TWO-DIMENSIONAL INCOMPRESSIBLE VISCOUS FLOW SIMULATION USING VELOCITY-VORTICITY DUAL RECIPROCITY BOUNDARY ELEMENT METHOD

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ABSTRACT

This paper describes a computational model based on the dual reciprocity boundary element method (DRBEM) for the solution of two-dimensional incompressible viscous flow problems. The model is based on the Navier-Stokes equations in velocity-vorticity variables. The model includes the solution of vorticity transport equation for vorticity whose solenoidal vorticity components are obtained by solving Poisson equations involving the velocity and vorticity components. Both the Poisson equations and the vorticity transport equations are solved iteratively using DRBEM and combined to determine the velocity and vorticity vectors. In DRBEM, all source terms, advective terms and time dependent terms are converted into boundary integrals and hence the computational domain of the problem reduces by one. Internal points are considered wherever solution is required. The model has been applied to simulate two-dimensional incompressible viscous flow problems with low Reynolds (Re) number in a typical square cavity. Results are obtained and compared with other models. The DRBEM model has been found to be reasonable and satisfactory.

Keywords : Navier-Stokes equations, Velocity-vorticity, Dual reciprocity boundary element method.

1. INTRODUCTION

For the solution of incompressible viscous flow problems, three formulations are well known in terms of: primitive variables of pressure and velocity, velocity-stream function and velocity-vorticity. The first two formulations have been thoroughly investigated by various researchers for two-dimensional and three-dimensional problems using various numerical methods like finite differences (FDM) [1], finite elements (FEM) [2] and boundary element methods (BEM) [3]. The third formulation in terms of velocity and vorticity also has been explained in the last decade, using various numerical schemes like FDM and FEM.

The velocity-vorticity form of Navier-Stokes equations has been established as an effective formulation for the solution of incompressible viscous flow problems. For the solution of Navier-Stokes flow problems using the velocity-vorticity formulation, the governing equations have been written as a system of advection-diffusion type and Poisson type equations for the components of the vorticity and velocity fields, respectively. The main advantage of this formulation includes the numerical separation of the kinematic and

kinetic aspects of the fluid flow from the pressure computation, which is determined afterwards from the known velocity and vorticity fields.

Boundary element method (BEM) has been established as a powerful numerical tool in the solution of various fluid flow problems [3,4]. In BEM, the governing partial differential equations describing the problem domain are transformed into integral equations relating only to the boundary values. Thus BEM is a boundary-oriented method and hence the computational dimension of the problem will be reduced by one, so that it is easier in discretization and data preparation for the problem considered. The main advantages of BEM are: reduction in computational dimensions, easiness in discretization and data preparation and direct solution to flux term at the boundaries. The major problems in using the general BEM techniques to solve the incompressible viscous flows are the difficulties in dealing with the convective and time dependent terms unless internal cells are defined [5]. Earlier time, to solve any problem using BEM, a fundamental solution for the problem had to be available. The fundamental solution had to take into account all terms in the governing equations to avoid domain integration, otherwise internal cells had to be defined [5].

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Even though fundamental solution is available for many problems, it does not ensure that one is available for any given case and it is inconvenient to alter a program by incorporating a new fundamental solution to study a slightly different problem. The use of cells to evaluate domain integrals implies internal discretization and hence BEM loses some of its attractions compared to other domain techniques like FDM and FEM. To avoid all these problems, it is desirable to have a boundary element technique which: enables a boundary only solution, does not depend on obtaining a new fundamental solution for each case and can be applied using a similar approach in each case like FEM.

Dual reciprocity boundary element method (DRBEM) employs a fundamental solution corresponding to a simpler equation and treats the remaining terms, through a procedure, which involves a series expansion using global approximating functions and the applications of the reciprocity principle [6] so that a boundary only solution is possible. The method is called dual reciprocity method because it utilizes twice the basic reciprocal theorem. The dual reciprocity BEM is essentially a generalized way of constructing particular solutions that can be used to solve non-linear and time-dependent problems as well as to represent any internal source distribution. The method can be applied to define sources over the whole domain or only part of it. The dual reciprocity boundary element method was introduced by Nardini and Brebbia [7] for elasto-dynamics problems and further extended for general engineering problems [6]. Eldho and Young [8] used dual reciprocity boundary element method for the solution of Stokes flow problems.

