On Fairness Notions in Distributed Systems

II. Equivalence-Completions and Their Hierarchies*

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This is the second part of a two-part paper in which we discuss the implementability of fairness notions in distributed systems where asynchronous processes interact via multiparty interactions. We focus here on equivalence-robust fairness notions where equivalent computations are either all fair or all unfair. Francez et al. (1992, Formal Aspects Comput. 4, 582–591) propose a notion of completion to transform a non-equivalence-robust fairness notion to an equivalence-robust one while maintaining several properties of the source. However, a completion may not preserve strong feasibility—a necessary and sufficient condition for a completion to be implementable. In this paper, we study the system requirement for a completion to be strongly feasible and determine the strongest implementable completion for every given fairness notion. Moreover, for most systems we obtain a fairness notion, which we refer to as SG⁺, such that SG⁺ is the strongest fairness notion that is both implementable and equivalence-robust. We also provide a comprehensive comparison of SG⁺ and several well-known fairness notions and their minimal and maximal completions. Finally, we show that if equivalence-robustness is dropped, then in general it is impossible to define a fairness notion that is implementable and stronger than all other implementable fairness notions, unless the system consists of only one interaction. This implies plenty of leeway in the design of fairness notions suitable for various applications.

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INTRODUCTION

This is the second part of a two-part paper in which we discuss the implementability of fairness notions in distributed systems where asynchronous processes interact via multiparty interactions. In Part I [5] we have presented a necessary and sufficient criterion for determining the implementability of fairness notions. We focus here on equivalence-robust fairness notions where equivalent computations are either all fair or all unfair.

1. EQUIVALENCE-ROBUSTNESS AND COMPLETIONS

Intuitively, equivalence-robustness ensures that different observations of the same partial-order computation obtain the same property of the system [6]. It thus serves as a natural bridge over the gap between interleaving semantics and partial-order semantics, which is highly desirable in distributed languages [3]. Furthermore, as we have shown in Part I, under strong feasibility equivalence-robustness suffices to guarantee the implementability of a fairness notion.

As it turns out, however, several important fairness notions are strongly feasible but are not equivalence-robust. For example, consider the notion of strong interaction fairness (SIF), which requires an interaction that is infinitely often enabled to be executed infinitely often. Assume a system \( IS \) with three interactions \( x_{12}, x_{13}, \) and \( x_{24} \) depicted in Fig. 1, where \( x_{12} \) involves \( p_1 \) and \( p_2 \), \( x_{13} \) involves \( p_1 \) and \( p_3 \), and \( x_{24} \) involves \( p_2 \) and \( p_4 \). SIF is strongly feasible for the system because a nonblocking scheduler satisfying SIF can be constructed by always choosing as the continuation the enabled interaction that is executed the least often; tie is broken arbitrarily. Then the computation \( \pi = (p_1 p_3 x_{13} p_2 p_4 x_{24})^\omega \)

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satisfies SIF, but its equivalent computation \( \psi = (p_1p_3p_2x_13p_4x_{24})^{\omega} \) does not because \( x_{12} \) is now enabled in every state immediately after \( p_2 \) is ready but it is never executed.

Francez et al. [3] propose a notion of completion to transform a non-equivalence-robust fairness notion to an equivalence-robust one while maintaining most properties of the source. To understand completions, consider Fig. 2. In this figure, run(\( IS \)) denotes the set of all possible computations of a system \( IS \), while \( C(IS) \) denotes the set of computations allowed by a fairness notion \( C \). Each partition represents an equivalence class induced by the equivalence relation considered above by permuting independent actions. Of these equivalence classes, \( X_i \)'s are said to be purely fair because they are contained in \( C(IS) \), while \( Y_j \)'s are purely unfair because they do not intersect \( C(IS) \). The classes \( Z_k \)'s are mixed as they contain both fair and unfair computations. A completion has to resolve the fairness of the mixed classes. Thus, the minimal completion (i.e., the strongest completion) can be obtained by treating all mixed classes as unfair, whereas the maximal completion (i.e., the weakest completion) can be obtained by treating all mixed classes as fair. A semantic comparison of the two can be found in [3]. In general, fewer liveness properties can be assumed for programs using the weakest completion, while the strongest completion has exactly the opposite characteristics.

Unfortunately, a completion may not necessarily preserve strong feasibility, meaning that it may not even be implementable. To see this, consider again the system shown in Fig. 1. The computation

\[
\pi = (p_1p_3x_{13}p_2p_4x_{24})^{\omega}
\]

is inevitable to all strongly feasible fairness notions of \( IS \), meaning that they must consider \( \pi \) as fair. This is because at any point of the computation at most one interaction is enabled. Thus according to the strong feasibility criterion, when only one interaction is enabled, there must be a continuation allowing the interaction to be executed. (Otherwise the system could be deadlocked if, say when \( x_{13} \) is enabled, the scheduling algorithm chooses to wait for more interactions to be enabled while \( p_2 \) and \( p_4 \) instead are busy doing their local actions forever.) So any completion of SIF, e.g., the minimal completion, that excludes the equivalence class of \( \pi \) would not be strongly feasible.

In this paper we determine, for any given fairness notion \( C \), the strongest strongly feasible completion of \( C \). Recall that strong feasibility is sufficient and necessary to guarantee the implementability of a completion. So we are looking for a strongest implementable completion of \( C \). Our results show that if no interaction contains an interaction (an interaction \( x \) contains \( y \) if \( x \neq y \) and the set of participants of \( y \) is a subset of \( x \)),\(^1\) then the strongest implementable completion of \( C \) exists; otherwise, in general no such completion is possible.

Furthermore, there exists a fairness notion, which we refer to as SG\(^+\), such that when interactions are not allowed to contain interactions, SG\(^+\) is the strongest implementable fairness notion satisfying equivalence-robustness. In other words, all other implementable and equivalence-robust fairness notions must be weaker than SG\(^+\), and all other equivalence-robust fairness notions that are stronger than SG\(^+\) or incomparable with SG\(^+\) must not be implementable. We also compare SG\(^+\) with several existing fairness notions. The results indicate that SG\(^+\) is equivalent to SIF\(^+\) and SPF\(^+\) and is stronger than WIF and WPF\(^+\), where SIF\(^+\), SPF\(^+\), and WPF\(^+\) are the maximal completions of SIF, SPF, and WPF, respectively. In particular, when interactions are CSP-like bipartition, SG\(^+\) is also equivalent to SPF. So SPF is the strongest equivalence-robust property one can observe from a CSP-like program executing in any asynchronous environment. Conversely, if interactions can contain interactions, then in general the strongest implementable and equivalence-robust fairness notion does not exist.

\(^1\) Note that although \( x \) contains \( y \), the establishment of \( y \) does not depend on the establishment of \( x \). In the literature, some languages (e.g., IP and Script) allow participants of an interaction \( v \) to establish a “subinteraction” within \( v \). So the subinteraction can be established only when \( v \) has been established.

![FIG. 1. A system of four processes \( p_1, p_2, p_3, \) and \( p_4 \) and three interactions \( x_{12}, x_{13}, \) and \( x_{24} \).](image-url)
Finally, we show that if equivalence-robustness is not required, then no system can have a strongest implementable fairness notion, unless the system consists of only one interaction. So, in general, for any implementable fairness notion $C$, there exists another implementable fairness notion $C'$ such that $C'$ is either strictly stronger than $C$ or is incomparable with $C$. This implies plenty of leeway in the design of fairness notions suitable for various applications.

The rest of the paper is organized as follows. Section 2 provides some preliminaries. Section 3 considers the implementability of completions. Section 4 presents a comprehensive analysis of several commonly used fairness notions and their minimal and maximal completions. The impossibility result of a strongest implementable fairness notion is the subject of Section 5. Section 6 concludes.

2. PRELIMINARIES

We shall follow the same notations as those given in Part I. In addition, we shall use $eq(\pi)$ to denote the equivalence class of $\pi$, i.e., the set of runs that are equivalent to $\pi$. Moreover, for notational simplicity we shall use a set $P$ in a run to denote an arbitrary sequence of ready transitions, one by each process in $P$, where each ready transition is of the form $p_i \cdot \{ x \in I \mid p_i \in P \}$. For example, assume that $P_{x12} = \{ p_1, p_2 \}$ and $P_{x23} = \{ p_2, p_3 \}$. Then $P_{x12} P_{x12} P_{x23}$ represents a partial run in which $p_1$ and $p_2$ become ready (in arbitrary order), they execute $x_{12}$, and then $p_2$ and $p_3$ become ready (again, in arbitrary order) and execute $x_{23}$. Likewise, $(P_{x12} \cup P_{x23}) x_{12} P_{x12} x_{23}$ represents a partial run in which $p_1$, $p_2$, and $p_3$ become ready, $p_1$ and $p_2$ execute $x_{12}$ and become ready again, and then $p_2$ and $p_3$ execute $x_{23}$.

In addition to the definitions given in Part I, the following are used in this article.

**DEFINITION 2.1.** A fairness notion $C_1$ for $\mathcal{S}$ is stronger than $C_2$ (or, alternatively, $C_2$ is weaker than $C_1$) if $C_1(\mathcal{S}) \subseteq C_2(\mathcal{S})$. $C_1$ is strictly stronger than $C_2$ if $C_1(\mathcal{S}) \subset C_2(\mathcal{S})$. $C_1$ and $C_2$ are incomparable if $C_1(\mathcal{S}) \not\subseteq C_2(\mathcal{S})$ and $C_2(\mathcal{S}) \not\subseteq C_1(\mathcal{S})$; they are equivalent if $C_1(\mathcal{S}) = C_2(\mathcal{S})$.

**DEFINITION 2.2.** [1]. A fairness notion $C$ is equivalence-robust for $\mathcal{S}$ iff $\forall \pi \in C(\mathcal{S}), eq(\pi) \subseteq C(\mathcal{S})$.

The following is a restatement of strongly feasibility.

**DEFINITION 2.3.** A fairness notion $C$ is strongly feasible for $\mathcal{S}$ iff there exists a nonblocking scheduler $S$ such that $r(S, A) \in C(\mathcal{S})$ for every adversary $A$.

An immediate consequence of these definitions is the following.

**LEMMA 2.1.** Let $\mathcal{S} = (P, I, M)$.

1. If $I \neq \emptyset$, then for any strongly feasible fairness notion $C$, $C(\mathcal{S}) \neq \emptyset$.
2. If $C_1$ is strongly feasible for $\mathcal{S}$ and $C_1(\mathcal{S}) \subseteq C_2(\mathcal{S})$, then $C_2$ is also strongly feasible for $\mathcal{S}$.
3. It may be the case that both $C_1$ and $C_2$ are strongly feasible for $\mathcal{S}$, but $C_1 \cap C_2$ is not, where the notion $C_1 \cap C_2$ is defined by $C_1(\mathcal{S}) \cap C_2(\mathcal{S})$.

**Proof.** The first two follow directly from the definitions. For the third, let $\mathcal{S} = \{(p_1, p_2, p_3), \{x_{12}, x_{23}\}, M^0\}$, where $P_{x12} = \{ p_1, p_2 \}$ and $P_{x23} = \{ p_2, p_3 \}$. Let $C_1$ be defined as follows: in a state where both $x_{12}$ and $x_{23}$ are enabled, $x_{12}$ must be scheduled for execution (i.e., $x_{12}$ has a higher priority than $x_{23}$) and $C_2$ be defined as “$x_{23}$ has a higher priority than $x_{12}$.” Then, both $C_1$ and $C_2$ are strongly
feasible, but $C_1 \cap C_2$ is not because no scheduler can make a move if the adversary lets the three processes be ready simultaneously.

