \frac{P_{-4}}{\mu} \right) \\ & + \mathbf{r} \left(\frac{P_{-2}}{\mu} + \frac{1}{2} \frac{P_{-3}}{\mu} + \frac{1}{3} \frac{P_{-4}}{\mu} \right), \end{aligned} \quad (4.16)$$

$$P^{(1)} = P_{-2} + P_{-3} + P_{-4}. \quad (4.17)$$

where

$$\chi_{-2} = \left[\frac{c}{2c+4\alpha} \right] \left(\frac{c}{r} \right)^2 \left[\frac{2V\alpha d}{b^2} - \frac{2Vcd}{b^2} \right] P_1^1(\cos \theta) \sin \omega, \quad (4.18 a)$$

$$\phi_{-3} = - \left[\frac{c}{6(c+4\alpha)} \right] \left(\frac{2\alpha Vcd}{b^2} - \frac{2Vc^2d}{b^2} \right) \left(\frac{c}{r} \right)^3 P_2^1(\cos \theta) \cos \omega, \quad (4.18 b)$$

$$\phi_{-4} = \left[\frac{-Vc^4 + V\alpha c^3}{4(c+6\alpha)b^2} \right] \left(\frac{c}{r} \right)^4 P_3(\cos \theta), \quad (4.18 c)$$

$$P_{-3} = \left[\frac{10\mu V(c+\alpha)d}{3(c+2\alpha)b^2} \right] \left(\frac{c}{r} \right)^3 P_2^1(\cos \theta) \cos \omega, \quad (4.18 d)$$

$$P_{-4} = -\frac{7}{2} \left(\frac{\mu}{c(c+2\alpha)b^2} \right) (V\alpha c^2 + Vc^3) \left(\frac{c}{r} \right)^4 P_3(\cos \theta), \quad (4.18 e)$$

$$\begin{aligned} \phi_{-2} = & \left\{ \left[\frac{-c^2 + \alpha c}{4(c+2\alpha)} \right] \left[V \left(1 - \frac{d^2}{b^2} \right) + W - U \right] \right. \\ & \left. + \frac{3Vc^4 - V\alpha c^3}{10(c+2\alpha)b^2} - \frac{4V\alpha c^3}{5(c+2\alpha)b^2} \right\} \left(\frac{c}{r} \right)^2 P_1(\cos \theta), \end{aligned} \quad (4.18 f)$$

$$\begin{aligned} P_{-2} = & \left\{ -\frac{\mu(3\alpha+3c)}{2c(c+2\alpha)} \left[V \left(1 - \frac{d^2}{b^2} \right) + W - U \right] \right. \\ & \left. - \frac{8\mu V\alpha c}{5(c+2\alpha)b^2} + \frac{\mu(3V\alpha c^2 + 5Vc^3)}{5c(c+2\alpha)b^2} \right\} \left(\frac{c}{r} \right)^2 P_1(\cos \theta). \end{aligned} \quad (4.18 g)$$

According to the vector identities

$$\mathbf{F}^{(1)} = -4\pi\nabla(r^3 P_{-2}^{(1)}), \quad (4.19)$$

$$\mathbf{T}^{(1)} = -8\pi\mu\nabla(r^3 \chi_{-2}^{(1)}), \quad (4.20)$$

the drag force and torque induced by the flow field $\mathbf{V}^{(1)}$ become

$$\begin{aligned} \mathbf{F}^{(1)} = -4\pi c \left\{ -\frac{\mu(3\alpha + 3c)}{2(c + 2\alpha)} \left[V \left(1 - \frac{d^2}{b^2} \right) + W - U \right] \right. \\ \left. - \frac{8\mu V \alpha c^2}{5(c + 2\alpha)b^2} + \frac{\mu(3V\alpha c^3 + 5Vc^4)}{5c(c + 2\alpha)b^2} \right\} \mathbf{e}_z, \end{aligned} \quad (4.21)$$

$$\mathbf{T}^{(1)} = -8\pi\mu c^2 \left[\frac{c}{2c + 4\alpha} \right] \left[\frac{2V\alpha d}{b^2} - \frac{2Vcd}{b^2} \right] \mathbf{e}_x. \quad (4.22)$$

