FINITE-ELEMENT ANALYSIS OF 3-D VISCOUS FLOW AND MIXED-CONVECTION PROBLEMS BY THE PROJECTION METHOD

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This article describes a finite-element procedure based on the projection method for solving the primitive variables form of the Navier-Stokes equations and energy equation in three dimensions. The proposed numerical algorithm involves the operator splitting technique, balance tensor diffusivity (BTD), the Runge-Kutta time-stepping method, and a bi-conjugate gradient iterative solver. Four numerical examples are chosen to test the capability of the method. Two benchmark flows, lid-driven cavity flow and laminar flow over a 3-D backward-facing step, are validated with numerical solutions available from the literature. Uniform flow over ripple beds is selected as the third example, to demonstrate the model capability for handling complex geometry. As a fourth attempt, a mixed-convection problem in a two-sided lid-driven and differentially heated cubic cavity is investigated. The physical properties are heavily dependent on the direction of the moving walls. The Richardson number ($Ri = Gr/Re^2$), in the range $0.01 \leq Ri \leq 10$, serves as a measure of the relative importance of forced- and natural-convection modes on the heat transfer. It is shown that the present numerical scheme is able to simulate both the viscous flow and convective heat transfer problems in three dimensions.

1. INTRODUCTION

A close look at the incompressible Navier-Stokes equations indicates that the continuity and the momentum equations are coupled through the velocity, but the pressure appears as a source term only in the momentum equations and does not prevail as an evolution equation. Since the pressure appears only in the momentum equations, it has to satisfy the mass conservation constraint implicitly in order to comply with the coupling between the continuity and the momentum equations. For the case of an incompressible fluid the satisfaction of the divergence-free condition makes the problem more complicated with regard to treating the pressure term in the momentum equations. The classical projection method, also called the fractional step method, proposed by Chorin [1] used the novel idea of decoupling the advection-diffusion mechanism from the incompressibility conditions to yield

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NOMENCLATURE

g	acceleration due to gravity	t	dimensionless time
Gr	Grashof number	T	dimensionless temperature
$Nu_{mean}(y)$	mean Nusselt number throughout the cavity	u, v, w	dimensionless velocities in the x, y , and z directions
$Nu_{overall}$	overall Nusselt number on the boundary	u^*, v^*, w^*	dimensionless middle velocities in the x, y , and z directions
P	pressure	x, y, z	dimensionless Cartesian coordinates
Pr	Prandtl number	Γ	boundary of the computational domain
Ra	Rayleigh number	Ω	computational domain
Re	Reynolds number		
Ri	Richardson number		

an approximate velocity field, which was made divergence-free by means of orthogonal projection of the apparent velocity field. Depending on the method of approximation of the advection term and the scheme used for the time discretization, the projection methods have different options and have undergone many modifications over the past years [2–5]. A coupled velocity–pressure algorithm proposed by Patankar and Spalding [6] was a predictor–corrector method in which an initial estimate for the velocity field from the momentum equations was corrected after satisfying the continuity constraint. The original SIMPLE algorithm proposed by Patankar and Spalding [6] has been further refined [7, 8] to produce faster convergence and more accurate numerical solutions.

In computational fluid dynamics, the fractional step methods have been proven to be effective [9–11]. In the present study, the operator splitting technique, balance tensor diffusivity (BTD), and the Runge-Kutta time-stepping method based on the projection method [1] are employed to solve 3-D viscous flow and heat transfer problems. In addition, the pressure Poisson equation is solved using a bi-conjugate gradient iterative scheme by storing only the nonzero values during computation. Laminar natural convection in differentially heated cavities in two and three dimensions has been studied extensively, for example, [12–15], among others. Mixed convection finds a wide range of engineering applications in many interdisciplinary fields, such as cooling of electronic devices, solar energy collection, crystal growth, compartment fires, and many other technical fields as found in computers and nuclear reactor technologies. There are many studies connected to the investigation of this problem [16–19]. The aim of the present study is to develop an effective numerical method for handling the flow problems along with heat and mass transfer processes. Numerical solutions for viscous flows are obtained over a range of Reynolds number for 3-D lid-driven cavity flow, backward-facing step flow, and flow over ripple beds problems. Another test heat transfer problem is a two-sided lid-driven cubic cavity filled with an incompressible fluid involving a mixed-convection process. The vertical lids have different constant temperatures and the other sides are adiabatic.