To solve the incompressible viscous flow problems, here DRBEM is used. In the DRBEM solution of Navier-Stokes equations, the fundamental solution of Laplace equation is used. The time dependent term, convective terms and source terms in the governing equations are approximated using the dual reciprocity method. The vorticity boundary conditions are exactly obtained by the DRBEM solution of the Poisson equation. An iterative scheme is used in the solution of the system of equations after the numerical discretization using DRBEM. The accuracy and feasibility of the DRBEM model has been demonstrated using the well-known model problem of flow in a driven square cavity.

2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The governing equations for mass and linear momentum transport in a domain ζ with boundary Γ occupied by incompressible Newtonian fluid can be written as [9]:

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \bar{u} \quad (1)$$

$$\nabla \cdot \bar{u} = 0 \quad (2)$$

where $\bar{u}$ is the velocity vector, p is the pressure, Re is the Reynolds number and t is the time. Equations (1) and (2) represent the Navier-Stokes equations in the pressure-velocity formulation.

The vorticity vector $\bar{\omega}$ can be expressed as:

$$\bar{\omega} = \nabla \times \bar{u} \quad (3)$$

By taking the curl of both sides of Eq. (1) and using Eqs. (2) and (3), one can obtain the vorticity transport equation as:

$$\frac{\partial \bar{\omega}}{\partial t} + \bar{u} \cdot \nabla \bar{\omega} = \bar{\omega} \cdot \nabla \bar{u} + \frac{1}{\text{Re}} \nabla^2 \bar{\omega} \quad (4)$$

By taking the curl of Eq. (3) and using Eq. (2) one gets,

$$\nabla^2 \bar{u} = -\nabla \times \bar{\omega} \quad (5)$$

which is vector form of Poisson's equation for $\bar{u}$. Equations (4) and (5) with $\bar{u}$ and $\bar{\omega}$ as velocity and vorticity vectors are known as velocity-vorticity formulation of the Navier-Stokes equations and can replace Eqs. (1) and (2) in which u and p are variables. For two-dimensional incompressible viscous flow problems, if (u, v) are the velocity vectors ($\bar{u}$) and ω is the corresponding vorticity, the governing Navier-Stokes equations in the velocity-vorticity formulation can be written as:

$$\frac{\partial \omega}{\partial t} + \bar{u} \cdot \nabla \omega = \frac{1}{\text{Re}} \nabla^2 \omega \quad (6)$$

$$\nabla^2 u = -\frac{\partial \omega}{\partial y}; \quad \nabla^2 v = \frac{\partial \omega}{\partial x} \quad (7)$$

A solution is sought in the domain ζ that satisfies the initial conditions,

$$\bar{u} = \bar{u}_0; \quad \bar{\omega} = \nabla \times \bar{u}_0 \quad \text{at } t = 0 \quad (8)$$

and the boundary conditions,

$$\bar{u} = \bar{u}_\Gamma; \quad \bar{\omega} = (\nabla \times \bar{u})|_\Gamma \quad \text{at } t \geq 0 \quad (9)$$

on the boundary Γ of ζ .

For the Navier-Stokes equations, solution of vorticity transport equation in ω (Eq. (6)), in combination with the Poisson Eqs. (7), gives the velocity and vorticity distribution all over the domain at the concerned time step.

3. NUMERICAL FORMULATION

In the solution of velocity-vorticity formulation of Navier-Stokes flow equations using the present model, the vorticity transport equation and Poisson type velocity equations are solved by DRBEM. In this section, the numerical formulation for DRBEM for

Poisson type velocity equations and advection diffusion type vorticity transport equation are briefly described.

3.1 DRBEM Formulation for Velocity Poisson Equation

Consider the Poisson type velocity equation in v and ω , Eq. (7),

$$\nabla^2 v = \frac{\partial \omega}{\partial x} = p \quad (10)$$

with velocity boundary conditions as,

$$v = \bar{v} \quad \text{on } \Gamma_1; \quad \bar{q} = \frac{\partial \bar{v}}{\partial n} \quad \text{on } \Gamma_2 \quad (11)$$

where n is the unit outward normal vector. Here, an iterative algorithm is used such that the right hand side of Eq. (11) is known in the current iteration and time step from the previous step by solving the vorticity transport Eq. (6).