As stated in §4*a*, since only solutions with an accuracy of $O(c/b)$ are pursued, any term smaller than $O(c/b)$ will be neglected. The solutions of $\mathbf{V}^{(1)}$ and $P^{(1)}$ are then reduced to

$$\mathbf{V}^{(1)} \approx \frac{1}{2\mu} \nabla(r^2 P_{-2}^{(1)}) + \frac{\mathbf{r}}{\mu} P_{-2}^{(1)}, \quad (4.23\ a)$$

$$P^{(1)} \approx P_{-2}^{(1)}, \quad (4.23\ b)$$

$$\mathbf{F}^{(1)} \approx -4\pi \left\{ -\frac{\mu c(3\alpha + 3c)}{2(c + 2\alpha)} \left[V \left(1 - \frac{d^2}{b^2} \right) + W - U \right] \right\} \mathbf{e}_z, \quad (4.23\ c)$$

with

$$P_{-2}^{(1)} \approx -\frac{\mu(3\alpha + 3c)}{2c(c + 2\alpha)} \left[V \left(1 - \frac{d^2}{b^2} \right) + W - U \right] \left(\frac{c}{r} \right)^2 P_1(\cos \theta). \quad (4.23\ d)$$

The torque remains unchanged.

(c) The second reflection

The boundary conditions for the second reflection field $\mathbf{V}^{(2)}$ on the tube surface $R = b$, as described in equation (4.5), are

$$\begin{aligned} V_R^{(2)} &= -V_R^{(1)}, & V_\phi^{(2)} + V_\phi^{(1)} &= -\alpha \frac{\partial(V_\phi^{(2)} + V_\phi^{(1)})}{\partial R}, \\ V_Z^{(2)} + V_Z^{(1)} &= -\alpha \frac{\partial(V_Z^{(2)} + V_Z^{(1)})}{\partial R}. \end{aligned}$$

To derive the field $\mathbf{V}^{(2)}$, $\mathbf{V}^{(1)}$ and $\partial\mathbf{V}^{(1)}/\partial R$ at $R = b$ need to be expressed in terms of (R, ϕ, Z) first, which yields

$$\begin{aligned} \mathbf{V}^{(1)}|_{R=b} &= \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \cos k\phi \int_0^\infty \alpha_k(v) \sin vZ \, dv \, \mathbf{e}_R \\ &+ \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \sin k\phi \int_0^\infty \beta_k(v) \sin vZ \, dv \, \mathbf{e}_\phi \\ &+ \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \cos k\phi \int_0^\infty \gamma_k(v) \cos vZ \, dv \, \mathbf{e}_Z, \end{aligned} \quad (4.24)$$

$$\begin{aligned}
\left. \frac{\partial \mathbf{V}^{(1)}}{\partial R} \right|_{R=b} &= \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \cos k\phi \int_0^{\infty} \alpha'_k(\lambda v) \sin vZ \, dv \, \mathbf{e}_R \\
&+ \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \sin k\phi \int_0^{\infty} \beta'_k(v) \sin vZ \, dv \, \mathbf{e}_{\phi} \\
&+ \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \cos k\phi \int_0^{\infty} \gamma'_k(v) \cos vZ \, dv \, \mathbf{e}_Z,
\end{aligned} \quad (4.25)$$

where

$$\left. \begin{aligned} \alpha_k(v) &= H \left[vbK_k(vb)I_k(vd) + vdK'_k(vb)I'_k(vd) + \frac{k^2}{vb}K_k(vb)I_k(vd) \right], \\ \beta_k(v) &= -Hk \left[\frac{d}{b}K_k(vb)I'_k(vd) + K'_k(vb)I_k(vd) \right], \\ \gamma_k(v) &= H[vbK'_k(vb)I_k(vd) + vdK_k(vb)I'_k(vd) + 2K_k(vb)I_k(vd)], \end{aligned} \right\} \quad (4.26 a)$$

$$\left. \begin{aligned} \alpha'_k(v) &= H \left[vK_k(vb)I_k(vd) + v^2bK'_k(vb)I_k(vd) + v^2dK''_k(vb)I'_k(vd) \right. \\ &\quad \left. - \frac{k^2}{vb^2}K_k(vb)I_k(vd) + \frac{k^2}{b}K'_k(vb)I_k(vd) \right], \\ \beta'_k(v) &= -Hk \left[\frac{vd}{b}K'_k(vb)I'_k(vd) - \frac{d}{b^2}K_k(vb)I'_k(vd) + vK''_k(vb)I_k(vd) \right], \end{aligned} \right\} \quad (4.26 b)$$

$$\begin{aligned} \gamma'_k(v) &= H[vK'_k(vb)I_k(vd) + v^2bK''_k(vb)I_k(vd) \\ &\quad + v^2dK'_k(vb)I'_k(vd) + 2vK'_k(vb)I_k(vd)]. \end{aligned} \quad (4.26 c)$$

with K_k and I_k denoting the modified k th-order Bessel functions of the first and second kind. By applying the vector identities between the spherical and cylindrical coordinates, the general solutions for $\mathbf{V}^{(2)}$, $\partial \mathbf{V}^{(2)}/\partial R$, and $P^{(2)}$ in the cylindrical coordinates can be expressed as