We have shown in Part I that SIF is strongly feasible. Then by Lemma 2.1, any fairness notion that is weaker than SIF is also strongly feasible. Examples include SPF, WPF, and WIF.

**Lemma 2.2.** The following four fairness notions are all strongly feasible: SIF, SPF, WPF, and WIF.

On the other hand, we have shown in Part I, both U-fairness and hyperfairness are not strongly feasible.

In general, to prove that $C$ is not strongly feasible for $\mathcal{I}$, we must show that for every nonblocking scheduler $S$, there is an adversary $A$ such that $r(S, A) \notin C(\mathcal{I})$. However, for most systems there exists an adversary such that every scheduler versus it must generate the same run. So if the run is not included in $C(\mathcal{I})$, then obviously $C$ cannot be strongly feasible. For example, if an adversary always lets at most one interaction be enabled, then any nonblocking scheduler versus the adversary must generate the same run: whichever interaction is enabled, it must be selected for execution. The resulting runs are called *singular* in Part I, and are inevitable to every strongly feasible fairness notion of $\mathcal{I}$. We shall use $SG(\mathcal{I})$ to denote the set of singular runs of $\mathcal{I}$. An immediate consequence of the definition is the following.

**Lemma 2.3.** For every strongly feasible fairness notion $C$, $SG(\mathcal{I}) \subseteq C(\mathcal{I})$.

Note that for every $\mathcal{I}$, $SG(\mathcal{I}) \neq \emptyset$, unless $\mathcal{I}$ contains more than one interaction and for every interaction $x$, there exists another interaction $y$ such that $y$ is contained in $x$. Moreover, if $SG(\mathcal{I}) \neq \emptyset$, then for every fairness notion $C$ it is never the case that $SG(\mathcal{I}) \subseteq C(\mathcal{I})$ and $SG(\mathcal{I}) \subseteq \bar{C}(\mathcal{I})$, where $\bar{C}$ is the complement of $C$ defined by $\bar{C}(\mathcal{I}) = run(\mathcal{I}) - C(\mathcal{I})$. Therefore, if $C$ is strongly feasible for $\mathcal{I}$, then $\bar{C}$ must not be strongly feasible. In other words, it is never the case that both $C$ and its complement are implementable.

**Corollary 2.4.** Let $\mathcal{I} = (P, I, M^\mathcal{I})$. If $\exists x \in I, \forall y \in I, y \neq x \Rightarrow P_y \not\subseteq P_x$, then for every fairness notion $C$, it is never the case that both $C$ and $\bar{C}$ are implementable for $\mathcal{I}$.

Recall from Lemma 4.1 of Part I that in the presence of equivalence-robustness, strong feasibility is sufficient and necessary to determine fairness implementability. As this result will be referenced several times in the paper, for ease of reference, we restate the lemma below.

**Lemma 2.5.** If $C$ is strongly feasible and equivalence-robust for $\mathcal{I}$, then $C$ is implementable for $\mathcal{I}$.

Moreover, since if $C$ is stronger than $C'$ then $C(\mathcal{I}) \subseteq C'(\mathcal{I})$, the fairness implementability criterion immediately implies the following lemma:

**Lemma 2.6.** Suppose $C$ is stronger than $C'$. If $C$ is implementable for $\mathcal{I}$, then so is $C'$; and if $C'$ is not implementable, then neither is $C$.

### 3. STRONG FEASIBILITY OF COMPLETIONS

In this section we consider the implementability of equivalence-robust fairness notions. In particular, we shall focus on *completions*—equivalence-robust fairness notions derived from a non-equivalence-robust one. We shall show that the process of completion may not preserve strong feasibility. Since in the presence of equivalence-robustness strong feasibility suffices to determine fairness implementability, completions are not necessarily implementable.

Furthermore, we shall also show that if no interaction of $\mathcal{I}$ contains an interaction, then there exists a fairness notion, denoted by $SG^+$, such that $SG^+$ is the strongest implementable and equivalence-robust fairness notion of $\mathcal{I}$. On the other hand, if $\mathcal{I}$ contains two interactions $x, y$ such that $P_x \not\subseteq P_y$, then in general there does not exist a strongest implementable and equivalence-robust fairness notion for $\mathcal{I}$.

#### 3.1. Definitions

**Definition 3.1 [3].** 1. An equivalence class $eq(\pi)$ in $run(\mathcal{I})$ is purely $C$-fair iff $eq(\pi) \subseteq C(\mathcal{I})$, it is purely $C$-unfair iff $eq(\pi) \cap C(\mathcal{I}) = \emptyset$, and it is $C$-mixed otherwise.
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2. A fairness notion \( \hat{C} \) is a completion of \( C \) iff the following three conditions are satisfied:
   (i) for every purely \( C \)-fair class \( eq(\pi) \), \( eq(\pi) \subseteq \hat{C}(IS) \)
   (ii) for every purely \( C \)-unfair class \( eq(\pi) \), \( eq(\pi) \cap \hat{C}(IS) = \emptyset \), and
   (iii) for every class \( C \)-mixed class \( eq(\pi) \), either \( eq(\pi) \subseteq \hat{C}(IS) \) or \( eq(\pi) \cap \hat{C}(IS) = \emptyset \).

It follows directly that a completion \( \hat{C} \) must be equivalence-robust. Two extreme completions of \( C \) arise naturally: the \textit{maximal} completion \( C^+ \), which treats every \( C \)-mixed class as fair, and the \textit{minimal} completion \( C^- \), which treats every \( C \)-mixed class as unfair [3]. Moreover, \( C^- (IS) \subseteq C(IS) \subseteq C^+(IS) \) for every \( C \). By Lemma 2.1, if \( C \) is strongly feasible then so is \( C^+ \). Since \( C^+ \) is equivalence-robust, by Lemma 2.5, if \( C \) is strongly feasible then \( C^+ \) must be implementable.

Lemma 3.1. \textit{If} \( C \) \textit{is strongly feasible for} \( IS \), \textit{then} \( C^+ \) \textit{must be implementable for} \( IS \).

On the other hand, since the process of minimal completion may exclude some runs that are inevitable to \( C^- \), \( C^- \) may be unimplementable. In fact, if \( C \) is not implementable then strong feasibility cannot be preserved by \( C^- \). Therefore, unlike maximal completions, minimal completions do not help us obtain an implementable fairness notion from an unimplementable one while pursuing equivalence-robustness.

Lemma 3.2. \textit{If} \( C \) \textit{is not implementable for} \( IS \), \textit{then neither is} \( C^- \). \textit{Moreover, if} \( C \) \textit{is not implementable, then} \( C^- \) \textit{must not be strongly feasible}.

\textit{Proof.} Since \( C^- (IS) \subseteq C(IS) \), by Lemma 2.6, if \( C \) is not implementable then neither is \( C^- \). Since \( C^- \) is equivalence-robust, by Lemma 2.5, it must not be strongly feasible. \( \blacksquare \)

3.2. Strongly Feasible Completions

As it turns out, the weakest completion (i.e., the maximal completion) of a strongly feasible fairness notion is also strongly feasible, while the strongest completion (i.e., the minimal completion) may not be. Since strongly feasible completions are implementable, and since stronger completions induce more liveness properties, given a strongly feasible fairness notion \( C \), one would wish to know what is the strongest, strongly feasible completion of \( C \), or does it even exist. To answer this, we first establish a more important theorem showing the existence of a strongest fairness notion that is both strongly feasible and equivalence-robust. For this, we need the following lemma.

Lemma 3.3. Let \( IS = (P, I, M) \) and assume that \( \forall x, y \in I, x \neq y \Rightarrow P_x \not\subseteq P_y \). The every \( \pi \in \text{run}(IS) \) satisfying SIF is equivalent to a singular run.

\textit{Proof.} Let \( \pi \in \text{run}(IS) \) be a run given by

\[ \pi = p_{1,1}, I_{1,1} \cdots p_{1,n_1}, I_{1,n_1}, x_1 p_{2,1}, I_{2,1} \cdots p_{2,n_2}, I_{2,n_2}, x_2 \cdots, \]

where \( x_1, x_2, \ldots \) are the sequence of interactions executed in \( \pi \). Assume that \( \pi \) satisfies SIF. Consider \( x_1 \). Suppose that we transform \( \pi \) into another run \( \pi_1 \) by the following procedure: For each \( p_{1,j}, I_{1,j}, 1 \leq j \leq n_1 \), if (1) \( p_{1,j} \notin P_{x_1} \) and (2) at some point in \( \pi \) (after the ready transition \( p_{1,j}, I_{1,j} \)) some interaction \( y \) involving \( p_{1,j} \) is enabled, then move the action \( p_{1,j}, I_{1,j} \) after \( x_1 \). Clearly, \( \pi \equiv \pi_1 \). Due to the restriction imposed on the structure of \( IS \), no subset of \( P_{x_1} \) is involved in any other interaction. So in \( \pi_1 \) at most one interaction is enabled at any point up to \( x_1 \).

Similarly, we can transform \( \pi_1 \) into another run \( \pi_2 \) by applying the above procedure to the ready transitions occurring between \( x_1 \) and \( x_2 \) so that \( \pi_1 \equiv \pi_2 \), and in \( \pi_2 \) at most one interaction is enabled at any point up to \( x_2 \). We claim that if we apply the procedure repeatedly for the rest of \( x_i \)'s, then we will obtain a run \( \pi_\infty \) such that \( \pi \equiv \pi_\infty \) and \( \pi_\infty \) is singular. To see this, observe that for any finite \( i \), we have \( \pi \equiv \pi_1 \equiv \cdots \equiv \pi_i \), and in \( \pi_i \) at most one interaction is enabled at any point up to \( x_i \). So it suffices to show that the equivalence relation between \( \pi \) and \( \pi_i \) is preserved when \( i \to \infty \).

\[ \cdots \]
Suppose otherwise that the equivalence relation does not hold. Then, it must be the case that some ready transition \( p_{k,j}, I_{k,j} \) in between \( x_{k-1} \) and \( x_k \) of run \( \pi_k \) has to be moved during the transformation from \( \pi_{k-1} \) to \( \pi_k \) for every \( l \geq k \), resulting in the extinction of \( p_{k,j}, I_{k,j} \) when the transformation is performed an infinite number of times. However, recall that if \( p_{k,j}, I_{k,j} \) has to be moved in the \( k \)th transformation (i.e., from \( \pi_{k-1} \) to \( \pi_k \), then (1) \( p_{k,j} \notin P_{x_k} \) and (2) later at some point in \( \pi_{k-1} \) (and thus in \( \pi \)) some interaction \( y \) involving \( p_{k,j} \) is enabled. Since in the remaining transformations \( p_{k,j}, I_{k,j} \) is continually moved, none of the interactions \( x_k, x_{k+1}, \ldots \) involves process \( p_{k,j} \), and there exists an infinitely number of points in \( \pi \) such that at each point some interaction involving \( p_{k,j} \) is enabled, but from \( x_k \) onward \( p_{k,j} \) never participates in any interaction. So \( \pi \) does not satisfy SPF, and thus \( \pi \) does not satisfy SIF. This contradicts the assumption that \( \pi \) satisfies SIF. Therefore, the equivalence relation between \( \pi \) and \( \pi_\infty \) is preserved during the transformations from \( \pi \) to \( \pi_\infty \). \( \blacksquare \)

Recall from Lemma 2.3 that for every strongly feasible fairness notion \( \mathcal{C} \), \( \text{SG}(\mathcal{I}) \subseteq \mathcal{C}(\mathcal{I}) \), where \( \text{SG}(\mathcal{I}) \) is the set of singular runs of \( \mathcal{I} \). Define fairness notion \( \text{SG}^+ \) to be the maximal completion of \( \text{SG} \), i.e.,

\[
\text{SG}^+(\mathcal{I}) = \bigcup_{\pi \in \text{SG}(\mathcal{I})} eq(\pi).
\]

Then, for every \( \mathcal{C} \) that is both strongly feasible and equivalence-robust, \( \text{SG}^+(\mathcal{I}) \subseteq \mathcal{C}(\mathcal{I}) \). Moreover, by Lemma 3.3, \( \text{SIF}(\mathcal{I}) \subseteq \text{SG}^+(\mathcal{I}) \). Since SIF is strongly feasible, by Lemma 2.1 \( \text{SG}^+ \) is also strongly feasible. Hence, by Lemma 2.5 \( \text{SG}^+ \) is implementable. The following theorem can thus be established.