Solution to Eq. (10) can be expressed as the sum of the solution of a homogeneous Laplace equation and a particular solution $\hat{v}$ as [6]:

$$v = \tilde{v} + \hat{v} \quad (12)$$

where $\tilde{v}$ is the solution to Laplace equation and $\hat{v}$ is a particular solution such that,

$$\nabla^2 \hat{v} = p \quad (13)$$

The DRBEM proposes the use of a series of particular solution $\hat{v}_j$ instead of a single $\hat{v}$. The number of $\hat{v}_j$ used is equal to the total number of nodes in the problem. If there are N boundary nodes and L internal nodes, there will be $N + L$ values of $\hat{v}_j$.

The following approximation of p is then proposed,

$$p \approx \sum_{j=1}^{N+L} \alpha_j f_j \quad (14)$$

where the α_j are a set of initially unknown coefficients and the f_j are approximating function. The particular solution $\hat{v}_j$ and the approximating function f_j are linked through the relation,

$$\nabla^2 \hat{v}_j = f_j \quad (15)$$

The expansion (14) may then be considered as valid over the whole problem domain as in the case of one large super-element. The functions f_j are geometrical dependent. Substituting (15) into (14) gives,

$$p = \sum_{j=1}^{N+L} \alpha_j (\nabla^2 \hat{v}_j) \quad (16)$$

Substituting (16) into (10) gives,

$$\nabla^2 v = \sum_{j=1}^{N+L} \alpha_j (\nabla^2 \hat{v}_j) \quad (17)$$

The source term p of Eq. (10) has been replaced in Eq. (17) by a summation of products of coefficients α_j and the Laplacian operator on the particular solutions $\hat{v}_j$. The procedure for developing the boundary element method for the Laplace equation [5] is applied now. Equation (17) can be multiplied by the fundamental solution v^* (for 2-D problems $v^* = -\ln(r) / 2\pi$) and integrating over the domain producing,

$$\int_{\Omega} (\nabla^2 v) v^* d\Omega = \sum_{j=1}^{N+L} \alpha_j \int_{\Omega} (\nabla^2 \hat{v}_j) v^* d\Omega \quad (18)$$

where r is the distance from the collocation point (k) to the other field points (i) given as, $r = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2}$.

Applying Green's second identity [5], produces the following integral equation for each source node i ,

$$C_i v_i + \int_{\Gamma} q^* v d\Gamma - \int_{\Gamma} v^* q d\Gamma = \sum_{j=1}^{N+L} \alpha_j \left(C_i \hat{v}_{ij} + \int_{\Gamma} q^* \hat{v}_j d\Gamma - \int_{\Gamma} v^* \hat{q}_j d\Gamma \right) \quad (19)$$

where C_i is the Green's constant, which can be calculated by surrounding the boundary, point i by a small circle of radius ε and taking each term in Eq. (10) in the limit as $\varepsilon \rightarrow 0$. Generally C_i can be represented as $\theta/2\pi$ where θ is the internal angle at point i in radians. The term $\hat{q}_j$ is defined as $\partial \hat{v}_j / \partial n$, q^* as $\partial u^* / \partial n$ where n is the unit outward normal to Γ .

Note that Eq. (19) involves no domain integrals. The source term p in Eq. (10) has been substituted by equivalent boundary integrals. This was done by first approximating p using Eq. (16), and then expressing both right and left-hand sides of the resulting expression as boundary integrals using a weighted residual technique or Green's second identity. Now the discretized form of the boundary integral equation can be written for a source node i as:

$$C_i v_i + \sum_{k=1}^N \int_{\Gamma_k} q^* v d\Gamma - \sum_{k=1}^N \int_{\Gamma_k} v^* q d\Gamma = \sum_{j=1}^{N+L} \alpha_j \left(C_i \hat{v}_{ij} + \sum_{k=1}^N \int_{\Gamma_k} q^* \hat{v}_j d\Gamma - \sum_{k=1}^N \int_{\Gamma_k} v^* \hat{q}_j d\Gamma \right) \quad (20)$$