$$\begin{aligned} \mathbf{V}^{(2)} &= -\mathbf{e}_R \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \cos k\phi \\ &\quad \times \int_0^{\infty} \left[\omega_k(v) \frac{kI_k(vR)}{vR} + \psi_k(v)I'_k(vR) + \pi_k(v)vRI''_k(vR) \right] \sin vZ \, dv \\ &- \mathbf{e}_{\phi} \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \sin k\phi \int_0^{\infty} \left[\pi_k(v) \frac{kI_k(vR)}{vR} - \pi_k(v)kI'_k(vR) \right. \\ &\quad \left. - \omega_k(v)I'_k(vR) - \psi_k(v) \frac{kI_k(vR)}{vR} \right] \sin vZ \, dv \\ &- \mathbf{e}_Z \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \cos k\phi \\ &\quad \times \int_0^{\infty} [\psi_k(v)I_k(vR) + \pi_k(v)vRI'_k(vR) + \pi_k(v)I_k(vR)] \cos vZ \, dv, \end{aligned} \quad (4.27)$$

$$\begin{aligned}
\frac{\partial V^{(2)}}{\partial R} = & -e_R \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \cos k\phi \\
& \times \int_0^{\infty} \left[\omega_k(v) \frac{kI'_k(vR)}{R} - \omega_k(v) \frac{kI_k(vR)}{vR^2} + v\psi_k(v)I''_k(vR) \right. \\
& \quad \left. + \pi_k(v)vI''_k(vR) + \pi_k(v)v^2RI'''_k(vR) \right] \sin vZ \, dv \\
& - e_{\phi} \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \sin k\phi \\
& \times \int_0^{\infty} \left[\pi_k(v) \frac{kI'_k(vR)}{R} - \pi_k(v) \frac{kI_k(vR)}{vR^2} - \pi_k(v)v kI''_k(vR) \right. \\
& \quad \left. - \omega_k(v)vI''_k(vR) - \psi_k(v) \frac{kI'_k(vR)}{R} \right. \\
& \quad \left. + \psi_k(v) \frac{kI_k(vR)}{vR^2} \right] \sin vZ \, dv \\
& - e_Z \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \cos k\phi \int_0^{\infty} [\psi_k(v)vI'_k(vR) + \pi_k(v)vI'_k(vR) \\
& \quad + \pi_k(v)v^2RI''_k(vR) + \pi_k(v)\lambda I'_k(vR)] \cos vZ \, dv
\end{aligned} \tag{4.28}$$

and

$$P^{(2)} = -\frac{2\mu}{\pi} \sum_{k=-\infty}^{\infty} \cos k\phi \int_0^{\infty} v^2 \pi_k(v) I_k(vR) \frac{\sin vZ}{v} \, dv, \tag{4.29}$$

where π_k , ψ_k and ω_k are functions of v to be determined from the following converted boundary conditions:

$$V_R^{(2)} = -V_R^{(1)}:$$

$$\begin{aligned}
\pi_k(v)\xi I''_k(\xi) + \omega_k(v) \frac{kI_k(\xi)}{\xi} + \psi_k(v)I'_k(\xi) \\
= H \left[\xi K_k(\xi)I_k(\xi\eta) + \xi\eta K'_k(\xi)I'_k(\xi\eta) + \frac{k^2}{\xi} K_k(\xi)I_k(\xi\eta) \right], \tag{4.30}
\end{aligned}$$

$$\begin{aligned}
V_{\phi}^{(2)} + V_{\phi}^{(1)} = & -\alpha \frac{\partial(V_{\phi}^{(2)} + V_{\phi}^{(1)})}{\partial R}: \\
& \pi_k(v) \left[-\frac{kI_k(\xi)}{\xi} + kI'_k(\xi) + Kn_b \left(-kI'_k(\xi) + \frac{kI_k(\xi)}{\xi} + \xi kI''_k(\xi) \right) \right] \\
& + \omega_k(v) [I'_k(\xi) + Kn_b \xi I''_k(\xi)] + \psi_k(v) \left[\frac{kI_k(\xi)}{\xi} + Kn_b \left(kI'_k(\xi) - \frac{kI_k(\xi)}{\xi} \right) \right] \\
& = Hk [\eta K_k(\xi)I'_k(\xi\eta) + K'_k(\xi)I_k(\xi\eta) \\
& \quad + Kn_b (\xi\eta K'_k(\xi)I'_k(\xi\eta) - \eta K_k(\xi)I'_k(\xi\eta) + \xi K''_k(\xi)I_k(\xi\eta))], \tag{4.31}
\end{aligned}$$