**Theorem 3.4.** Let \( \mathcal{I} = (P, l, M) \) and assume that \( \forall x, y \in l, x \neq y \Rightarrow P_x \not\subseteq P_y \). Then, \( \text{SG}^+ \) is strongly feasible and equivalence-robust for \( \mathcal{I} \), and for every other strongly feasible and equivalence-robust fairness notion \( \mathcal{C} \), \( \text{SG}^+(\mathcal{I}) \subseteq \mathcal{C}(\mathcal{I}) \).

To illustrate \( \text{SG}^+ \), consider the following example taken from [1]:

\[
\begin{align*}
p_1 :& b_1 := \text{true}; \\
& \ast [b_1; p_2 ! 0 \rightarrow b_1 := \text{false}] \\
p_2 :& b_2 := \text{true}; \\
& \ast [b_2; p_1 ? x \rightarrow b_2 := \text{false} \\
& \quad \Box b_2; p_3 ? x \rightarrow \text{skip}]; \\
p_3 ! 0 \\
p_3 :& b_3 := \text{true}; \\
& \ast [b_3; p_2 ! 0 \rightarrow \text{skip} \\
& \quad \Box b_3; p_2 ? y \rightarrow b_3 := \text{false}]
\end{align*}
\]

In this system, \( p_1 \) and \( p_2 \) may interact, and \( p_2 \) and \( p_3 \) may interact. In particular, \( p_2 \) and \( p_3 \) may establish two possible interactions, one to deliver a value from \( p_3 \) to \( p_2 \) and the other in the opposite direction. Although the two interactions contain each other, the program does not allow them to be enabled simultaneously. So Theorem 3.4 can be applied to the system so that \( \text{SG}^+ \) is the strongest implementable and equivalence-robust fairness notion for the system. From the program, it can be seen that the system may not terminate as \( p_2 \) and \( p_3 \) may repeatedly establish an interaction forever. However, any run of the system satisfying \( \text{SG}^+ \) must terminate.

From Theorem 3.4, we can derive the following corollary.

**Corollary 3.5.** Let \( \mathcal{I} = (P, l, M) \) and assume that \( \forall x, y \in l, x \neq y \Rightarrow P_x \not\subseteq P_y \). Then for every strongly feasible fairness notion \( \mathcal{C} \), the completion \( \mathcal{C}^+ \) defined by

\[
\mathcal{C}^+(\mathcal{I}) = \text{SG}^+(\mathcal{I}) \cup \{ \pi \in E \mid E \text{ is a purely } \mathcal{C} \text{-fair equivalence class in run}(\mathcal{I}) \}
\]

is the strongest implementable completion of \( \mathcal{C} \).
Proof. Since $\mathcal{C}$ is strongly feasible, by Lemma 2.3, $\text{SG}(\mathcal{I}) \subseteq \mathcal{C}(\mathcal{I})$. So every equivalence class in $\text{SG}^+ (\mathcal{I})$ must be purely $\mathcal{C}$-fair or $\mathcal{C}$-mixed. So by definition $\mathcal{C}^+$ is a completion of $\mathcal{C}$. Moreover, since $\text{SG}^+ (\mathcal{I}) \subseteq \mathcal{C}^+(\mathcal{I})$ and since $\text{SG}^+$ is strongly feasible, by Lemma 2.1, $\mathcal{C}^+$ is also strongly feasible. By Lemma 2.5, therefore, $\mathcal{C}^+$ is an implementable completion of $\mathcal{C}$.

To show that $\mathcal{C}^+$ is the strongest implementable completion of $\mathcal{C}$, let $\hat{\mathcal{C}}$ be any other implementable completion of $\mathcal{C}$. Then, by Theorem 3.4 and by the definition of completions, $\mathcal{C}^+(\mathcal{I}) \subseteq \hat{\mathcal{C}}(\mathcal{I})$. Hence, $\mathcal{C}^+$ is the strongest strongly feasible completion of $\mathcal{C}$.

Note that Theorem 3.4 does not depend on the arity of interactions, and so it holds as well if interactions are strictly bipartied. In particular, if every pair of processes share at most one interaction, then $\text{SG}^+$ is the strongest strongly feasible and equivalence-robust fairness notion one can get for bipartite interaction systems.

**Corollary 3.6.** For every $\mathcal{I} = (\mathcal{P}, l, M)$ such that $\forall x \in l, |P_x| = 2$ and $\forall x, y \in l, x \neq y \Rightarrow P_x \neq P_y$, $\text{SG}^+$ is the strongest fairness notion for $\mathcal{I}$ that is both strongly feasible and equivalence-robust.

### 3.3. Non-Strongly-Feasible Completions

On the other hand, if some interaction contains an interaction, then the strongest strongly feasible and equivalence-robust fairness notion may not exist. Before proving this, we first show that if $\mathcal{I}$ in addition contains at least two processes, then not all strongly feasible fairness notions of $\mathcal{I}$ have a strongest implementable completion. Note that for this we shall consider interaction systems with programs of type $M^\delta$. It can be seen that even if interactions may contain interactions, if the associated program $M$ guarantees that at any time no enabled interaction $x$ contains an interaction that is also enabled simultaneously, then $\text{SG}^+$ is still the strongest strongly feasible and equivalence-robust fairness notion for the system.

**Theorem 3.7.** Let $\mathcal{I} = (\mathcal{P}, l, M^\delta)$ be an interaction system satisfying the following conditions:

1. $|\mathcal{P}| > 1$ and
2. $\exists x, y \in l, x \neq y, P_x \subseteq P_y$, and $\forall z \in l, P_z \subseteq P_y \Rightarrow P_z \subseteq P_x$.

Then, there exists a fairness notion $\mathcal{C}$ which is strongly feasible but does not have a strongest strongly feasible completion.

**Proof.** Let $x, y \in l$ be two interactions satisfying condition (2). Consider first that $|P_y| \geq 2$. Let $p_1$ and $p_2$ be two arbitrary processes in $P_y$. Clearly, $\text{run}(\mathcal{I})$ contains runs of the form

$$p_1p_2(p_y - (p_1, p_2))z_1p_c_1z_2p_c_2z_3p_c_3\ldots$$

Let $E_x$ and $E_{\hat{\pi}}$ be two subsets of $\text{run}(\mathcal{I})$ defined by

$$E_x = \{ \pi \mid \pi \text{ is equivalent to some run of form (1) where } \forall i, z_i = x \}$$

$$E_{\hat{\pi}} = \{ \pi \mid \pi \text{ is equivalent to some run of form (1) where } \exists i, z_i \neq x \}.$$

The two sets are obviously not empty and disjoint. Note that in the presence of $p_1$ and $p_2$, each run $\pi$ in $E_x$ and $E_{\hat{\pi}}$ has at least one equivalent run different from $\pi$, i.e., $|eq(\pi)| \geq 2$ (because the first ready transitions of the two processes can be arbitrarily permuted).

Let $S$ be a nonblocking scheduler which selects an arbitrary enabled interaction for execution, except:

- If initially the adversary schedules the sequence of ready transitions $p_1p_2(p_y - (p_1, p_2))$, then $x$ will always be chosen for execution whenever it is enabled.
- If initially the adversary schedules the sequence of ready transitions $p_2p_1(p_y - (p_1, p_2))$, then $y$ will always be chosen for execution whenever it is enabled.

Define fairness notion $\mathcal{C}$ to be the following:

$$\mathcal{C}(\mathcal{I}) = \{ \pi \mid \text{there is an adversary } A \text{ of } \mathcal{I} \text{ such that } r(S, A) = \pi \}.$$
Then \( C \) is strongly feasible because \( S \) generates only \( C \)-fair runs. Observe that every run of the form 
\[ p_1 p_2 (P_x - (p_1, p_2)) \] 
belongs to \( C(\mathcal{I}S) \) but its equivalent run \( p_2 p_1 (P_x - (p_1, p_2)) \) does not. So every equivalence class \( eq(\pi) \subseteq E_x \) is \( C \)-mixed. (In fact, there is only one equivalence class in \( E_x \).) On the other hand, for every \( \pi \) of form (1) such that for some \( i, z_i \neq x, \pi \) must not belong to \( C(\mathcal{I}S) \) because no adversary versus \( S \) can generate \( \pi \). So every equivalence class \( eq(\pi) \subseteq E_x \) is either \( C \)-mixed or is purely \( C \)-unfair.

Recall from Lemma 2.1 that if \( C_1 \) is strongly feasible and \( C_1(\mathcal{I}S) \subseteq C_2(\mathcal{I}S) \), then \( C_2 \) must also be strongly feasible. From the above description of \( C \), it can be seen that there exists a strongly feasible fairness notion \( C' \) for \( \mathcal{I}S \) satisfying the following two conditions:

1. \( \forall \pi \in \text{run}(\mathcal{I}S) - (E_x \cup E_\delta), \pi \in C'(\mathcal{I}S) \), and
2. \( \forall \pi \in E_x \cup E_\delta, eq(\pi) \cap C'(\mathcal{I}S) \neq \emptyset \) and \( eq(\pi) \cap \overline{C'(\mathcal{I}S)} \neq \emptyset \).

For example, \( C' \) can be obtained by extending the above \( C \) to include every run in \( \text{run}(\mathcal{I}S) - (E_x \cup E_\delta) \) and including one run from every purely \( C \)-unfair class of \( E_x \).

Now, since each equivalence class \( eq(\pi) \subseteq E_x \cup E_\delta \) is \( C' \)-mixed, every completion \( \hat{C} \) of \( C' \) must decide the fairness of \( eq(\pi) \). Moreover, if \( \hat{C} \) is strongly feasible, then \( \hat{C} \) cannot treat all the \( eq(\pi) \)'s in \( E_x \cup E_\delta \) as unfair; otherwise, no nonblocking scheduler can generate a fair run if it faces the following adversary. Initially, the adversary schedules the sequence of partial run \( p_1 p_2 (P_y - (p_1, p_2)) \); subsequently, whichever interaction is chosen by the scheduler, the adversary in response simply schedules the processes of the interaction to be ready again.