The numerical scheme to solve the 3-D Navier-Stokes equations using the finite-element technique is described by the following procedures. Section 2 presents the governing equations for viscous flow and heat transfer problems. Section 3 proposes the finite-element numerical procedure for the solution of the governing

Navier-Stokes equations in primitive variables form. The numerical results for four examples, lid-driven cavity flow, laminar flow over a 3-D backward-facing step, uniform flow over ripple beds, and mixed convection in a cubic cavity, are discussed in Section 4. The accuracy of the results predicted by the present model are verified and compared with other numerical schemes. Section 5 contains our main conclusions.

2. GOVERNING EQUATIONS

Assuming incompressibility and the Boussenesq approximation for the fluid medium, the governing equations for mixed convection can be expressed by means of the dimensionless three-dimensional Navier-Stokes equations in primitive variables form and energy equation as follows.

Continuity equation:

$$\nabla \cdot \vec{u} = 0 \quad (1)$$

Momentum equation:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \vec{u} - \frac{\text{Gr}}{\text{Re}^2} T \vec{g} \quad (2)$$

Energy equation:

$$\frac{\partial T}{\partial t} + (\vec{u} \cdot \nabla) T = \frac{1}{\text{Re Pr}} \nabla^2 T \quad (3)$$

where

$$\text{Gr} = \frac{g \beta L^3 \Delta T}{\nu^2} \quad \text{Re} = \frac{V_p L}{\nu} \quad \text{Pr} = \frac{\nu}{\alpha} \quad \text{Ri} = \frac{\text{Gr}}{\text{Re}^2}$$

for a Cartesian coordinate frame in which x - y represents the horizontal plane and z represents the vertical direction. The following parameters are used to define the non-dimensional quantities in the above governing equations and later physical explanations. The $\vec{u}$ are the velocity vectors in the x , y , and z directions. P is the pressure, T is the temperature. Gr is the Grashof number, Re is the Reynolds number, and Pr is the Prandtl number. The Richardson number, defined as $\text{Ri} = \text{Gr}/\text{Re}^2$, is a characteristic number for the mixed-convection process that indicates the relative dominance of the natural-and forced-convection effects. The Nusselt number (Nu) is one of the important dimensionless parameters to be computed for heat transfer analysis in mixed-convection flow. The mean Nusselt number throughout the cavity is defined as

$$\text{Nu}_{\text{mean}}(y) = \int_0^1 \frac{\partial T(x, y, z)}{\partial x} \Big|_{x=0 \text{ or } x=1} dz \quad (4)$$

The overall Nusselt number on the boundary at $x = 0$ or $x = 1$ is defined as

$$\text{Nu}_{\text{overall}} = \int_0^1 \text{Nu}_{\text{mean}}(y) dy \quad (5)$$

3. FINITE-ELEMENT SCHEME

In the present model, the balancing tensor diffusivity [9] is employed to discretize the Navier-Stokes equations and energy equation with the operator splitting scheme. Equations (2) and (3) can be written in a segregated form as

$$u^{n+1} = u^n - \Delta t \left[(u^n) \frac{\partial u^n}{\partial x} + (v^n) \frac{\partial u^n}{\partial y} + (w^n) \frac{\partial u^n}{\partial z} + \frac{\partial p^n}{\partial x} - \frac{1}{\text{Re}} (\nabla^2 u) - BTDu^n \right] \quad (6)$$

$$v^{n+1} = v^n - \Delta t \left[(u^n) \frac{\partial v^n}{\partial x} + (v^n) \frac{\partial v^n}{\partial y} + (w^n) \frac{\partial v^n}{\partial z} + \frac{\partial p^n}{\partial y} - \frac{1}{\text{Re}} (\nabla^2 v) - BTDv^n \right] \quad (7)$$