$$\begin{aligned}
V_Z^{(2)} + V_Z^{(1)} &= -\alpha \frac{\partial(V_Z^{(2)} + V_Z^{(1)})}{\partial R}: \\
\pi_k(v) &[-\xi I'_k(\xi) - I_k(\xi) - Kn_b(\xi I'_k(\xi) + \xi^2 I''_k(\xi) + \xi I'_k(\xi))] \\
&+ \psi_k(v)[-I_k(\xi) - Kn_b \xi I'_k(\xi)] \\
&= -H[\xi K'_k(\xi) I_k(\xi \eta) + \xi \eta K_k(\xi) I'_k(\xi \eta) \\
&\quad + 2K_k(\xi) I_k(\xi \eta) + Kn_b(\xi K'_k(\xi) I_k(\xi \eta) \\
&\quad + \xi^2 K''_k(\xi) I_k(\xi \eta) + \xi^2 \eta K'_k(\xi) I'_k(\xi \eta) + 2\xi K'_k(\xi) I_k(\xi \eta)].
\end{aligned} \tag{4.32}$$

In the above equations,

$$\xi = vb, \quad \eta = \frac{d}{b}, \quad Kn_b = \frac{2 - \sigma}{\sigma} \frac{\lambda}{b}.$$

Using the above expressions and the identity

$$I''_k(\xi) = -\frac{I'_k(\xi)}{\xi} + \left(1 + \frac{k^2}{\xi^2}\right) I_k(\xi), \tag{4.33}$$

π_k , ψ_k and ω_k can then be determined, respectively, as

$$\begin{aligned}
\pi_k(v) &= \frac{1}{\xi^2 f(\xi)} \{g_3[-k^2(-1 + Kn_b) + I'_k \xi^2(I'_k - I'_k Kn_b + I_k Kn_b \xi)] \\
&\quad + \xi(I_k + I'_k Kn_b \xi)\{-I_k g_2 k + g_1[I'_k \xi + I_k Kn_b(k^2 + I_k I'_k \xi + \xi^2)]\}\},
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
\psi_k(v) &= \frac{1}{\xi^2 f(\xi)} \{\xi\{g_2\{k^2 + \xi[\xi + I'_k(2I_k + I'_k \xi + 2I'_k Kn_b \xi)]\} \\
&\quad + g_1 k\{2 - Kn_b\{2 + 4I_k I'_k \xi \\
&\quad + Kn_b\{k^2 + \xi[\xi + I'_k(-2I_k + 3I'_k \xi)]\}\}\} \\
&\quad - g_3 k(-1 + Kn_b)[k^2 + (1 + I'_k{}^2)\xi^2]\}\},
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
\omega_k(v) &= \frac{1}{\xi^2 f(\xi)} \{g_3\{(I'_k - I_k \xi)\xi^2[I'_k(-1 + Kn_b) - I_k Kn_b \xi] \\
&\quad + k^2(-1 + Kn_b - Kn_b \xi^2)\} \\
&\quad - \xi[1 + k^2 Kn_b - I_k I'_k(1 + Kn_b)\xi + Kn_b \xi^2] \\
&\quad \times \{-g_2 k + g_1[k^2 Kn_b + I_k I'_k(-1 + Kn_b)\xi + Kn_b \xi^2]\}\},
\end{aligned} \tag{4.36}$$

with

$$g_1(\xi, \eta) = H \left[\xi K_k(\xi) I_k(\xi \eta) + \xi \eta K'_k(\xi) I'_k(\xi \eta) + \frac{k^2}{\xi} K_k(\xi) I_k(\xi \eta) \right], \tag{4.37 a}$$

$$\begin{aligned}
g_2(\xi, \eta) &= Hk[\eta K_k(\xi) I'_k(\xi \eta) + K'_k(\xi) I_k(\xi \eta) \\
&\quad + Kn_b(\xi \eta K'_k(\xi) I'_k(\xi \eta) - \eta K_k(\xi) I'_k(\xi \eta) + \xi K''_k(\xi) I_k(\xi \eta))],
\end{aligned} \tag{4.37 b}$$

$$\begin{aligned}
g_3(\xi, \eta) &= -H[\xi K'_k(\xi) I_k(\xi \eta) + \xi \eta K_k(\xi) I'_k(\xi \eta) + 2K_k(\xi) I_k(\xi \eta) \\
&\quad + Kn_b(\xi K'_k(\xi) I_k(\xi \eta) + \xi^2 K''_k(\xi) I_k(\xi \eta) \\
&\quad + \xi^2 \eta K'_k(\xi) I'_k(\xi \eta) + 2\xi K'_k(\xi) I_k(\xi \eta)],
\end{aligned} \tag{4.37 c}$$