After introducing the interpolation functions and integrating over each boundary element, the above equation can be written in terms of nodal values as:

$$C_i v_i + \sum_{k=1}^N H_{ik} v_k - \sum_{k=1}^N G_{ik} q_k = \sum_{j=1}^{N+L} \alpha_j \left(C_i \hat{v}_{ij} + \sum_{k=1}^N H_{ik} \hat{v}_{kj} - \sum_{k=1}^N G_{ik} \hat{q}_{kj} \right) \quad (21)$$

The index k is used for the boundary nodes, which are the field points. After application to all boundary

nodes using a collocation technique, Eq. (21) can be expressed in matrix form as,

$$Hv - Gq = \sum_{j=1}^{N+L} \alpha_j (H \hat{v}_j - G \hat{q}_j) \quad (22)$$

Here the C_i terms have been incorporated onto the principal diagonal of H . If each of the vector $\hat{v}_j$ and $\hat{q}_j$ is considered to be one column of the matrices $\hat{V}$ and $\hat{Q}$ respectively, then Eq. (22) can be written as:

$$Hv - Gq = (H\hat{V} - G\hat{Q}) \alpha \quad (23)$$

Equation (23) is the basis for the application of the dual reciprocity boundary element method (DRBEM) and involves discretization of the boundary only. Internal nodes may be defined in the number and the location desired by the user. From Eq. (23),

$$p = \sum_{j=1}^{N+L} f_j \alpha_j \quad (24)$$

By taking the value of b at $(N + L)$ points, one can express (24) as a matrix form as:

$$p = F \alpha \quad (25)$$

where each column of F consists of vectors f_j containing the values of the function f_j at the $(N + L)$ collocation points. Therefore,

$$\alpha = F^{-1} p \quad (26)$$

Thus the right hand side vector of (26) is a known vector. Applying the boundary conditions to (23) gives a linear system of equations of the form:

$$A x = y \quad (27)$$

where x contains the unknown boundary values of v and q . Gauss elimination scheme is used to solve this system of equations to get the boundary unknowns. After finding the boundary unknowns, the function values at internal nodes can be found from,

$$v_i = - \sum_{k=1}^N H_{ik} v_k + \sum_{k=1}^N G_{ik} q_k + \sum_{j=1}^{N+L} \alpha_j \left(\hat{v}_{ij} + \sum_{k=1}^N H_{ik} \hat{v}_{kj} - \sum_{k=1}^N G_{ik} \hat{q}_{kj} \right) \quad (28)$$

The particular solution $\hat{v}$, its normal derivative and the corresponding approximating function f used in DRBEM analysis are limited by the formulation except that the resulting F matrix should be non-singular. The generally used f function is:

$$f = 1 + r + r^2 + \dots + r^m \quad (29)$$

Correspondingly [6],

$$\hat{v} = \frac{r^2}{6} + \frac{r^3}{12} + \dots + \frac{r^{m+2}}{(m+2)^2 + (m+2)} \quad (30)$$

$$\hat{q} = \left(r_x \frac{\partial x}{\partial n} + r_y \frac{\partial y}{\partial n} + r_z \frac{\partial z}{\partial n} \right) \left(\frac{1}{3} + \frac{r}{4} + \dots + \frac{r^m}{m+3} \right) \quad (31)$$

Other than the boundary only solution, the main advantage of using DRBEM in the solution of the velocity Poisson equation is the exact determination of the vorticity boundary conditions. The vorticity boundary conditions, which are essential to solve the vorticity transport equation, are directly obtained as the velocity normal derivative from the solution of Eq. (20).

3.2 DRBEM Formulation for Vorticity Transport Equation

Consider the vorticity transport equation,

$$\nabla^2 \omega = \text{Re} \left(\frac{\partial \omega}{\partial t} + \vec{u} \cdot \nabla \omega \right) = \text{Re} d(x, y, \omega, t) \quad (32)$$

As mentioned earlier, an iterative procedure is used to solve Eq. (32). In the current iteration it will be assumed that the values of (u, v) are known from the previous iteration. The fundamental solution of Laplace equation is used in the solution of Eq. (32). Appropriate boundary conditions ω or q ($\partial \omega / \partial n$) (Dirichlet condition or Neumann boundary condition) and initial conditions (prescribed ω) should be prescribed for the problem.