$$w^{n+1} = w^n - \Delta t \left[(u^n) \frac{\partial w^n}{\partial x} + (v^n) \frac{\partial w^n}{\partial y} + (w^n) \frac{\partial w^n}{\partial z} + \frac{\partial p^n}{\partial z} - \frac{1}{\text{Re}} (\nabla^2 w) - BTDw^n - \text{Ri} T \right] \quad (8)$$

$$T^{n+1} = T^n - \Delta t \left[(u^n) \frac{\partial T^n}{\partial x} + (v^n) \frac{\partial T^n}{\partial y} + (w^n) \frac{\partial T^n}{\partial z} - \frac{1}{\text{Re Pr}} (\nabla^2 T) - BTDT^n \right] \quad (9)$$

where

$$\begin{aligned} BTDu^n &= \frac{\Delta t}{2} \left\{ \frac{\partial}{\partial x} \left[(u)^2 \frac{\partial u^n}{\partial x} \right] + \frac{\partial}{\partial x} \left[(u)(v) \frac{\partial u^n}{\partial y} \right] + \frac{\partial}{\partial x} \left[(u)(w) \frac{\partial u^n}{\partial z} \right] \right. \\ &\quad + \frac{\partial}{\partial y} \left[(v)^2 \frac{\partial u^n}{\partial y} \right] + \frac{\partial}{\partial y} \left[(v)(u) \frac{\partial u^n}{\partial x} \right] + \frac{\partial}{\partial y} \left[(v)(w) \frac{\partial u^n}{\partial z} \right] \\ &\quad \left. + \frac{\partial}{\partial z} \left[(w)^2 \frac{\partial u^n}{\partial z} \right] + \frac{\partial}{\partial z} \left[(w)(u) \frac{\partial u^n}{\partial x} \right] + \frac{\partial}{\partial z} \left[(w)(v) \frac{\partial u^n}{\partial y} \right] \right\} \\ BTDv^n &= \frac{\Delta t}{2} \left\{ \frac{\partial}{\partial x} \left[(u)^2 \frac{\partial v^n}{\partial x} \right] + \frac{\partial}{\partial x} \left[(u)(v) \frac{\partial v^n}{\partial y} \right] + \frac{\partial}{\partial x} \left[(u)(w) \frac{\partial v^n}{\partial z} \right] \right. \\ &\quad + \frac{\partial}{\partial y} \left[(v)^2 \frac{\partial v^n}{\partial y} \right] + \frac{\partial}{\partial y} \left[(v)(u) \frac{\partial v^n}{\partial x} \right] + \frac{\partial}{\partial y} \left[(v)(w) \frac{\partial v^n}{\partial z} \right] \\ &\quad \left. + \frac{\partial}{\partial z} \left[(w)^2 \frac{\partial v^n}{\partial z} \right] + \frac{\partial}{\partial z} \left[(w)(u) \frac{\partial v^n}{\partial x} \right] + \frac{\partial}{\partial z} \left[(w)(v) \frac{\partial v^n}{\partial y} \right] \right\} \\ BTDw^n &= \frac{\Delta t}{2} \left\{ \frac{\partial}{\partial x} \left[(u)^2 \frac{\partial w^n}{\partial x} \right] + \frac{\partial}{\partial x} \left[(u)(v) \frac{\partial w^n}{\partial y} \right] + \frac{\partial}{\partial x} \left[(u)(w) \frac{\partial w^n}{\partial z} \right] \right. \\ &\quad + \frac{\partial}{\partial y} \left[(v)^2 \frac{\partial w^n}{\partial y} \right] + \frac{\partial}{\partial y} \left[(v)(u) \frac{\partial w^n}{\partial x} \right] + \frac{\partial}{\partial y} \left[(v)(w) \frac{\partial w^n}{\partial z} \right] \\ &\quad \left. + \frac{\partial}{\partial z} \left[(w)^2 \frac{\partial w^n}{\partial z} \right] + \frac{\partial}{\partial z} \left[(w)(u) \frac{\partial w^n}{\partial x} \right] + \frac{\partial}{\partial z} \left[(w)(v) \frac{\partial w^n}{\partial y} \right] \right\} \\ BTDT^n &= \frac{\Delta t}{2} \left\{ \frac{\partial}{\partial x} \left[(u)^2 \frac{\partial T^n}{\partial x} \right] + \frac{\partial}{\partial x} \left[(u)(v) \frac{\partial T^n}{\partial y} \right] + \frac{\partial}{\partial x} \left[(u)(w) \frac{\partial T^n}{\partial z} \right] \right. \\ &\quad + \frac{\partial}{\partial y} \left[(v)^2 \frac{\partial T^n}{\partial y} \right] + \frac{\partial}{\partial y} \left[(v)(u) \frac{\partial T^n}{\partial x} \right] + \frac{\partial}{\partial y} \left[(v)(w) \frac{\partial T^n}{\partial z} \right] \\ &\quad \left. + \frac{\partial}{\partial z} \left[(w)^2 \frac{\partial T^n}{\partial z} \right] + \frac{\partial}{\partial z} \left[(w)(u) \frac{\partial T^n}{\partial x} \right] + \frac{\partial}{\partial z} \left[(w)(v) \frac{\partial T^n}{\partial y} \right] \right\} \end{aligned}$$