$$\begin{aligned}
f(\xi) = & \frac{I_k k^4 (-1 + Kn_b) Kn_b}{\xi^2} \\
& + \frac{k^2 \{2I_k (-1 + Kn_b) + I'_k (-1 + Kn_b)(1 + 2Kn_b)\xi + I_k Kn_b [-2 + I_k'^2 (-1 + Kn_b) + Kn_b]\xi^2\}}{\xi^2} \\
& + [I'_k (-1 + Kn_b) - I_k Kn_b \xi] \{\xi + I'_k [2I_k + I'_k (\xi + 2Kn_b \xi)]\}. \quad (4.37 d)
\end{aligned}$$

The pressure field $P^{(2)}$ can then be expressed as follows:

$$\begin{aligned}
P^{(2)} = & -\frac{2\mu}{\pi} \sum_{k=-\infty}^{\infty} \cos k\phi \int_0^{\infty} v^2 \pi_k(v) I_k(vR) \frac{\sin vZ}{v} dv \\
= & -\frac{2\mu}{\pi} \frac{1}{b^2} \sum_{k=-\infty}^{\infty} \cos k\phi \int_0^{\infty} \xi^2 \pi_k(\xi) I_k\left(\xi \frac{R}{b}\right) \frac{\sin \xi(Z/b)}{\xi} d\xi. \quad (4.38)
\end{aligned}$$

To calculate the pressure drop of this reflection, $\Delta P^{(2)}$, due to the presence of the sphere, we have

$$\Delta P^{(2)} = -2 \lim_{Z \rightarrow \infty} [P^{(2)}]_{R=b}. \quad (4.39)$$

By applying the Dirichlet theorem,

$$\lim_{Z/b \rightarrow \infty} \int_0^{\infty} [\xi^2 \pi_k(\xi) I_k(\xi)] \frac{\sin \xi(Z/b)}{\xi} d\xi = \frac{1}{2} \pi [\xi^2 \pi_k(\xi) I_k(\xi)]_{\xi=0+}, \quad (4.40)$$

and

$$[\xi^2 \pi_k(\xi) I_k(\xi)]_{\xi=0+} = \begin{cases} \frac{-4H[1 - \eta^2 + Kn_b(3 - \eta^2) + 2Kn_b^2]}{1 + 5Kn_b + 4Kn_b^2} & \text{for } k = 0, \\ 0 & \text{for } k \neq 0, \end{cases} \quad (4.41)$$

we finally have

$$\begin{aligned}
\Delta P^{(2)} = & -\frac{8\mu H[1 - \eta^2 + Kn_b(3 - \eta^2) + 2Kn_b^2]}{b^2(1 + 5Kn_b + 4Kn_b^2)} \\
= & \frac{12c(c/b + Kn_b)\mu}{b^2(1 + 5Kn_b + 4Kn_b^2)(c/b + 2Kn_b)} [1 - \eta^2 + Kn_b(3 - \eta^2) + 2Kn_b^2] \\
& \times [V(1 - \eta^2) + W - U]. \quad (4.42)
\end{aligned}$$

(d) The third and fourth reflections

The drag force and torque produced by the third reflection field will be derived in this section. Rather than determining the velocity field explicitly first as in §§ 4 b, c, a more direct method will be used here. From the modified Faxen's laws for the slip boundary condition (Keh & Chen 1996), the expressions for the drag and torque on the sphere due to the flow field are

$$\mathbf{F} = 6\pi\mu c \left(\frac{c + \alpha}{c + 2\alpha} \right) ([\mathbf{V}_{\infty}]_o - \mathbf{U}) + \pi\mu c^3 \left(\frac{c - \alpha}{c + 2\alpha} \right) [\nabla^2 \mathbf{V}_{\infty}]_o, \quad (4.43)$$

$$\mathbf{T} = 8\pi\mu c^2 \left(\frac{c - \alpha}{c + 2\alpha} \right) \left(\frac{1}{2} [\nabla \times \mathbf{V}_{\infty}]_o - \boldsymbol{\omega} \right), \quad (4.44)$$

where $\mathbf{U}$ is the translational velocity of the sphere and $\mathbf{V}_\infty$ is an arbitrary Stokes flow without any immersed object, which is also the velocity field far from the sphere. The subscript 'o' means the value in the bracket is evaluated at the origin of the sphere. Since $\mathbf{V}^{(2)}$ does not produce any drag and torque, as shown in § 4 *c*, $\mathbf{V}_\infty$ in equations (4.43) and (4.44) can be replaced by $\mathbf{V}^{(2)}$ and $\mathbf{U}$ is zero. Equations (4.43) and (4.44) then become