We can define two completions \( \hat{C} \) and \( \hat{C}' \) which treats all equivalence classes in \( E_x \) as fair and those in \( E_\delta \) as unfair, and \( \hat{C}' \), which is defined in the other way. We argue that both are strongly feasible. To see this, consider first a nonblocking scheduler \( S \), which selects an arbitrary enabled interaction for execution, except that \( x \) must be chosen whenever it is enabled. Then for every run \( \pi \) generated by \( S \), if \( \pi \) is equivalent to some run of form (1), then all of the \( z_i \)'s must be instances of \( x \). This is because at any point in \( \pi \) only the processes in \( P_x \) can be ready. So for every \( z_i \), \( P_x \subseteq P_{z_i} \). By condition (2), \( x \) is enabled whenever \( z_i \) is. Since \( S \) prefers \( x \) to any other interaction containing \( x, z_i \), \( x \) can be ready. So for every run in \( \hat{C}' \), all of its strongly feasible completions \( \hat{C}' \) are instances of interactions such that \( \forall \pi \in \text{run}(\mathcal{I}S) - (E_x \cup E_\delta) \), \( \pi \in C'(\mathcal{I}S) \), and including one run from every purely \( C \)-unfair class of \( E_x \).

For the strong feasibility of \( \hat{C}' \) consider a nonblocking scheduler \( S \), which selects an arbitrary enabled interaction for execution, except that \( y \) must be chosen whenever \( y \) is enabled. Since every run in \( \hat{C}' \) contains a state in which \( y \) is enabled, no adversary versus \( S \) can generate a run in \( \hat{C}' \). So every run generated by \( S \) satisfies \( \hat{C}' \). Hence, \( \hat{C}' \) is also strongly feasible.

Observe that the two completions \( \hat{C} \) and \( \hat{C}' \) are incomparable. So neither of them can be the strongest implementable completion of \( C' \). Moreover, recall that for every strongly feasible completion \( \hat{C} \) of \( C' \), \( \hat{C}(\mathcal{I}S) \) must contain some equivalence classes in \( E_x \cup E_\delta \). So if \( \hat{C} \) contains all equivalence classes in \( E_x \cup E_\delta \), then \( \hat{C} \) must be weaker than \( \hat{C}' \) and \( \hat{C}' \); and if \( \hat{C}(\mathcal{I}S) \) contains only part of them, then \( \hat{C}' \) must be incomparable with either \( \hat{C}' \) or \( \hat{C}' \). Therefore, \( C' \) does not have a strongest strongly feasible completion.

In the above proof, we have assumed that \( |P_0| \geq 2 \). If \( |P_0| = 1 \), then \( P_x = P_y \). Since \( |P| > 1 \), either there exists an interaction \( u \) involving more than one process, or there exists two interactions \( u_1, u_2 \) such that \( |P_{u_1}| = |P_{u_2}| = 1 \) and \( P_{u_1} \neq P_{u_2} \). In the former case, we can modify form (1) to

\[ p_1 p_2 (P_u - (p_1, p_2)) u_1 \cdots u_k P_z z_1 z_2 z_3 P_z z_3 \cdots \]

where \( p_1 \) and \( p_2 \) are two arbitrary process in \( P_u \) and \( u_1, \ldots, u_k \) are instances of interactions such that no interaction is enabled immediately after \( u_k \). In the latter case, we instead consider the form

\[ p_1 u_1 p_2 u_2 P_y z_1 z_2 P_z z_3 P_z z_3 \cdots \]

where \( P_{u_1} = \{p_1\} \) and \( P_{u_2} = \{p_2\} \). In either case, we can define \( E_x \) and \( E_\delta \) analogously and show that there exists a strongly feasible fairness notion \( C' \) such that all its strongly feasible completions intersect (a) both \( E_x \) and \( E_\delta \), (b) only \( E_x \), or (c) only \( E_\delta \). So \( C' \) does not have a strongest strongly feasible completion.
Note that in Theorem 3.7, if $P$ consists of only one process, then for any $\pi \in \text{run}(\mathcal{I})$, the equivalence class $eq(\pi)$ consists of $\pi$ itself. So for any fairness notion $\mathcal{C}$, there is only one completion, i.e., $\mathcal{C}$ itself. Therefore, it holds trivially that if $\mathcal{C}$ is strongly feasible for $\mathcal{I}$, then $\mathcal{C}$ is the strongest strongly feasible completion of $\mathcal{C}$.

We now show that if some interaction contains an interaction, then there may not exist a strongest fairness notion for $\mathcal{I}$ that is both strongly feasible and equivalence-robust.

**Theorem 3.8.** Let $\mathcal{I} = (P, I, M')$ be an interaction system satisfying the following condition:

$$\exists x, y \in I, x \neq y, P_x \subseteq P_y, \quad \text{and} \quad \forall z \in I, P_z \subseteq P_y \Rightarrow P_z \subseteq P_z.$$

Then, there does not exist a fairness notion $\mathcal{C}$ such that (1) $\mathcal{C}$ is strongly feasible and equivalence-robust and (2) $\mathcal{C}$ is the strongest fairness notion for $\mathcal{I}$ that satisfies Condition (1).

**Proof.** The proof is similar to that of Theorem 3.7, except that we do not need processes $p_1$ and $p_2$ to make some equivalence classes of $\text{run}(\mathcal{I})$ consist of more than one run. Let $E_x$ and $E_x$ be two subsets of $\text{run}(\mathcal{I})$ defined by

$$E_x = \{\pi \mid \pi \text{ is equivalent to some run of the form } P_y(x P_x)^w\},$$

$$E_x = \{\pi \mid \pi \text{ is equivalent to some run of the form } P_y z_1 P_y z_2 P_z \ldots \text{ where } \exists i, z_i \neq x\}.$$

Then, using an argument similar to Theorem 3.7, we can show that any strongly feasible and equivalence-robust fairness notion $\mathcal{C}$ for $\mathcal{I}$ must intersect either $E_x$ or $E_x$. Furthermore, there are two incomparable fairness notions (which are strongly feasible and equivalence-robust) $\mathcal{C}_x$ and $\mathcal{C}_x$ such that the following two conditions are satisfied: (1) $E_x \subseteq C_x(\mathcal{I})$ and $C_x(\mathcal{I}) \cap E_x = \emptyset$ and (2) $E_x \subseteq C_x(\mathcal{I})$ and $C_x(\mathcal{I}) \cap E_x = \emptyset$. All other strongly feasible and equivalence-robust fairness notions for $\mathcal{I}$ must be either weaker than $\mathcal{C}_x$ and $\mathcal{C}_x$ or incomparable with one of them. Therefore, the strongest strongly feasible and equivalence-robust fairness notion for $\mathcal{I}$ does not exist. $\blacksquare$

Note that like Theorem 3.4, both Theorems 3.7 and 3.8 hold as well if interactions are strictly bipartied.

To illustrate Theorem 3.8, let $\mathcal{I} = (\{p\}, \{x, y\}, M')$, where $P_x = P_y = \{p\}$. Let $F_x = \{(p x)^w\}$ and $F_y = \{(p y)^w\}$. Then both are strongly feasible and equivalence-robust (and so are implementable). However, the two fairness notions are incomparable. Note that each fairness notion has only one completion, i.e., itself. So each has a strongest implementable completion.

### 3.4. A Patch

The readers may have noticed that Theorem 3.8 alone is not enough to determine whether there exists a strongest implementable and equivalence-robust fairness notion for all $\mathcal{I} = (P, I, M')$ where some interaction contains an interaction. This is because Theorem 3.8 concerns only the case where some interaction $y \in I$ contains a minimal interaction $x$ (where an interaction $u$ is minimal if for every interaction $v$ contained in $u$, $P_v = P_u$) such that for all other minimal interactions $w$ contained in $y$, $P_w = P_y$. Clearly, $y$ may contain two minimal interactions $x$ and $z$ such that neither of them contains the other. As we shall see, these systems do have a strongest implementable and equivalence-robust fairness notion, which is, by no surprise, $SG^+$.

**Lemma 3.9.** Let $\mathcal{I} = (P, I, M')$ and assume that

$$\forall x, y \in I, x \neq y, P_x \subseteq P_y \Rightarrow \exists z, w \in I, P_z \subseteq P_y, P_w \subseteq P_y, P_z \not\subseteq P_w, \text{ and } P_w \not\subseteq P_z.$$

Then a run $\pi \in \text{run}(\mathcal{I})$ is equivalent to a singular run if it satisfies the following conditions:

1. No interaction containing an interaction is ever executed in $\pi$.
2. For every $x$ not containing any interaction, if $x$ is enabled infinitely often, then it is executed infinitely often.

Note that in Theorem 3.7, if $P$ consists of only one process, then for any $\pi \in \text{run}(\mathcal{I})$, the equivalence class $eq(\pi)$ consists of $\pi$ itself. So for any fairness notion $\mathcal{C}$, there is only one completion, i.e., $\mathcal{C}$ itself. Therefore, it holds trivially that if $\mathcal{C}$ is strongly feasible for $\mathcal{I}$, then $\mathcal{C}$ is the strongest strongly feasible completion of $\mathcal{C}$. 

We now show that if some interaction contains an interaction, then there may not exist a strongest fairness notion for $\mathcal{I}$ that is both strongly feasible and equivalence-robust.

**Theorem 3.8.** Let $\mathcal{I} = (P, I, M')$ be an interaction system satisfying the following condition:

$$\exists x, y \in I, x \neq y, P_x \subseteq P_y, \quad \text{and} \quad \forall z \in I, P_z \subseteq P_y \Rightarrow P_z \subseteq P_z.$$

Then, there does not exist a fairness notion $\mathcal{C}$ such that (1) $\mathcal{C}$ is strongly feasible and equivalence-robust and (2) $\mathcal{C}$ is the strongest fairness notion for $\mathcal{I}$ that satisfies Condition (1).

**Proof.** The proof is similar to that of Theorem 3.7, except that we do not need processes $p_1$ and $p_2$ to make some equivalence classes of $\text{run}(\mathcal{I})$ consist of more than one run. Let $E_x$ and $E_x$ be two subsets of $\text{run}(\mathcal{I})$ defined by

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Then, using an argument similar to Theorem 3.7, we can show that any strongly feasible and equivalence-robust fairness notion $\mathcal{C}$ for $\mathcal{I}$ must intersect either $E_x$ or $E_x$. Furthermore, there are two incomparable fairness notions (which are strongly feasible and equivalence-robust) $\mathcal{C}_x$ and $\mathcal{C}_x$ such that the following two conditions are satisfied: (1) $E_x \subseteq C_x(\mathcal{I})$ and $C_x(\mathcal{I}) \cap E_x = \emptyset$ and (2) $E_x \subseteq C_x(\mathcal{I})$ and $C_x(\mathcal{I}) \cap E_x = \emptyset$. All other strongly feasible and equivalence-robust fairness notions for $\mathcal{I}$ must be either weaker than $\mathcal{C}_x$ and $\mathcal{C}_x$ or incomparable with one of them. Therefore, the strongest strongly feasible and equivalence-robust fairness notion for $\mathcal{I}$ does not exist. $\blacksquare$

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To illustrate Theorem 3.8, let $\mathcal{I} = (\{p\}, \{x, y\}, M')$, where $P_x = P_y = \{p\}$. Let $F_x = \{(p x)^w\}$ and $F_y = \{(p y)^w\}$. Then both are strongly feasible and equivalence-robust (and so are implementable). However, the two fairness notions are incomparable. Note that each fairness notion has only one completion, i.e., itself. So each has a strongest implementable completion.