Combing the continuity equation (1) and momentum equations (2), the pressure Poisson equation can be derived to correct the velocity equation as

$$\nabla^2 p^{n+1} = \frac{1}{\Delta t} \left(\frac{\partial u^{*n+1}}{\partial x} + \frac{\partial v^{*n+1}}{\partial y} + \frac{\partial w^{*n+1}}{\partial z} \right) \quad (10)$$

The u^* , v^* , w^* are the middle velocity variables.

The pressure boundary conditions can be written as

$$\frac{\partial p}{\partial n} = \frac{1}{\text{Re}} \vec{n} \cdot \nabla^2 \vec{u} + \vec{n} \cdot \vec{f}$$

In the present model, the pressure Poisson equation is solved using the finite-element method (FEM) and the bi-conjugate gradient method. The velocity and temperature values are solved using the Jacobi iteration technique. As obviously seen, the Jacobi iteration technique is effective for handling time derivative equations without a matrix solver. The spatial discretization of the equations is performed by the standard Galerkin finite-element method using eight-node trilinear elements. The resulting finite-element equations can be expressed as follows.

Figure 1. Velocity profiles at horizontal and vertical line of 3-D cavity flow.

Figure 2. Numerical results for 3-D cavity flow at $Re = 3,200$: (a) flow pattern at $y = 0.5$; (b) pressure contour at $y = 0.5$.

For step 1,

$$M_{ij}u_j^{*n+1} = M_{ij}u_j^n - \Delta t \left(C_{ij} + \frac{1}{Re} K_{ij} + \hat{K}_{ij} \right) u_j^n + \Delta t F_i^u \quad (11a)$$

$$M_{ij}v_j^{*n+1} = M_{ij}v_j^n - \Delta t \left(C_{ij} + \frac{1}{Re} K_{ij} + \hat{K}_{ij} \right) v_j^n + \Delta t F_i^v \quad (11b)$$

$$M_{ij}w_j^{*n+1} = M_{ij}w_j^n - \Delta t \left(C_{ij} + \frac{1}{\text{Re}} K_{ij} + \hat{K}_{ij} \right) w_j^n + (\Delta t \cdot \text{Ri}) M_{ij} T_j + \Delta t F_i^w \quad (11c)$$

For step 2,

$$K_{ij} P_j^{n+1} = -(G_{ij}^x u_j^{*n+1} + G_{ij}^y v_j^{*n+1} + G_{ij}^z w_j^{*n+1}) \quad (12)$$

For step 3,

$$M_{ij} u_j^{n+1} = M_{ij} u_j^{*n+1} - \Delta t G_{ij}^x P_j^{n+1} \quad (13a)$$

$$M_{ij} v_j^{n+1} = M_{ij} v_j^{*n+1} - \Delta t G_{ij}^y P_j^{n+1} \quad (13b)$$

$$M_{ij} w_j^{n+1} = M_{ij} w_j^{*n+1} - \Delta t G_{ij}^z P_j^{n+1} \quad (13c)$$