$$\mathbf{F}^{(3)} = 6\pi\mu c \left(\frac{c+\alpha}{c+2\alpha} \right) [\mathbf{V}^{(2)}]_o + \pi c^3 \left(\frac{c-\alpha}{c+2\alpha} \right) [\nabla P^{(2)}]_o, \quad (4.45)$$

$$\mathbf{T}^{(3)} = 4\pi\mu c^2 \left(\frac{c-\alpha}{c+2\alpha} \right) [\nabla \times \mathbf{V}^{(2)}]_o. \quad (4.46)$$

To the order of (c/b) , equations (4.34) and (4.35) can be approximated as

$$\mathbf{F}^{(3)} = -6\mu \frac{c}{b} \left(\frac{c+\alpha}{c+2\alpha} \right) HP(\eta) \mathbf{e}_Z, \quad (4.47)$$

$$\mathbf{T}^{(3)} = 4\mu c \left(\frac{c}{b} \right)^2 \left(\frac{c-\alpha}{c+2\alpha} \right) HQ(\eta) \mathbf{e}_x, \quad (4.48)$$

with

$$P(\eta) = \left\{ \sum_{k=-\infty}^{\infty} \int_0^\infty [\psi_k(\xi) I_k(\xi\eta) + \pi_k(\zeta) \xi \eta I'_k(\xi\eta) + \pi_k(\zeta) I_k(\xi\eta)] d\xi \right\} / H, \quad (4.49)$$

$$Q(\eta) = \left\{ \sum_{k=-\infty}^{\infty} \int_0^\infty \left[2\xi \pi_k(\zeta) I'_k(\xi\eta) - \frac{k\omega_k(\xi) I_k(\xi\eta)}{\eta} \right] d\xi \right\} / H. \quad (4.50)$$

To determine the velocity field $\mathbf{V}^{(4)}$ and the pressure field $P^{(4)}$ in the fourth reflection, the formula

$$4\pi\nabla(r^3 P_{-2}^{(3)}) = -\mathbf{F}^{(3)} \quad (4.51)$$

is needed. Taking the inner product of the above equation with $\mathbf{r}$, we have

$$4\pi r \frac{\partial}{\partial r} (r^3 P_{-2}^{(3)}) = -\mathbf{r} \cdot \mathbf{F}^{(3)}, \quad (4.52)$$

i.e.

$$P_{-2}^{(3)} = -\frac{1}{4\pi r^3} \mathbf{r} \cdot \mathbf{F}^{(3)} \quad (\text{by the Euler theorem}). \quad (4.53)$$

Then

$$\begin{aligned} P_{-2}^{(3)} &= \frac{3\mu}{2\pi r^2} \frac{c}{b} \left(\frac{c+\alpha}{c+2\alpha} \right) HP(\eta) \cos \theta \\ &= \frac{3}{2\pi} \frac{c}{b} \left(\frac{c+\alpha}{c+2\alpha} \right) P(\eta) P_{-2}^{(1)} \\ &= JP_{-2}^{(1)}, \end{aligned} \quad (4.54)$$

where

$$J = \frac{3}{2\pi} \frac{c}{b} \left(\frac{c+\alpha}{c+2\alpha} \right) P(\eta), \quad (4.55)$$

of order (c/b) .

Figure 5. The first wall correction factor (K_1) calculated by approximations of matrix A of different orders with $Kn = 0.05$; n represents an $(n \times n)$ matrix.

With equation (4.54), the value of $\mathbf{V}^{(3)}$ and $P^{(3)}$ far from the sphere can be approximated by

$$\mathbf{V}^{(3)} \approx \frac{1}{2\mu} \nabla(r^2 P_{-2}^{(3)}) + \frac{\mathbf{r}}{\mu} P_{-2}^{(3)} = J\mathbf{V}^{(1)}, \tag{4.56}$$

$$P^{(3)} \approx P_{-2}^{(3)} = JP^{(1)}. \tag{4.57}$$

This can be shown from Lamb’s general solution, where P_{-2} is the only harmonic term of order $(1/r)$. This approximation is accurate to order (c/b) . Since the third reflection field is a J of order smaller than the first reflection field (Happel & Brenner 1983, pp. 96–157 and 286–321), it is a reasonable to take

$$\mathbf{V}^{(4)} = J\mathbf{V}^{(2)}, \tag{4.58}$$

$$P^{(4)} = JP^{(2)}. \tag{4.59}$$

The total pressure drop caused by the fourth reflected field is then

$$\Delta P^{(4)} = J\Delta P^{(2)}. \tag{4.60}$$

In summary, the drag and torque experienced by a sphere travelling parallel to the centreline of a micro-tube are approximated by the following expressions:

$$\mathbf{F} = \mathbf{F}^{(1)} + \mathbf{F}^{(3)} + \mathbf{F}^{(5)} + \dots \approx \mathbf{F}^{(1)} + J\mathbf{F}^{(1)} + J^2\mathbf{F}^{(1)} + \dots = \frac{\mathbf{F}^{(1)}}{1 - J}, \tag{4.61}$$