Using the dual reciprocity principle as in previous section, solution to Eq. (32) can be expressed as the sum of the solution of a homogeneous Laplace equation and a particular solution $\hat{\omega}$ as:

$$\omega = \tilde{\omega} + \hat{\omega} \quad (33)$$

where $\tilde{\omega}$ is the solution to Laplace equation and $\hat{\omega}$ is a particular solution such that,

$$\nabla^2 \hat{\omega} = d \quad (34)$$

The DRBEM proposes the use of a series of particular solution $\hat{\omega}_j$ instead of a single $\hat{\omega}$. The number of $\hat{\omega}_j$ used is equal to the total number of nodes in the problem. If there are N boundary nodes and L internal nodes, there will be $N + L$ values of $\hat{\omega}_j$.

Following approximation of d is then proposed,

$$d \approx \sum_{j=1}^{N+L} \alpha_j f_j \quad (35)$$

where the α_j are a set of initially unknown coefficients and the f_j are approximating function. As explained in Section 3.1, Eq. (32) can be written as:

$$\nabla^2 \omega = \text{Re} \sum_{j=1}^{N+L} \alpha_j (\nabla^2 \hat{\omega}_j) \quad (36)$$

The source term d of (32) has been replaced in Eq. (36) by a summation of products of coefficients α_j and the Laplacian operating on the particular solution $\hat{\omega}_j$.

Similar to the procedure in Section 3.1, using the two-dimensional fundamental solution of Laplace equation $(-\ln r/2\pi)$ and using the BEM procedure, the boundary integral equation is written as:

$$C_i \omega_i + \int_{\Gamma} q^* \omega d\Gamma - \int_{\Gamma} u^* q d\Gamma = \text{Re} \sum_{j=1}^{N+L} \alpha_j \left(C_i \hat{\omega}_j + \int_{\Gamma} q^* \hat{\omega}_j d\Gamma - \int_{\Gamma} u^* \hat{q}_j d\Gamma \right) \quad (37)$$

After the boundary discretization and integration, one can write the final system of equations in matrix form as:

$$H \omega - G q = \text{Re}(H \hat{\omega} - G \hat{Q}) \alpha \quad (38)$$

In Eq. (35), the time dependent term and advective terms are approximated as:

$$d \approx \sum_{j=1}^{N+L} f_j(x, y, z) \alpha_j(t) \quad (39)$$

Here f_j are known functions of geometry and α_j are unknown functions of time. Then Eq. (39) can be written in matrix form as:

$$d = F \alpha \quad (40)$$

or

$$\alpha = F^{-1} d \quad (41)$$

Therefore, one can write Eq. (38) as,

$$H \omega - G q = \text{Re}(H \hat{\omega} - G \hat{Q}) F^{-1} d \quad (42)$$

Putting,

$$S = (H \hat{\omega} - G \hat{Q}) F^{-1} \quad (43)$$

Substituting for d , Eq. (42) can be written as,

$$H \omega - G q = \text{Re} \left[S \left(\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} \right) \right] \quad (44)$$

Setting,

$$\omega = F \beta \quad (45)$$

or

$$\beta = F^{-1} \omega \quad (46)$$

Differentiating (45) with respect to x and y ,

$$\begin{aligned} \frac{\partial \omega}{\partial x} &= \frac{\partial F}{\partial x} \beta \\ \frac{\partial \omega}{\partial y} &= \frac{\partial F}{\partial y} \beta \end{aligned} \quad (47)$$

Therefore,

$$\begin{aligned} \frac{\partial \omega}{\partial x} &= \frac{\partial F}{\partial x} F^{-1} \omega \\ \frac{\partial \omega}{\partial y} &= \frac{\partial F}{\partial y} F^{-1} \omega \end{aligned} \quad (48)$$

$$H \omega - G q = \text{Re} S \left[\frac{\partial \omega}{\partial t} + \left(u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} \right) F^{-1} \omega \right] \quad (49)$$

Putting

$$\omega' = \frac{\partial \omega}{\partial t}; \quad E = u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} \quad (50)$$