For step 4,

$$M_{ij} T_j^{n+1} = M_{ij} T_j^n - \Delta t \left(C_{ij} + \frac{1}{\text{Re Pr}} K_{ij} + \hat{K}_{ij} \right) T_j^n + \Delta t F_i^T \quad (14)$$

Figure 3. Comparison of 3-D numerical and experimental results for the reattachment length up to $\text{Re} = 800$.

where

$$\begin{aligned}
M_{ij} &= \iiint N_i N_j d\Omega \\
C_{ij} &= \iiint \left[(u^n) N_i \frac{\partial N_j}{\partial x} + (v^n) N_i \frac{\partial N_j}{\partial y} + (w^n) N_i \frac{\partial N_j}{\partial z} \right] d\Omega \\
G_{ij}^x &= \iiint N_i \frac{\partial N_j}{\partial x} d\Omega \\
G_{ij}^y &= \iiint N_i \frac{\partial N_j}{\partial y} d\Omega \\
G_{ij}^z &= \iiint N_i \frac{\partial N_j}{\partial z} d\Omega \\
K_{ij} &= \iiint \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} + \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right) d\Omega \\
\hat{K}_{ij} &= \frac{\Delta t}{2} \iiint \left[\frac{\partial N_i}{\partial x} (u)^2 \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial x} (u)(v) \frac{\partial N_j}{\partial y} + \frac{\partial N_i}{\partial x} (u)(w) \frac{\partial N_j}{\partial z} \right. \\
&\quad + \frac{\partial N_i}{\partial y} (v)^2 \frac{\partial N_j}{\partial y} + \frac{\partial N_i}{\partial y} (v)(u) \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} (v)(w) \frac{\partial N_j}{\partial z} \\
&\quad \left. + \frac{\partial N_i}{\partial z} (w)^2 \frac{\partial N_j}{\partial z} + \frac{\partial N_i}{\partial z} (w)(u) \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial z} (w)(v) \frac{\partial N_j}{\partial y} \right] d\Omega \\
F_i^u &= \int N_i \left(\frac{\partial u^n}{\partial x} n_x + \frac{\partial u^n}{\partial y} n_y + \frac{\partial u^n}{\partial z} n_z \right) d\Gamma \\
F_i^v &= \int N_i \left(\frac{\partial v^n}{\partial x} n_x + \frac{\partial v^n}{\partial y} n_y + \frac{\partial v^n}{\partial z} n_z \right) d\Gamma \\
F_i^w &= \int N_i \left(\frac{\partial w^n}{\partial x} n_x + \frac{\partial w^n}{\partial y} n_y + \frac{\partial w^n}{\partial z} n_z \right) d\Gamma \\
F_i^T &= \int N_i \left(\frac{\partial T^n}{\partial x} n_x + \frac{\partial T^n}{\partial y} n_y + \frac{\partial T^n}{\partial z} n_z \right) d\Gamma
\end{aligned}$$

in which N_i and N_j are the shape functions.

After assembling the system and applying the initial and boundary conditions as described in Eqs. (11a), (11b), (11c), (13a), (13b), (13c), and (14), the system of the equations is solved by Jacobi iteration. In Jacobi iteration, the linear system of equations is solved iteratively using the values from the previous iteration. If a linear system of equations of the form $MX = R$ is to be solved, the system is modified as

$$\overline{M}X^{k+1} = R - (M - \overline{M})X^k \quad (15)$$

where M is the consistent coefficient matrix, X is the unknown vector, R is the known vector, k is the iteration number, and $\overline{M}$ is a lumped matrix. Generally speaking, three iterations [20] can achieve satisfactory accuracy. The bi-conjugate gradient iterative solver [21] is employed to solve the linear system of the pressure Poisson equation (12).

Figure 4. Plots of streamlines at x - z plane for 3-D backward-facing step flow at different times for $Re = 800$.