Figure 6. The velocity field for flow situation with $U = 0.5$, $V_c = 1$, $Kn = 0.05$ and $c/b = 0.4$ (a) calculated by the solutions in the cylindrical coordinates and (b) calculated by the solutions in the spherical coordinates.

accurate to $O(c/b)$,

$$\begin{aligned}\Delta P &= \Delta P^{(2)} + \Delta P^{(4)} + \Delta P^{(6)} + \dots \approx \Delta P^{(2)} + J\Delta P^{(2)} + J^2\Delta P^{(2)} + \dots \\ &= \frac{\Delta P^{(2)}}{1 - J},\end{aligned}\quad (4.62)$$

accurate to $O(c/b)$, and

$$\begin{aligned}\mathbf{T} &\approx \mathbf{T}^{(1)} + \mathbf{T}^{(3)} \\ &= \left\{ -8\pi\mu c^2 \left[\frac{c}{2c + 4\alpha} \right] \left[\frac{2V\alpha d}{b^2} - \frac{2Vcd}{b^2} \right] + 4\mu c \left(\frac{c}{b} \right)^2 \left(\frac{c - \alpha}{c + 2\alpha} \right) HQ(\eta) \right\} \mathbf{e}_x,\end{aligned}\quad (4.63)$$

accurate to $O(c/b)^2$.

In equation (4.62), the pressure drops due to the fact that the odd terms are equal to zero. This can be proved by a direct evaluation of ΔP from the odd reflection fields.

5. Results and discussion

(a) Sphere moving along the centreline of a micro-tube

The effects on the flow characteristics due to slip boundary conditions of a flow passing a single sphere moving along the centreline of a micro-tube, as derived analytically in § 4, will be presented and discussed. In order that the analysis is valid, the Knudsen number based on the radius of the micro-tube, the radius of the sphere or the gap between the micro-tube and sphere should be between 0.01 and 0.1. However, only the Knudsen number based on the radius of the micro-tube, Kn_b , is indicated in the figures and, simply for convenience, the subscript ‘ b ’ will be omitted.

When solving the flow field by the stream-function method, calculations with an approximation of matrix A of finite dimensions are necessitated. Calculations of K_1 , the first wall-correction factor, with approximation of matrix A with (2×2) , (10×10) and (30×30) dimensions, are shown in figure 5. It can be seen that a (10×10)

Figure 7. The velocity distribution at $X = 0$ with $U = 0.5$, $V = 1$, $Kn = 0.05$ and $c/b = 0.4$.

Figure 8. Variation of pressure drop with c/b due to the existence of the sphere.

matrix seems to be a good approximation of A . However, in order to acquire a higher accuracy within a reasonable computation time, a (30×30) matrix of A is used for calculations in the present study.

The velocity fields calculated by the solutions in the cylindrical coordinate system and the spherical coordinate system, respectively, are shown in figure 6. The closeness between these two flow fields verifies the validity of the solutions and the solution procedures used in this study.

Figure 9. Variations of K_2/K_1 with c/b .

The velocity distributions at $X = 0$ with slip and no-slip boundary conditions are shown in figure 7. A closer comparison between these two velocity distributions is also included; the velocities at the spherical surface are made to coincide for the purpose of comparison. It then becomes obvious that, due to the slip boundary conditions, the shear stress at the spherical surface is less than that with no-slip boundary conditions. As a result, the pressure drop due to the existence of the sphere is smaller for the flow situation with slip boundary conditions than it is for that with no-slip boundary conditions and the pressure drop becomes smaller and smaller as Kn_b increases. This phenomenon is shown in figure 8. The rapid increase in the pressure drop over the sphere for $c/b \geq 0.5$, when the most part of the cross-section of the tube is blocked by the sphere, is also indicated in this figure.