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**Lemma 3.9.** Let $\mathcal{I} = (P, I, M')$ and assume that

$$\forall x, y \in I, x \neq y, P_x \subseteq P_y \Rightarrow \exists z, w \in I, P_z \subseteq P_y, P_w \subseteq P_y, P_z \not\subseteq P_w, \text{ and } P_w \not\subseteq P_z.$$

Then a run $\pi \in \text{run}(\mathcal{I})$ is equivalent to a singular run if it satisfies the following conditions:

1. No interaction containing an interaction is ever executed in $\pi$.
2. For every $x$ not containing any interaction, if $x$ is enabled infinitely often, then it is executed infinitely often.
Proof. Let \( \pi \in \text{run}(\mathcal{I}) \) be a run given by

\[
\pi = p_1, I_1, p_1, I_1, \ldots, p_{1,n_1}, I_{1,n_1}, x_1, p_2, I_2, \ldots, p_{2,n_2}, I_{2,n_2}, x_2, \ldots,
\]

where \( x_1, x_2, \ldots \) are the sequence of interactions executed in \( \pi \). Assume that \( \pi \) satisfies conditions 1 and 2. We can use the method described in Lemma 3.3 to transform \( \pi \) into an equivalent run \( \pi_1 \) by moving each \( p_{1,j}, I_{1,j}, 1 \leq j \leq n_1 \), to the end of \( x_1 \), where \( p_{1,j}, I_{1,j} \) satisfies the following two conditions: (a) \( p_{1,j} \notin \mathcal{P}_x \), and (b) at some point in \( \pi \) (after the ready transition \( p_{1,j}, I_{1,j} \)) some interaction \( y \) involving \( p_{1,j} \) is enabled. By condition 1 of the lemma \( \pi_1 \) is singular up to \( x_1 \).

Like Lemma 3.3, the transformation can be done ad infinitum. Let \( \pi_\infty \) denote the resulting run. Unlike Lemma 3.3, however, some ready transition \( p_{k,j}, I_{k,j} \) in between \( x_{k-1} \) and \( x_k \) may be kept moving forever in the rest of the transformation from \( \pi_k \) to \( \pi_\infty \). (If no ready transition is moved indefinitely then \( \pi \equiv \pi_\infty \) and \( \pi_\infty \) is singular; hence we are done.) If this happens, then by the transformation some interaction \( y \) involving process \( p_{k,j} \) is enabled infinitely often but from \( x_k \) onward \( y \) is never executed. Note that by the conditions of the lemma, \( y \) must contain two interactions \( w \) and \( z \) such that \( P_w - P_z \neq \emptyset \), \( P_z - P_w \neq \emptyset \), and neither of them contains an interaction. Since both \( w \) and \( z \) are enabled infinitely often in \( \pi_\infty \), they are executed infinitely often in \( \pi \). Since they do not contain any interaction, by condition 2 they are executed infinitely often in \( \pi \).

We can then modify the transformation such that starting from \( \pi_k \) the ready transition \( p_{k,j}, I_{k,j} \) will not be moved in the rest of the transformation. Without loss of generality assume that no other ready transition is kept moving forever in the new transformation. (If there is one, then we can use the same method to freeze that transition too.) Let \( \pi'_\infty \) be the resulting run. It is clear that \( \pi \equiv \pi'_\infty \) because no ready transition is kept moving forever.

We claim that \( \pi'_\infty \) is singular. By the transformation, it suffices to show that \( y \) will never be enabled even if we stop moving \( p_{k,j}, I_{k,j} \) from \( x_k \) onward. For this, suppose otherwise \( y \) is enabled in some state \( s \) in \( \pi'_\infty \). So \( w \) and \( z \) are also enabled in \( s \). Recall that \( w \) and \( z \) are executed infinitely often in \( \pi \) (and thus in \( \pi'_\infty \)). Let \( u \) be the first interaction that is executed after \( s \). By condition 1, \( P_u \) cannot contain \( P_w \) and \( P_z \). So either \( P_w - P_u \neq \emptyset \) (when \( u \neq w \)) or \( P_z - P_u \neq \emptyset \) (when \( u \neq z \)). Since both \( w \) and \( z \) are executed infinitely often, by the transformation either the ready transitions by the processes in \( P_w - P_u \) or the ready transitions by the processes in \( P_z - P_u \) will be moved after \( s \). Hence, \( y \) cannot be enabled in \( s \); contradiction. The lemma is thus proven.

To illustrate Lemma 3.9, assume that \( \mathcal{I} = ([p_1, p_2, p_3], \{x_2, x_3, x_{23}, x_{123}\}, M') \), where \( P_{x_2} = \{p_2\}, P_{x_3} = \{p_3\}, P_{x_{23}} = \{p_2, p_3\}, \) and \( P_{x_{123}} = \{p_1, p_2, p_3\} \) (see Fig. 3c). Then the run

\[
\pi = p_1p_2p_3(x_2p_2x_2p_2x_3p_1)w
\]

can be transformed into

\[
\rho = p_1(p_2x_2p_2x_2p_3x_3)w
\]

which is singular and is equivalent to \( \pi \).

Theorem 10. Let \( \mathcal{I} = (P, l, M') \) and assume that

\[
\forall x, y \in l, x \neq y, P_x \subseteq P_y \Rightarrow \exists z, w \in l, P_z \subseteq P_y, P_w \subseteq P_y, P_z \not\subseteq P_w, \text{ and } P_w \not\subseteq P_z.
\]

Then, \( SG^+ \) is the strongest strongly feasible and equivalence-robust fairness notion for \( \mathcal{I} \).

Proof. By Lemma 2.3, for every strongly feasible and equivalence-robust fairness notion \( C, SG^+(\mathcal{I}) \subseteq C(\mathcal{I}) \). As by definition \( SG^+ \) is equivalence-robust, to prove the theorem it suffices to show that \( SG^+ \) is strongly feasible for \( \mathcal{I} \). Moreover, by Lemma 3.9, it suffices to show that there exists a nonblocking scheduler for \( \mathcal{I} \) satisfying conditions 1 and 2 of the lemma. Such a scheduler can be easily obtained by modifying the SIF-scheduler presented in Part I (Fig. 6) so that no interaction containing an interaction is ever selected for execution. ■
Figure 3 illustrates some instances of \( IS = (P, I, M^\forall) \) which permit some interactions to contain an interaction but have a strongest implementable and equivalence-robust fairness notion.

**Corollary 3.11.** Let \( IS = (P, I, M^\forall) \) and assume that

\[
\forall x, y \in I, x \neq y, P_x \subseteq P_y \Rightarrow \exists z, w \in I, P_z \subseteq P_y, P_z \not\subseteq P_w, \text{ and } P_w \not\subseteq P_z.
\]

Then for every strongly feasible fairness notion \( C \), the completion \( C^* \) defined by

\[
C^*(IS) = SG^+(IS) \cup \{ \pi \in E \mid E \text{ is a purely } C\text{-fair equivalence class in } \text{run}(IS) \}
\]

is the strongest implementable completion of \( C \).

### 4. COMPARISONS OF SG^+ WITH OTHER FAIRNESS NOTIONS

In this section we compare \( SG^+ \) with the following well-known fairness notions and their completions:

- **Strong interaction fairness (SIF):** An interaction that is infinitely often enabled is executed infinitely often.
- **Strong process fairness (SPF):** A process that is infinitely often ready for an enabled interaction engages in an interaction infinitely often.
- **Weak process fairness (WPF):** A process that is continuously ready for an enabled interaction (not necessarily the same interaction) will eventually engage in an interaction.
- **Weak interaction fairness (WIF):** An interaction that is continuously enabled will eventually be executed.

The comparison is intended to be comprehensive so that we know how these fairness notions differ for various systems. In particular, we shall divide the comparison into two subsections—one for systems involving strictly biparty interactions (à la CSP and Ada), and the other for those involving multiparty interactions of arbitrary arity. Recall that \( SG^+ \) is the strongest implementable and equivalence-robust fairness notion for systems where interactions cannot contain interactions. In the biparty case, an interaction \( x \) cannot be contained in another interaction \( y \) if \( P_x \not\subseteq P_y \). This means that if interaction names only serve to identify the participants, then \( SG^+ \) is the strongest implementable and equivalence-robust fairness notion for biparty interaction systems.\(^3\) We shall therefore use \( IB \) to denote a set of biparty interactions such that \( \forall x \in IB, |P_x| = 2 \) and \( \forall y \in IB - \{x\}, P_x \not\subseteq P_y \).

Recall from Lemma 2.2 that the above four fairness notions are all strongly feasible. Their equivalence-robustness is summarized in Table 1. It is clear that for every \( IS \) the following relation holds [1, 2]:

\[
\text{SIF}(IS) \subseteq \text{SPF}(IS) \subseteq \text{WPF}(IS) \subseteq \text{WIF}(IS).
\]

In particular, depending on the instances of \( IS \), the fairness notions may be identical or strictly different.

To study the structure of interactions that distinguishes these fairness notions and their minimal and maximal completions, we shall associate \( IS \) with a program of type \( M^\forall \).

\(^3\) In practice an interaction name usually identifies the set of participants, while the interaction body determines the content of communication, which can vary dynamically, and in some cases can even involve nondeterministic choices among guarded commands.
4.1. Biparty Interaction Systems

We will establish some lemmas that are useful in classifying the relationship between SG and SIF, SPF, WPF, WIF, and their minimal and maximal completions. We begin with the comparison of $SIF^–$, SIF, and $SIF^+$. By definition of completions, for every $\mathcal{I}$ we have $SIF^–(\mathcal{I}) \subseteq SIF(\mathcal{I}) \subseteq SIF^+(\mathcal{I})$. Since SIF is not equivalence-robust, there exists some $\mathcal{I}$ that distinguishes $SIF^–$, SIF, and $SIF^+$. The following lemma shows when they are distinct.

**Lemma 4.1.** For every $\mathcal{I} = (P, I, M^r)$, $SIF^–(\mathcal{I}) \subseteq SIF(\mathcal{I}) \subseteq SIF^+(\mathcal{I})$. In particular, $SIF^–(\mathcal{I}) \subseteq SIF(\mathcal{I}) \subseteq SIF^+(\mathcal{I})$ if $\exists x, y, z \in I, P_x \cap P_y \not\subseteq P_z$ and $P_x \cap P_z \not\subseteq P_y$.

**Proof.** To see the proper subset relation, consider the run

$$\pi = ((P_x - P_z) \cup P_y)(y P_z z P_y)^o.$$  

If only $y$ and $z$ are enabled infinitely often in $\pi$, then $\pi \in SIF(\mathcal{I})$. If some other interaction $w$ is enabled infinitely often, then due to the restriction imposed on $\mathcal{I}$, $w \neq x$ because $x$ is never enabled in $\pi$. So either $P_w \subseteq (P_x - P_z) \cup P_y$ or $P_w \not\subseteq (P_x - P_y) \cup P_z$. If $P_w \subseteq (P_x - P_z) \cup P_y$, then let

$$\pi' = ((P_x - P_z) \cup P_y)(w P_w y P_z z P_y)^o;$$

otherwise let

$$\pi' = ((P_x - P_z) \cup P_y)(w P_w z P_y)^o.$$  

In either case, if $\pi'$ is still not in SIF, then similarly there must be another interaction $u$ such that either $P_u \subseteq (P_x - P_z) \cup P_y$ or $P_u \not\subseteq (P_x - P_z) \cup P_y$ and $u$ is enabled infinitely often but is never executed. Then we can use the above method to obtain another run $\pi''$ such that $u$ is executed infinitely often in $\pi''$. So without loss of generality assume that $\pi \in SIF(\mathcal{I})$.