4. NUMERICAL EXAMPLES

The numerical algorithm developed based on the projection FEM has been applied to four test problems to verify the accuracy and the capability of the model. The present model has been used to simulate various viscous flows and convective heat transfer problems such as a lid-driven cavity flow, flow over a backward-facing step, channel flow over ripple beds, and mixed convection in a cubic cavity.

4.1. 3-D Lid-Driven Cavity Flow

The lid-driven cavity flow problem is always considered a benchmark problem for testing new numerical schemes. The present finite-element procedure is validated

Table 1. Nusselt number of the isothermal walls at $x = 0$ and $x = 1$ of 3-D mixed convection in a cubic cavity

Ri	0.01		0.1		1		10	
Position	$x = 0$	$x = 1$	$x = 0$	$x = 1$	$x = 0$	$x = 1$	$x = 0$	$x = 1$
$Nu_{overall}$	14.825	16.938	7.90	8.498	1.122	1.417	1.338	1.369
$Nu_{mean} (y = 0.5)$	13.144	16.192	7.975	8.788	1.1306	1.392	1.421	1.45

Figure 5. Separation-region “footprint” for 3-D backward-facing step flow at $Re = 800$.

by applying it to a cubic lid-driven cavity, in which the top wall is assumed to move parallel to the x axis with a constant velocity of unity (dimensionless). The fluid in the cavity is assumed to be an incompressible viscous fluid. In the present work, results were obtained for the lid-driven cavity flow for $Re = 100, 400, 1,000$, and $3,200$, using a nonuniform mesh size of $25 \times 25 \times 25$.

The results obtained by the present numerical method for 3-D cavity flow are presented in the form of velocity vectors and pressure contours at the mid-planes of the principal axes in order to understand the flow patterns at different Reynolds numbers. Figure 1 shows the comparison of the present results for the distribution of u velocity along the vertical centerline of the cavity, with the results

Figure 6. Local mesh generation of 3-D rippled beds.

Figure 7. Streamline plots at central plane ($y = 5$) for flow over 3-D rippled beds at $Re = 1,000$.

Figure 8. 3-D physical model for coordinate system and boundary conditions.

of Jiang et al. [22] for $Re = 100, 400$, and $1,000$. It can be observed that the present predictions agree well with those of Jiang et al. [22] for $Re = 100, 400$, and $1,000$. The x - w velocity profiles shown in Figure 1 for $Re = 100, 400$, and $1,000$ exhibit the expected trends on the effect of increasing the Reynolds number. The efficiency of the present numerical scheme can be well appreciated by noting that the present results are obtained using an entirely different numerical scheme compared to Jiang et al. [22], who used a least-squares finite-element method. The increase in Reynolds number from 100 to 1,000 has resulted in the boundary-layer thinning near the moving wall by the sharp velocity gradient as observed in Figure 1. Also it develops a dip in the u - z velocity profile before reaching the maximum negative u velocity.

The most important feature of 3-D cavity flow is the end wall effect, which is not observed in the case of 2-D cavity flows. The end wall effect produces different flow patterns, resulting in various eddies and vortices. These flow structures can be well understood by the velocity and pressure contours projected on the principal mid-planes. The velocity vectors and the pressure contours at the $y = 0.5$ plane for $Re = 3,200$ are shown in Figure 2. These results indicate the formation of vortices originally closer to the initial vortices and the secondary vortices gain more strength and start moving toward the corners as the Reynolds number increases. These flow patterns can be clearly visualized from the above figures. In general, it is observed that the flow pattern becomes more complicated as the Reynolds number becomes greater than 1,000.

Figure 9. Velocity vectors at $y = 0.5$ for 3-D mixed convection in a cubic cavity for Richardson numbers (a) $Ri = 0.01$, (b) $Ri = 0.1$, (c) $Ri = 1$, and (d) $Ri = 10$.