Variations of the wall-correction factors with c/b are shown in figures 9–11. Figure 9 elucidates that the ratio K_2/K_1 is always less than unity and decreases as c/b increases. That means the effect due to K_2 becomes less and less important than that due to K_1 as c/b increases or as the radius of the sphere becomes larger and larger with a fixed size of micro-tube. It is also indicated in this figure that K_2/K_1 possesses a larger value for flow situations with slip boundary conditions than those with no-slip boundary conditions. In other words, the K_1 effect or the influence on the flow field due to the migration motion of the sphere in the micro-tube is, relatively, less important for flow situations with slip boundary conditions than that with no-slip boundary conditions.

Instead of the relative importance between K_1 and K_2 , the variations of K_1 or K_2 with c/b are shown in figures 10 and 11. From these two figures, it can be seen that both K_1 and K_2 increase with c/b . That means that, as the size of the sphere increases, the effect on the flow field through K_1 and K_2 becomes more and more important. However, K_1 increases more rapidly than K_2 , which thus makes the ratio K_2/K_1 decrease as c/b increases. Also elucidated in figure 10 is the fact that the K_1 -effect on the flow field becomes less and less important when Kn_b increases, i.e. when the slip boundary conditions become more and more important, as indicated in figure 9 and discussed previously.

Figure 10. Variations of K_1 with c/b for different value of Kn .

Figure 11. Variations of K_1 and K_2 with c/b for $Kn = 0.05$.

(b) Sphere moving parallel to the centreline of a micro-tube

Results based on the solutions obtained by the reflection method in §4 for the flow passing a single sphere moving parallel to the centreline of a micro-tube will be presented and discussed. The range of Knudsen numbers based on the tube radius

Figure 12. Variations of resultant force versus η with $c/b = 0.05$.

lies between 0.01 and 0.1 so that the Navier–Stokes equations with slip boundary conditions are valid.

The resultant force on the sphere with $O(c/b)$ is, according to equation (4.49),

$$F_{\text{resultant}} = \frac{6\pi\mu c(-U + V(1 - \eta^2) + W)(c + \alpha)}{(c + 2\alpha) - (3(c + \alpha)/2\pi)(c/b)P(\eta)},$$

which is positive in the negative Z -direction, with $\eta = d/b$ indicating the off-centre distance. It should be noted that, without the inertia effect, the resultant force is only in the Z -direction for all η . The variation of $F_{\text{resultant}}$ with η is shown in figure 12, which indicates that, with slip boundary conditions, the resultant force becomes smaller. However, the difference of $F_{\text{resultant}}$ between those with and without slip boundary conditions decreases as the sphere is located closer to the tube wall, i.e. as η increases. The corresponding pressure drop is shown in figure 13. The consistency between figures 12 and 13 is obvious.

The terminal velocity of the sphere can be derived by letting $F_{\text{resultant}} = 0$, which gives

$$U_{\text{terminal}} = V(1 - \eta^2) + W.$$

It turns out that U_{terminal} is the same as the velocity of the flow itself computed at the centre of the sphere.

In addition to the drag force, there exists a torque on the sphere as shown in equation (4.63). When the terminal velocity is reached, the last term in equation (4.63) vanishes. Then, the torque on the sphere depends implicitly on the Knudsen number through the parameter α ; it decreases as the Knudsen number increases.

A lot of information can also be learned from figure 13.

- (i) The pressure drop over the sphere is smaller with slip boundary conditions than with no-slip boundary conditions. However, the difference diminished as η increases.

Figure 13. Variations of ΔP versus η with $c/b = 0.05$ and $c/b = 0.1$.

- (ii) The pressure drop increases with the ratio of c/b . It means that, with a larger size of sphere which blocks more space in the flow passage, the pressure drop over the sphere decreases more.
- (iii) The off-centreline effect, i.e. the η -effect, on the pressure drop over the sphere become less significant with a smaller value of c/b .

6. Conclusions

The flow passing a single sphere moving in a micro-tube is investigated theoretically in the present study. The stream-function method as proposed by Haberman & Sayre (1958) is used to obtain analytic solutions for the flow field with a single sphere moving along the centreline of a micro-tube, while the reflection method as proposed by Happel & Brenner (1957) is for that with a single sphere moving parallel to the centreline of a micro-tube. The results indicate that, as the slip boundary conditions become more and more important, the shear stresses and, consequently, the pressure drop over the sphere become smaller and smaller. As the size of the sphere increases, the effect on the flow field through the wall-correction factors becomes more important. However, the wall-correction factors decrease as Kn_b increases. When the sphere is not moving along the centreline, both the resultant force on the sphere and the pressure drop over the sphere decrease as the sphere is located closer to the tube wall, i.e. as η increases. The differences of $F_{\text{resultant}}$ and ΔP between the flow situations with and without slip boundary conditions diminish as η increases. Because of the

asymmetry, there exists a torque on the sphere moving parallel to the centreline of the micro-tube.

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