Consider the run

$$\rho = ((P_x - P_z) \cup P_y)((P_z - P_y) y (P_z \cap P_y) z P_y)^o.$$  

It is easy to see that $\rho \equiv \pi$. However, $\rho$ does not satisfy SIF because $x$ is now enabled infinitely often in $\rho$ but it is never executed. By the definition of minimal completions, $\pi \in SIF(\mathcal{I}) - SIF^–(\mathcal{I})$; and by the definition of maximal completions $\rho \in SIF^+(\mathcal{I}) - SIF(\mathcal{I})$.  

**Figures 4b and 4c illustrate some instances of $\mathcal{I}$ for which $SIF^–(\mathcal{I}) \subseteq SIF(\mathcal{I}) \subseteq SIF^+(\mathcal{I})$.**

**Lemma 4.2.** For every $\mathcal{I} = (P, I, M^r)$, $SIF(\mathcal{I}) \subseteq SPF(\mathcal{I})$. In particular, $SIF(\mathcal{I}) \subseteq SPF(\mathcal{I})$ if $\exists x, y, \ldots, z_0 \in I$ such that $\forall i \leq n, P_{y_i} \cap P_z \not\subseteq P_z$ and $\bigcup_{i} (P_{y_i} \cap P_z) = P_z$.

---

4 It can also be shown that $SIF^–(\mathcal{I}) \subseteq SIF(\mathcal{I}) \subseteq SIF^+(\mathcal{I})$ only if $\exists x, y, z \in I, P_x \cap P_y \not\subseteq P_z$ and $P_x \cap P_z \not\subseteq P_y$. Since we do not need this result in the main theorems of this section, we omit the proof.
of the lemma such that for all other interactions $x$ the run $\text{SPF}_C(j) \in IS(\text{SPF}_C(j))$ without loss of generality assume that $C$ shall see shortly, $\text{SIF}_C(\text{SPF}_C(j))$ and equivalence-robust. By Theorem 3.4, $\text{SG}_C(j)$ infinitely often, …

Moreover, since in the biparty case SPF is equivalence-robust, it follows that SPF, SPF$^+$, SIF$^+$, and SG$^+$ are all equivalent when interactions cannot contain interactions. Moreover, since in the biparty case SPF is equivalence-robust, it follows that SPF, SPF$^+$, SIF$^+$, and SG$^+$ are all equivalent.

**Lemma 4.3.** For every $\mathbb{IS} = (P, I, M)$, if $\forall x, y \in I, x \neq y \Rightarrow P_x \not\subseteq P_y$, then $\text{SIF}(\mathbb{IS}) \subseteq \text{SG}^+(\mathbb{IS})$, and $\text{SPF}(\mathbb{IS}) \subseteq \text{SG}^+(\mathbb{IS})$.

**Proof.** This follows directly from Lemma 3.3, and note that the proof of Lemma 3.3 can also be used to show that every run $\pi \in \text{SPF}(\mathbb{IS})$ is equivalent to a singular run. ■

**Lemma 4.4.** For every $\mathbb{IS} = (P, I, M)$, if $\forall x, y \in I, x \neq y \Rightarrow P_x \not\subseteq P_y$, then $\text{SIF}^+(\mathbb{IS}) = \text{SPF}^+(\mathbb{IS}) = \text{SG}^+(\mathbb{IS})$.

**Proof.** By Lemma 4.3 and the fact that SG$^+$ is equivalence-robust, SIF$^+(\mathbb{IS}) \subseteq \text{SG}^+(\mathbb{IS})$ and SPF$^+(\mathbb{IS}) \subseteq \text{SG}^+(\mathbb{IS})$. On the other hand, by Lemmas 2.2 and 2.1, SIF$^+$ and SPF$^+$ are strongly feasible and equivalence-robust. By Theorem 3.4, $\text{SG}^+(\mathbb{IS}) \subseteq \text{SIF}^+(\mathbb{IS})$ and $\text{SG}^+(\mathbb{IS}) \subseteq \text{SPF}^+(\mathbb{IS})$. Hence $\text{SIF}^+(\mathbb{IS}) = \text{SPF}^+(\mathbb{IS}) = \text{SG}^+(\mathbb{IS})$. ■

We now consider the notion of WPF and its completions.

**Lemma 4.5.** For every $\mathbb{IS} = (P, I, M^\prime)$, $\text{SPF}^-(\mathbb{IS}) \subseteq \text{WPF}^-(\mathbb{IS})$. In particular, $\text{SPF}^-(\mathbb{IS}) \subseteq \text{WPF}^-(\mathbb{IS})$ if $\exists x, y \in I$ such that (1) $P_x \cap P_y \neq \emptyset$, and (2) $\exists p \in P_x - P_y$ such that $\forall z \in I, P_z \subseteq P_x - P_y \Rightarrow p \not\in P_z$.

**Proof.** Since SPF$^-(\mathbb{IS}) \subseteq \text{WPF}(\mathbb{IS})$, SPF$^-(\mathbb{IS}) \subseteq \text{WPF}^-(\mathbb{IS})$. To see the proper subset condition, without loss of generality assume that $x$ and $y$ are two interactions satisfying conditions (1) and (2) of the lemma such that for all other interactions $x'$ and $y'$ satisfying the same conditions, $|P_x \cup P_y| \leq |P_{x'} \cup P_{y'}|$.

Consider the run

$$\pi = (P_x \cup P_y)(y P_y)^\omega.$$ 

Let $p$ be the process satisfying condition (2) of the lemma. Since $p \in P_x - P_y$ and since $x$ is enabled infinitely often, $\pi$ does not satisfy SPF. So $\pi$ does not satisfy SPF$^-$ either. Moreover, if some interaction

\[ \text{Fig. 4.} \text{ Instances of interaction systems } \mathbb{IS} = (P, I, M^\prime) \text{ for which } \text{SIF}(\mathbb{IS}) \subseteq \text{SPF}(\mathbb{IS}) \text{. Moreover, SIF is not equivalence-robust for } (b) \text{ and } (c). \]
\( z_1 \) is continuously enabled in \( \pi \), then \( P_{z_1} \subseteq P_x - P_y \). By condition (2) of the lemma, \( p \notin P_{z_1} \). Then in the run

\[
\pi' = (P_x \cup P_y)(y_1 P_{z_1} P_{z_1})^\omega.
\]

\( z_1 \) is not continuously enabled. Still, \( \pi' \) does not satisfy SPF because \( p \) is ready for an enabled interaction (i.e., \( x \)) infinitely often but it never engages in any interaction. So \( \pi' \) does not satisfy SPF\(^{-}\) either. If some other interaction \( z_2 \) is still continuously enabled in \( \pi' \), then \( P_{z_2} \subseteq P_x - P_y - P_{z_1} \), and \( p \notin P_{z_2} \). The above method can be used again to obtain a run \( \pi'' \) such that \( \pi'' \) does not satisfy SPF and \( z_2 \) is not continuously enabled. As there are only a finite number of interactions in \( \mathcal{I} \), without loss of generality let \( \rho = (P_x \cup P_y)(y_{z_2} P_{z_1} P_{z_1} P_{z_1} \ldots P_{z_1})^\omega \) be a run that does not satisfy SPF and SPF\(^{-}\) and that does not have an interaction that is continuously enabled. Note that by the construction, for every two different interactions \( a, b \) executed in \( \rho \), \( P_a \cap P_b = \emptyset \).

We argue that all runs in eq(\( \rho \)) satisfy WPF. To see this, suppose otherwise some run \( \psi \) in eq(\( \rho \)) violates WPF. Then, given that no interaction is continuously enabled in \( \rho \) (and thus in \( \psi \)), there must exist two interactions \( u_1 \) and \( v_1 \), where \( P_{u_1} \cap P_{v_1} \subseteq P_x - P_y \), such that (a) \( P_{u_1} \cap P_{v_1} \neq \emptyset \), (b) \( v_1 \) is executed infinitely often in \( \psi \), and (c) some process \( q \in P_{u_1} - P_{v_1} \) is continuously ready for an enabled interaction but it never executes any interaction. Moreover, immediately after \( v_1 \) is executed \( q \) must still be ready for an enabled interaction. Let \( u_2 \) be the smallest such interaction so that there is no interaction \( a \) such that \( P_a \subseteq P_{u_2} \). Because no interaction is continuously enabled in \( \rho \), \( u_2 \) will subsequently be disabled due to the execution of some interaction \( v_2 \). So \( P_{u_1} \cap P_{u_2} = \emptyset \) and \( P_{v_2} \cap P_{u_2} \neq \emptyset \). Since any two different interactions executed in \( \rho \) are disjoint, \( P_{u_1} \cap P_{v_1} = \emptyset \). Given that \( P_{u_1}, P_{u_2}, P_{v_1}, P_{v_2} \subseteq P_x \cup P_y \), we have \( P_{u_2} \cup P_{v_2} \subseteq P_x \cup P_y \).

However, because \( q \) is not involved in an interaction \( a \) such that \( P_a \subseteq P_{u_2} \), it is not involved in any interaction \( b \) such that \( P_b \subseteq P_{u_2} - P_{v_2} \). Then \( u_2 \) and \( v_2 \) satisfy the lemma conditions on \( \mathcal{I} \); but this then contradicts our earlier assumption that \( |P_{u_1} \cup P_{v_1}| \leq |P_{u_2} \cup P_{v_2}| \). Therefore, all runs in eq(\( \rho \)) satisfy WPF. Hence \( \rho \in \text{WPF}^{-}(\mathcal{I}) \). ■

Figure 5 depicts some instances of \( \mathcal{I} \) for which SPF\(^{-}(\mathcal{I}) \subseteq \text{WPF}^{-}(\mathcal{I}) \). The following lemma on the non-equivalence-robustness of WPF is somewhat complex.

**Lemma 4.6.** For every \( \mathcal{I} = (P, I, M^I) \), SPF\(^{-}(\mathcal{I}) \subseteq \text{WPF}(\mathcal{I}) \subseteq \text{WPF}^{+}(\mathcal{I}) \). In particular, SPF\(^{-}(\mathcal{I}) \subseteq \text{WPF}(\mathcal{I}) \subseteq \text{WPF}^{+}(\mathcal{I}) \) if \( \exists x_i, y_i \in I, \) where \( 0 \leq i \leq n - 1 \) and \( n > 1 \), and \( \exists p \in P \) such that (1) \( p \in \bigcap P_{x_i}, p \notin \bigcup P_{y_i} \) if \( \forall i, P_{y_i} \cap P_{x_i} \neq \emptyset \), \( P_{y_i} \cap P_{x_{i+1}} = \emptyset \), and (3) \( \forall u \in I, P_u \subseteq \bigcup P_{x_i} - \bigcup P_{y_i} \Rightarrow \exists \nu \in I, P_{x_i} \cup P_{y_i} = \emptyset, p \notin P_{x_i} \), and \( \exists i, P_{x_i} \cap P_{y_i} = \emptyset \). (Note that in the lemma additions and subtractions on indices of \( x \) and \( y \) are to be interpreted modulo \( n \)).