4.2. Laminar Flow over 3-D Backward-Facing Step

The second test case was selected to verify the present algorithm for a free boundary flow problem. The 3-D laminar flow over a backward-facing step with the experiments of Armaly [23] et al. was examined. The nonuniform discretization of the computational domain results in 19,200 trilinear hexahedral elements with 21,777 nodes. The most important issue is the determination of the reattachment point for the 3-D backward-facing step flow. Figure 3 shows the comparison of the reattachment lengths of the primary separation region for these symmetric plane solutions with the experimental data of Armaly et al. [23]. The present 3-D results show excellent agreement with the primary reattachment data of Armaly et al. [23]. Figure 4 shows the x - z plane streamline distribution at different time levels from $t = 20$ to $t = 270$. It was found that the steady-state solution was obtained

Figure 10. Temperature profiles in the symmetry plane $y = 0.5$ and $z = 0.5$ for 3-D mixed convection in a cubic cavity for Richardson numbers $0.01 \leq Ri \leq 10$.

at $t = 270$. Figure 5 highlights how the primary separation bubble continues to grow and develop at $t = 40$ and $t = 200$.

4.3. Laminar Flow over 3-D Ripple Beds

To demonstrate that the present model has good capability to handle complex geometries, the laminar flow over 3-D ripple beds is simulated. The computation for a flow over ripple beds in a channel is assumed with the upstream dimensionless depth of 2. The dimensionless units of the channel are assumed to be 30 in the x direction and 10 in the y direction. The computation of the present case is computed with a total number of 67,200 elements and 75,375 nodes (201, 15, and 25 divisions in the x , y , and z directions, respectively) with a time step $\Delta t = 0.01$ and $Re = 1,000$. In the three-dimensional Navier-Stokes equations system, the intensity of the nonlinearity, the diffusion, and the convective effects are related to the magnitude of the Reynolds number. Therefore, finer meshes have to be employed while solving high Reynolds number flows. Accurate predictions for different Reynolds number flows can be obtained only by using an appropriate mesh. The results obtained are also strongly associated with the way in which the mesh size is adopted. We

Figure 11. Contour map of temperature T at $y = 0.5$ for 3-D mixed convection in a cubic cavity for Richardson numbers (a) $Ri = 0.01$, (b) $Ri = 0.1$, (c) $Ri = 1$, and (d) $Ri = 10$.

require a higher density mesh in the regions near the gradient contours, where the boundary layer is developed as the Reynolds number increases. Figure 6 shows the local nonuniform mesh distribution for the flow over ripple beds. The streamline at the central plane ($y = 5$) for $Re = 1,000$ is shown in Figure 7. Although the analytic results are not available, the results obtained by the present model could be adopted to simulate the flow over 3-D ripple beds without difficulties.

4.4. Mixed Convection in a Cubic Cavity Flow

The capability of the present numerical scheme based on the projection FEM method was tested for mixed convection in a differentially heated cubic cavity. Mixed-convection heat transfer and viscous flow phenomena take place due to the combination of the buoyancy force and the inertial force generated by the moving end walls. The buoyancy force generated due to the temperature difference between

Figure 12. Contour map of temperature T at $z = 0.5$ for 3-D mixed convection in a cubic cavity for Richardson numbers (a) $Ri = 0.01$, (b) $Ri = 0.1$, (c) $Ri = 1$, and (d) $Ri = 10$.

the end walls is computed using the Boussinesq approximation. The geometry and the boundary conditions for mixed convection in a cubic cavity are depicted in Figure 8. Temperature values equal to 0 and 1 are enforced at the $x = 0$ and $x = 1$ walls, respectively, in the x direction. Numerical results obtained for the test problem are discussed in the following contents.