One can write Eq. (49) as,

$$H \omega - G q = \text{Re} S [\omega' + E F^{-1} \omega] \quad (51)$$

Now substituting,

$$M = \text{Re} S E F^{-1}; \quad N = -\text{Re} S \quad (52)$$

The final system of equation can be written as:

$$N \omega' + (H - M) \omega = G q \quad (53)$$

Using a two level time integration for ω and q and a difference scheme for the time derivative

$$\begin{aligned} \omega &= (1 - \theta_u) \omega^m + \theta_u \omega^{m+1} \\ q &= (1 - \theta_q) q^m + \theta_q q^{m+1} \\ \omega' &= \frac{1}{\Delta t} (\omega^{m+1} - \omega^m) \end{aligned} \quad (54)$$

Here θ_u and θ_q are weighting factors which position the values of ω and q , respectively, between time levels m and $m+1$. Substituting (54) into (53) gives:

$$\begin{aligned} &\left(\frac{N}{\Delta t} + \theta_u (H - M) \right) \omega^{m+1} - \theta_q G q^{m+1} \\ &= \left[\frac{N}{\Delta t} - (1 - \theta_u)(H - M) \right] \omega^m + (1 - \theta_q) G q^m \end{aligned} \quad (55)$$

The right hand side of (55) is known at time $(m+1)\Delta t$, since it involves values, which have been specified as initial conditions or calculated previously. Upon imposing the boundary conditions at time $(m+1)\Delta t$, one can form system of linear equations. The linear system of equations, are solved using the Gauss elimination scheme to find the unknown function values initially over the boundary and then at the internal nodes considered.

The approximating functions ' f ' given in Section 3.1 and corresponding particular solutions and derivatives given in Eqs. (30) and (31) are used here also. As mentioned earlier, here an iterative scheme is used in the solution of the Navier-Stokes equations and the derivatives of the vorticity ω in x - and y -direction should be determined before the velocity Poisson equations are solved so that the right hand sides of Poisson equations are known in the each iteration. Here a special DRBEM technique is used to determine the derivatives of ω . Once the vorticity values are known by solving the linear system of equations, then the derivatives of vorticity can be easily calculated from Eq. (48).

4. SOLUTION PROCEDURE

Here an iterative scheme is used in the solution of the velocity-vorticity formulation of Navier-Stokes equations. In most of the incompressible viscous flow problems, the vorticity boundary conditions are seldom prescribed and hence should be determined from computation. In the present model, the velocity Poisson equations are initially solved to get the vorticity boundary conditions that are used in the solution of the vorticity transport equations. The solution procedure adopted here includes the following iterative steps:

For the time step $n = 1$,

- (1) Assume some initial values for vorticity distribution and derivatives, say zero.
- (2) Solve the velocity Poisson Eqs. (7) using DRBEM
 - Get the unknown boundary velocity or velocity flux values
 - Calculate the velocity distribution and velocity derivatives at all nodal points
 - Determine the new vorticity boundary values using the definition of vorticity in Eq. (3)
 - Solve the vorticity transport Eq. (6) using DRBEM
 - Get the unknown vorticity values throughout the domain
 - Calculate the derivatives of the vorticity vectors using Eq. (48) to use in the velocity Poisson equation
 - Relax the vorticity values, if necessary
- (3) Check for the convergence of the velocity and vorticity components in the present iteration, for example,

$$\frac{|v_n^{k+1} - v_n^k|}{|v_n^k|} \leq 0.001$$

If convergence criterion is satisfied, then stop and proceed to the next time step, otherwise go to step (2).

In the successive time step, use the velocity and

vorticity components from the previous time step as initial conditions and use the iterative procedure, steps (2) to (4). The procedure is repeated until the prescribed time step is reached.

5. MODEL RESULTS AND DISCUSSION

The feasibility and accuracy of DRBEM model has been tested using the classical benchmark problem of 'driven flow in a square cavity'. The cavity flow problem is of continuing interest because it offers a relatively simple model on which numerical techniques may be examined and verified with other numerical schemes. Present model results are compared with a series solution, FDM, FEM and BEM models in two-dimensions.

The model problem consists of a square cavity with a moving top lid with constant velocity, totally filled with an incompressible viscous fluid. Here the low Reynolds number flow is considered. The flow inside the cavity is initially at rest. Figure 1 shows the geometry and boundary conditions of the problem. No slip and impermeability conditions were imposed on all walls, with the velocity at the upper wall set equal to unity. Due to the computational limitations (an IBM Pentium II PC with 64 MB RAM is used in the study), the present analysis is limited to a maximum of computational mesh points of 21×21 (on the boundary and internal nodes). The analyses are performed for the Reynolds numbers of 0, 1, 10 and 100.