**Proof.** To see the proper subset conditions, consider the run

\[
\pi = \left( \bigcup P_{x_i} \cup \bigcup P_{y_i} \right) (y_0 P_{y_0} y_1 P_{y_1} \ldots y_{n-1} P_{y_{n-1}})^\omega.
\]

Observe that before each instance of \( y_j \) process \( p \) is ready for all \( x_i \)’s; and since \( P_{y_j} \cap P_{x_{j+1}} = \emptyset \), after the instance \( p \) is ready for at least \( x_{j+1} \) (which exists because \( n > 1 \)). So \( p \) is continuously ready for an enabled interaction (starting from the point the first interaction is to be executed). Since \( p \) never executes any interaction, \( \pi \notin \text{WPF}(\mathcal{I}) \). Now consider the run

\[
\rho = \left( \bigcup P_{x_i} - \bigcup P_{y_i} \right) (P_{y_0} y_0 P_{y_1} y_1 \ldots y_{n-1} P_{y_{n-1}})^\omega
\]

![FIG. 5. Instances of interaction systems $\mathcal{I} = (P, I, M^I)$ for which SPF\(^{-}(\mathcal{I}) \subseteq \text{WPF}^{-}(\mathcal{I})$.](image)
which is obtained from \( \pi \) by deferring for each \( j \) the readiness of \( \bigcup P_{y_j} - P_{y_j} \) before each instance of \( y_j \) until the instance is executed. Since the deferred actions are independent of the instance of \( y_j \), and since no action is deferred indefinitely, \( \rho \) is equivalent to \( \pi \). However, since \( \forall i . P_{y_i} \cap P_{x_i} \neq \emptyset \), right after each instance of \( y_j \) none of the \( x_i \)'s is enabled. So the \( x_i \)'s can no longer cause \( p \) to be continuously ready for an enabled interaction.

If \( \rho \in \text{WPF}(\mathcal{IS}) \) then we are done because \( \rho \in \text{WPF}(\mathcal{IS}) \), while \( \pi \in \text{WPF}^+(\mathcal{IS}) - \text{WPF}(\mathcal{IS}) \). If \( \rho \notin \text{WPF}(\mathcal{IS}) \), then some process is still continuously ready for an enabled interaction but it never engages in any interaction. Observe that after each instance of \( y_j \) only the set of processes \( \bigcup P_{x_i} - \bigcup P_{y_j} \) are ready for interaction. Moreover, the processes are continuously ready for interaction throughout \( \rho \). So if some process is continuously ready for an enabled interaction but never engages in any interaction, then the process must be continuously ready for the same interaction, say \( u \), and \( P_u \subseteq \bigcup P_{x_i} - \bigcup P_{y_j} \). By condition (3) there exists some \( v \) (where \( v \) could be \( u \)) and some \( k \) such that 

\[
\text{WPF}(\mathcal{IS}) \neq \emptyset, p \notin P_v, \text{ and } P_x \cap P_e = \emptyset.
\]

Let \( \pi' \) and \( \rho' \) be two equivalent runs given by

\[
\pi' = \left( \bigcup P_{x_i} \cup \bigcup P_{y_j} \cup P_v \right) \left( y_0 P_{y_0} y_1 P_{y_1} \cdots y_{n-1} P_{y_{n-1}} v P_v \right)^\omega
\]

\[
\rho' = \left( \bigcup P_{x_i} - \bigcup P_{y_j} - P_v \right) \left( y_0 P_{y_0} y_1 P_{y_1} \cdots y_{n-1} P_{y_{n-1}} v P_v \right)^\omega.
\]

Since \( P_{e} \cap P_{x_i} = \emptyset \), \( p \) is still continuously ready for an enabled interaction in \( \pi' \) even if \( v \) is executed infinitely often. So \( \pi' \notin \text{WPF}(\mathcal{IS}) \). Moreover, since \( P_{e} \cap P_{x_i} = \emptyset \), \( u \) is not continuously enabled in \( \rho' \). So either \( \rho' \notin \text{WPF}(\mathcal{IS}) \), in which case we are done, or there exists another \( u' \), \( P_{u'} \subseteq \bigcup P_{x_i} - \bigcup P_{y_j} - P_v \), such that \( u' \) is continuously enabled in \( \rho' \). In the latter case, we can apply the above method again to obtain two equivalent runs \( \pi'' \) and \( \rho'' \) such that \( \pi'' \notin \text{WPF}(\mathcal{IS}) \) and \( u' \) (and \( u \)) are not continuously enabled in \( \rho'' \). Given that there are only a finite number of interactions and a finite number of processes in \( \mathcal{IS} \), eventually we will establish the lemma.

Figure 6 depicts some instances of \( \mathcal{IS} \) for which \( \text{WPF}^{-}(\mathcal{IS}) \subseteq \text{WPF}(\mathcal{IS}) \subseteq \text{WPF}^{+}(\mathcal{IS}) \). Note that the non-equivalence-robustness must be intrigued by at least four interactions. In Fig. 6c, all the six interactions are needed in making \( p_1 \) be continuously ready for an enabled interaction but never engage in any interaction.

**Lemma 4.7.** For every \( \mathcal{IS} \), \( \text{WPF}^{+}(\mathcal{IS}) = \text{WIF}(\mathcal{IS}) \).

**Proof.** We shall show that for every \( \pi \in \text{run}(\mathcal{IS}) \) it is never the case that \( \text{eq}(\pi) \subseteq \text{WIF}(\mathcal{IS}) \) - \( \text{WPF}(\mathcal{IS}) \). Since both \( \text{WPF}^{+} \) and \( \text{WIF} \) are equivalence-robust, and since \( \text{WPF}(\mathcal{IS}) \subseteq \text{WPF}^{+}(\mathcal{IS}) \subseteq \text{WIF}(\mathcal{IS}) \), we therefore have \( \text{WPF}^{+}(\mathcal{IS}) = \text{WIF}(\mathcal{IS}) \).

Let \( \psi \) be any run in \( \text{WIF}(\mathcal{IS}) - \text{WPF}(\mathcal{IS}) \). Then, there must exist a process \( p \) such that from some point onward (say \( t_0 \)) \( p \) is continuously ready for an enabled interaction, but \( p \) never executes any interaction thereafter. Moreover, since \( \psi \) satisfies \( \text{WIF} \), \( p \) cannot be ready for the same interaction continuously.
So there exists an infinite number of points \( t_1, t_2, \ldots \) such that (1) \( p \) is continuously ready for some interaction \( x_i \) at the time between \( t_{i-1} \) and \( t_i \) (inclusive), (2) \( x_i \) becomes enabled at some point between \( t_{i-2} \) and \( t_{i-1} \), and is disabled at \( t_i \), and (3) \( x_i \neq x_{i+1} \) for each \( i > 0 \). Suppose that \( x_i \) is disabled due to the execution of some interaction \( y_i \). Since \( x_i \) remains enabled while \( y_i \) is executed, \( P x_i C_1 \) \( \subseteq \) \( P y_i D \). That is, the enabledness of \( x_i C_1 \) (due to the readiness of some processes in \( P x_i C_1 \)) is independent of the execution of \( y_i \).

Consider the run \( \hat{\psi} \) obtained from \( \psi \) by deferring, for each \( i \), the enabledness of \( x_i C_1 \) until \( y_i \) is executed. Then, \( \psi \equiv \psi' \). Note that the transformation from \( \psi \) to \( \psi' \) does not cause any new interaction to be enabled, nor does it extend the duration of an interaction’s enabledness. So the transformation cannot cause any new process to be continuously ready for an enabled interaction. However, for each \( i \), right after \( y_i \) is executed in \( \psi' \), \( p \) is not ready for \( x_i C_1 \) (and not ready for \( x_i \) either). If there exist infinitely many \( i \)’s such that \( p \) is not ready for any interaction immediately after each \( y_i \) is executed, then \( p \) is not continuously ready for an enabled interaction. Otherwise, there exists some \( i_0 \) such that for all \( i \geq i_0 \) there still exists another interaction \( x'_{i+1} \) which remains enabled right after \( y_i \) is executed. We can also use the above method again to break the overlap of the enabledness of \( x_i \) and \( x'_{i+1} \). Since there is only a finite number of interactions, we can obtain a run equivalent to \( \psi \) such that \( p \) is not continuously ready for an enabled interaction.

Similarly, if there is some other process \( q \) in \( \psi' \) (and thus in \( \psi \)) that is continuously ready for an enabled interaction, then we can use the same method again to transform \( \psi' \) into \( \psi'' \) so that \( q \) is not continuously ready for an enabled interaction. Since there are only a finite number of processes, we can transform \( \psi \) into an equivalent run satisfying WPF. Therefore, for every run \( \psi \in \text{WIF}(\mathcal{IS}) \cap \text{WPF}(\mathcal{IS}) \), \( \text{eq}(\psi) \cup \text{WPF}(\mathcal{IS}) \neq \emptyset \).

We have finished the comparison of \( \text{SG}^+ \) with the four fairness notions SIF, SPF, WPF, and WIF, and their minimal/maximal completions. The following theorem summarizes the results.

**Theorem 4.8.** Let \( \mathcal{IS} = (P, I^B, M^I) \) be a given biparty interaction system. Then the following relation holds:

\[
\text{SIF}^-(\mathcal{IS}) \subseteq \text{SIF}(\mathcal{IS}) \subseteq \text{SIF}^+(\mathcal{IS}) = \text{SG}^+(\mathcal{IS}) = \text{SPF}^-(\mathcal{IS}) = \text{SPF}(\mathcal{IS}) = \text{SPF}^+(\mathcal{IS}) \\
\subseteq \text{WPF}^-(\mathcal{IS}) \subseteq \text{WPF}(\mathcal{IS}) \subseteq \text{WPF}^+(\mathcal{IS}) = \text{WIF}(\mathcal{IS}).
\]

In particular, there exists an \( \mathcal{IS} \) for which all the stronger-than relations “\( \subseteq \)" become strict.

Figure 7a depicts the relationship between these fairness notions. In this figure \( A \rightarrow B \) means \( A \) is stronger than \( B \). The relation “\( \rightarrow\)” is transitive. Note that since \( \text{SG}^+ \) is the strongest implementable and

![Diagram of fairness notions](image-url)

**FIG. 7.** The hierarchy of various fairness notions when interactions cannot contain interactions. The fairness notions in boldface are implementable, while the others are not.
equivalence-robust fairness notion, all fairness notions weaker than $\text{SG}^+$ are also implementable, and all equivalence-robust fairness notions stronger than $\text{SG}^+$, e.g., $\text{SIF}^-$, are unimplementable. Moreover, although $\text{SIF}$ is not equivalence-robust, as we have seen in Part I, it is also unimplementable.