In order to understand the effect of Ri on the viscous flow and heat transfer phenomena, a parametric study of Ri varying from 0.01 to 10 was carried out. The working incompressible fluid is assumed to have $Pr = 0.71$ and a constant Grashof number value of $Gr = 10^4$. Three kinds of heat transfer regimes were observed according to the magnitude of Ri : a forced-convection-dominated regime, a mixed-convection regime, and a buoyancy-dominated (or natural-convection) regime. The mean and overall Nusselt numbers were considered as the comparison parameters to study the effect of Ri on the mixed-convection phenomenon. Table 1 shows the comparison of these Nusselt numbers for different Richardson numbers varying from 0.01 to 10. With increase in Ri value from 0.01 to 0.1, the Nusselt

Figure 13. Contour map of temperature T at $x = 0.5$ for 3-D mixed convection in a cubic cavity for Richardson numbers (a) $Ri = 0.01$, (b) $Ri = 0.1$, (c) $Ri = 1$, and (d) $Ri = 10$.

number decreases. As Ri is increased from 0.1 to 1, a further big decrease in the Nusselt number is observed. Further increase in the value of Ri from 1 to 10 results in only a marginal increase in the Nusselt number. Figures 9a–9d represent the velocity vectors plotted on the x – z plane at the $y = 0.5$ symmetric plane of the cavity for $Ri = 0.01$, 0.1, 1, and 10, respectively.

At very low values of Richardson number, 0.01 and 0.1, the forced convection due to the driven force dominates the flow structure. At this order of Ri , the viscous force of the fluid is dominant compared to the buoyancy force. As the Richardson number increases to $Ri = 1$, the viscous and buoyancy forces balance each other, which then results in a mixed convection. When the Richardson number further increases to $Ri = 10$, the buoyancy force becomes the dominant mechanism to drive the convection of the fluid, and flow is in the regime of natural convection. In order to capture the 3-D effect, the temperature variations at central planes along the x direction at $y = 0.5$ and $z = 0.5$ are shown in Figure 10. More pronounced effect of heat transport by fluid convection closer to the hot and the cold walls is observed.

Figure 14. Distribution of the mean Nusselt number in the y direction for 3-D mixed convection in a cubic cavity for Richardson numbers (a) $Ri = 0.01$, (b) $Ri = 0.1$, (c) $Ri = 1$, and (d) $Ri = 10$.

Moreover, since the middle walls are assumed to be adiabatic for heat transfer, the temperature gradient increases steeply near the boundary as the Richardson number decreases. The characteristics of the mixed-convection phenomenon can be well understood by plotting the temperature contours on the various symmetric mid-planes along the principal axes. Figures 11a–11d represent the temperature contours plotted on the x – z plane at $y = 0.5$ of the cavity for $Ri = 0.01, 0.1, 1$, and 10 , respectively. It is interesting to note that the x – z plane at $y = 0.5$ is the principal plane for the heat transfer phenomena, because this plane consists of the axes of the temperature differential (x coordinate) and the gravitational direction (z coordinate). One can observe the rapid changes occurring conspicuously in the temperature contours, which indicate the effect of the increase in the Richardson number on the heat transfer process. Figures 12a–12d show similar temperature contours on the x – y plane at $z = 0.5$ for various Richardson numbers. Figures 13a–13d depict the temperature contours on the y – z plane at $x = 0.5$ for various Richardson numbers. Since the Nusselt number is another characteristic parameter of concern in heat transfer in

convective heat transfer problems, its variation along the y direction is also computed and is plotted in Figures 14a–14d for different values of the Richardson number.

5. CONCLUSIONS

This study has been concentrated on the application of the FEM to obtain numerical solutions of 3-D lid-driven cavity flow, backward-facing step flow, uniform flow over ripple beds, and mixed convection in two-sided lid-driven differentially heated cavities. The present predictions are found to compare well with the available numerical results for 3-D lid-driven cavity and backward-facing step flow. Moreover, the model has been successfully applied to simulate the 3-D channel flows over ripple beds, which demonstrate the capability to deal with complex 3-D topography. The important observations made on the mixed convection in a cavity are summarized as follows.

1. The governing parameter affecting heat transfer is the Richardson number. For $Ri > 1$, the heat transfer is dominated by natural convection. When $Ri < 1$, the flow and heat transfer are dominated by forced convection. The mixed regime is obtained when $Ri = 1$.
2. The mean value of the Nusselt number plotted along the y direction indicates that the mean value of the Nusselt number on the isothermal hot wall at $x = 1$ is higher than the value of the Nusselt number on the isothermal cold wall at $x = 0$, for values of Richardson number ranging from 0.01 to 10.

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