Figures 2 and 3 show the u and v velocity profiles along the vertical and horizontal centerlines of the cavity, respectively for $Re = 0$ with 21×21 mesh. The results are compared with a series solution [10], FEM solution [11] and BEM solution [12] and are in good agreement, bearing in mind that the mesh density is coarse. As a convergence test for the grid, analysis has been done at $Re = 1$ for the uniform meshes of 11×11 and 21×21 . Figure 4 shows the x -component velocity profiles (u) on the vertical centerline with different meshes. The results seem to be varying slightly for both the meshes.

Fig. 1 Test problem domain with boundary conditions

Fig. 2 Velocity profile (u) along vertical centerline for $Re = 0$

Fig. 3 Velocity profile (v) along horizontal centerline for $Re = 0$

Fig. 4 Velocity profile (u) on vertical centerline at $x = 0.5$ of the square cavity for $Re = 1$, Mesh convergence

Figures 5 and 6 show the u and v velocity profiles along the vertical and horizontal centerlines of the cavity, respectively for $Re = 10$ with 11×11 mesh. Figure 7 shows the vorticity distribution along the domain for $Re = 10$. Figures 8 and 9 show the u and v velocity profiles along the vertical and horizontal centerlines of the cavity, respectively for $Re = 100$ with 21×21 mesh. The velocity variations are compared with a series solution of Burggraf [10], FDM solution of Ghia, *et al.* [13], FEM solution of Young and Lin [11] and BEM solution of Young, *et al.* [14]. The results mostly agree with all the model results, considering the coarse nature of the mesh used in the present analysis. Figure 10 shows the vorticity distribution along the domain and Fig. 11 shows the velocity vector field.

DRBEM has been proved to be a feasible method to solve incompressible viscous flow problems governed by Poisson equation and advection-diffusion type vorticity transport equation. In the model, the necessary vorticity boundary conditions are determined by the DRBEM solution of the velocity Poisson

Fig.5 Velocity profile (u) on the centerline at $x = 0.5$ of a square cavity for $Re = 10$

Fig. 6 Velocity profile (v) on the centerline at $y = 0.5$ of a square cavity for $Re = 10$

Fig.7 Vorticity profile (ω) in the square cavity for $Re = 10$

Fig. 10 Vorticity profile (ω) in the square cavity for $Re = 100$

Fig. 8 Velocity profile (u) along vertical centerline for $Re = 100$

Fig. 11 Profile of the flow vectors for $Re = 100$

Fig. 9 Velocity profile (v) along horizontal centerline for $Re = 100$

equations together with the no-slip boundary conditions. Hence the present model in which the velocity Poisson equations and vorticity transport equation are solved using DRBEM, and the iterative scheme combining both models provides the best way to treat the velocity-vorticity formulation for two-dimensional incompressible viscous flow problems.

The merit of the DRBEM model is that, it provides the best way to treat the velocity-vorticity formulation for two-dimensional Navier-Stokes flow by DRBEM without domain discretization. For the computation, only the boundary discretization is necessary and the internal points are defined wherever solution is required. Hence the computational domain of the problem reduces by one, that is, the two-dimensional problem is solved in one dimension effectively. In general, the boundary conditions of vorticity are not known *a priori*, so that the solution of vorticity transport equation is rather difficult. However the DRBEM solution of the

velocity Poisson equations will automatically render the necessary vorticity boundary conditions. This study demonstrates that even with a relatively coarse mesh, the solutions are stable and reasonable as indicated by the case studies.

6. CONCLUDING REMARKS

Here a model based on dual reciprocity boundary element method for solving the velocity-vorticity Navier-Stokes equations is presented. The Poisson type velocity equations and the vorticity transport equation are solved using the DRBEM. The vorticity boundary conditions for the solution of vorticity transport equation are exactly obtained from the DRBEM solution of velocity Poisson equations. The use of DRBEM enables one to have a boundary only solution for the problem. The DRBEM model has been applied for the solution of low Reynolds number incompressible viscous flow problems in two- dimensions. The results are presented for the classical driven cavity flow problem. A comparison with other models demonstrates the feasibility of the DRBEM model.

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