**Example.** Consider the Producers–Consumers problem, in which there are two producers $\text{producer}_1$ and $\text{producer}_2$ and two consumers $\text{consumer}_1$ and $\text{consumer}_2$. The data produced by a producer can be consumed by either of the consumers. The following is a CSP program for the problem, where $i = 1, 2$:

$$\text{producer}_i :: \text{compute(data)};$$
$$\star [\text{consumer}_1 ! \text{data} \rightarrow \text{compute(data)}]$$
$$\square \text{producer}_2 ! \text{data} \rightarrow \text{compute(data)}]$$

$$\text{consumer}_i :: \star [\text{producer}_1 ? \text{data} \rightarrow \text{digest(data)}]$$
$$\square \text{producer}_2 ? \text{data} \rightarrow \text{digest(data)}]$$

There are four biparty interactions in this program, each of which involves a producer and a consumer. They have the structure shown in Fig. 6a. So by our results in this section, the stronger-than relations (i.e., the arrow $\rightarrow$) in Fig. 7a are all strict; that is, for this problem

$$\text{SIF}^- (\text{IS}) \subset \text{SIF} (\text{IS}) \subset \text{SG}^+ (\text{IS}) \subset \text{WPF}^- (\text{IS}) \subset \text{WPF} (\text{IS}) \subset \text{WIF} (\text{IS}).$$

Therefore, any implementation of CSP’s input/output guards which guarantees only WIF cannot prevent the following behavior, which continuously blocks $\text{producer}_1$ from sending its data to either consumer and so does not satisfy WPF (although it does satisfy WIF):

- all processes are ready (for communication/interaction), and then the repeat of the following forever
- $\text{producer}_2$ sends data to $\text{consumer}_1$
- $\text{producer}_2$ and $\text{consumer}_1$ ready
- $\text{producer}_2$ sends data to $\text{consumer}_2$
- $\text{producer}_2$ and $\text{consumer}_2$ ready
- $\ldots$

Similarly, the following behavior which satisfies WIF and WPF but does not satisfy WPF$^-$ and SPF is also possible:

- all processes are ready, and then the repeat of the following forever
- $\text{producer}_2$ sends data to $\text{consumer}_1$
- $\text{producer}_2$ ready
- $\text{producer}_2$ sends data to $\text{consumer}_2$
- $\text{producer}_2$ ready
- $\text{consumer}_1$ and $\text{consumer}_2$ ready
- $\ldots$

In the absence of $\text{consumer}_1$ (say, it terminates prematurely), the following scenario which satisfies WIF, WPF, and WPF$^-$ but not SPF is also possible:

- all processes are ready, and then the repeat of the following forever
- $\text{producer}_2$ sends data to $\text{consumer}_2$
- $\text{producer}_2$ and $\text{consumer}_2$ ready
- $\ldots$

Since in the biparty case SPF (which is equivalent to $\text{SG}^+$) is implementable, a good implementation should be able to avoid all the above unfair scenarios. On the other hand, since SPF is also the strongest implementable and equivalence-robust fairness notion, the strongest equivalence-robust property one can observe from the program executing in any asynchronous environment is that no process is forever blocked from communicating with its partners if it has infinitely many such opportunities.
4.2. Multiparty Interaction Systems

We now consider multiparty interactions of arbitrary arity. Recall that in Theorem 4.8 the following relations hold as well even if interactions are multipartied (but cannot contain interactions):

1. SIF⁻(I$S$) $\subseteq$ SIF(I$S$) $\subseteq$ SIF⁺(I$S$) = SG⁺(I$S$) = SPF⁺(I$S$)
2. SIF(I$S$) $\subseteq$ SPF(I$S$)
3. SPF⁻(I$S$) $\subseteq$ WPF⁻(I$S$) $\subseteq$ SPF⁺(I$S$) $\subseteq$ WPF⁺(I$S$) = WIF(I$S$)
4. SPF⁺(I$S$) $\subseteq$ WPF⁺(I$S$).

Since SPF becomes non-equivalence-robust in the multiparty case, we need to resolve the relationship between its completions and the other fairness notions.

**Lemma 4.9.** For every I$S$ = (P, I, M$^\epsilon$), SPF⁻(I$S$) $\subseteq$ SPF(I$S$) $\subseteq$ SPF⁺(I$S$). In particular, SPF⁻(I$S$) $\subseteq$ SPF⁺(I$S$) if $\exists$ x, y, z $\in$ I, $\exists$ p$_1$, p$_2$, p$_3$ $\in$ P, such that (1) p$_1$ $\in$ P$^x$ $-$ P$^y$ $-$ P$^z$, p$_2$ $\in$ P$^z$ $-$ P$^y$, and p$_3$ $\in$ P$^y$ $-$ P$^z$; and (2) $\forall$ u $\in$ I, p$_1$ $\in$ P$^x$ $\Rightarrow$ (P$^u$ $\not\subseteq$ (P$^x$ $-$ P$^y$) $\cup$ P$^z$) and (P$^u$ $\not\subseteq$ (P$^x$ $-$ P$^y$) $\cup$ P$^z$).

**Proof.** To see the proper subset conditions, it suffices to find a run $\pi$ $\in$ SPF(I$S$) such that some run equivalent to $\pi$ does not satisfy SPF. Consider the run

$$\pi = ((P^x - P^z) \cup P^y)(yP^wzP^z)^\omega.$$

Due to the two conditions $p_2$ $\in$ P$^z$ $\cap$ P$^x$ $-$ P$^y$ and $p_3$ $\in$ P$^y$ $\cap$ P$^z$ $-$ P$^x$, x is never enabled in $\pi$. So if no other interaction is enabled infinitely often but is never executed, then $\pi$ $\in$ SPF⁺(I$S$). Moreover, the run

$$\rho = ((P^x - P^z) \cup P^y)((P^z - P^y) y (P^z \cap P^y) zP^z)^\omega$$

is equivalent to $\pi$ but x is now enabled just before each instance of y is to be executed. So $p_1$, which belongs to P$^z$, is now ready for an enabled interaction infinitely often. Since $p_1$ never executes any interaction, $\rho$ $\not\in$ SPF⁺(I$S$).

If $\pi$ $\not\in$ SPF⁺(I$S$), then some interaction w is enabled infinitely often but some process q in P$^w$ never executes any interaction. Then either P$^w$ $\subseteq$ (P$^x$ $-$ P$^y$) $\cup$ P$^z$ or P$^w$ $\subseteq$ (P$^x$ $-$ P$^z$) $\cup$ P$^y$. Due to condition (2) imposed on I$S$, $p_1$ $\not\in$ P$^w$. Assume that P$^w$ $\subseteq$ (P$^x$ $-$ P$^z$) $\cup$ P$^y$. (The other case can be proved similarly.) Then in the run

$$\pi' = ((P^x - P^z) \cup P^y)(yP^wzP^y)^\omega$$

q has executed w infinitely often. So either $\pi'$ $\in$ SPF⁺(I$S$), or similarly there exists another interaction u, $u \neq w$, such that u is enabled infinitely often but some process in P$^u$ never executes any interaction. In the former case, we can find a run $\rho'$ similar to $\rho$ such that $\rho' \equiv \pi'$ but $\rho'$ does not satisfy SPF because $p_1$ is infinitely often ready for an enabled interaction (i.e., x) but it never engages in any interaction. In the latter case, given that I and P are finite, we can use the above method repeatedly to find two equivalent runs such that one satisfies SPF while the other does not.

Figure 8 illustrates some instances of I$S$ for which SPF is not equivalence-robust. Note that all of them consists of a multiparty interaction involving more than two processes.

![Figure 8](https://example.com/fig8.png)
**Lemma 4.10.** For every $\mathcal{I} = (P, I, \mathbf{M}^I)$, $\text{SIF}^-(\mathcal{I}) \subseteq \text{SPF}^- (\mathcal{I})$. In particular, $\text{SIF}^-(\mathcal{I}) \subseteq \text{SPF}^- (\mathcal{I})$ if $\exists x, y_1, \ldots, y_n \in I$ such that $\forall i \leq n$, $P_{y_i} \cap P_x \neq \emptyset$ and $\bigcup_i (P_{y_i} \cap P_x) = P_x$.

**Proof.** Since $\text{SIF}^-(\mathcal{I}) \subseteq \text{SPF}^- (\mathcal{I})$, $\text{SIF}^- (\mathcal{I}) \subseteq \text{SPF}^- (\mathcal{I})$. To see the proper subset condition, recall the following run in Lemma 4.2,

$$\pi = (P_{y_1} \cup \cdots \cup P_{y_n})(y_1 P_{y_1} \cdots y_n P_{y_n})^\omega,$$

which is in $\text{SPF}^- (\mathcal{I}) - \text{SIF}^- (\mathcal{I})$. So $\pi \notin \text{SIF}^- (\mathcal{I})$ either. Since every process in $\pi$ executes some interaction $y_i$ infinitely often, $\text{eq}(\pi) \subseteq \text{SPF}^- (\mathcal{I})$. So $\pi \in \text{SPF}^- (\mathcal{I}) - \text{SIF}^- (\mathcal{I}).$}

Figure 4 also illustrates some instances of $\text{SIF}^- (\mathcal{I})$ for which $\text{SIF}^- (\mathcal{I}) \subseteq \text{SPF}^- (\mathcal{I})$.

**Lemma 4.11.** There exists a system $\mathcal{I} = (P, I, \mathbf{M}^I)$, where $\forall x, y \in I$, $x \neq y \Rightarrow P_x \not\subseteq P_y$, such that

$$\text{SIF}^-(\mathcal{I}) \not\subseteq \text{SPF}^- (\mathcal{I}) \quad \text{and} \quad \text{SPF}^- (\mathcal{I}) \not\subseteq \text{SIF}^- (\mathcal{I}).$$

**Proof.** Let $\mathcal{I}$ be the interaction system with $P$ and $I$ shown in Fig. 9. Consider the run

$$\pi_1 = p_5(p_1 p_3 x_{13} p_2 p_4 x_{24})^\omega.$$ 

Then $\pi_1 \in \text{SIF}^- (\mathcal{I})$. However, $\pi_1 \notin \text{SPF}^- (\mathcal{I})$ because its equivalent run

$$\pi_2 = p_5(p_1 p_3 p_2 p_4 x_{13} x_{24})^\omega$$

does not satisfy SPF due to the fact that $p_3$ is now ready for an enabled interaction (i.e., $x_{345}$) infinitely often but it never executes any interaction. So $\text{SIF}^- (\mathcal{I}) \not\subseteq \text{SPF}^- (\mathcal{I})$.

On the other hand, the run

$$\rho = (p_1 p_3 p_2 p_4 x_{13} x_{24})^\omega$$

and all of its equivalent runs satisfy SPF, and so they also satisfy SPF$^-$. However, $\rho \notin \text{SIF}^- (\mathcal{I})$ because $x_{12}$ is enabled infinitely often but is never executed. So $\text{SPF}^- (\mathcal{I}) \not\subseteq \text{SIF}^- (\mathcal{I})$.}

**Lemma 4.12.** There exists a system $\mathcal{I} = (P, I, \mathbf{M}^I)$, where $\forall x, y \in I$, $x \neq y \Rightarrow P_x \not\subseteq P_y$, such that

1. $\text{SG}^+(\mathcal{I}) \not\subseteq \text{WPF}^- (\mathcal{I})$ and $\text{WPF}^- (\mathcal{I}) \not\subseteq \text{SG}^+(\mathcal{I})$ and
2. $\text{SG}^+(\mathcal{I}) \not\subseteq \text{WPF}^- (\mathcal{I})$ and $\text{WPF}^- (\mathcal{I}) \not\subseteq \text{SG}^+(\mathcal{I})$.

**Proof.** Let $\mathcal{I}$ be the interaction system with $P$ and $I$ shown in Fig. 10. Consider the two runs

$$\pi_1 = p_5(p_1 p_3 x_{13} p_2 p_4 x_{24} p_6 p_8 x_{68} p_7 p_9 x_{79})^\omega$$

$$\pi_2 = p_5 p_1 p_3 p_2 p_4 p_6 p_8 p_7 p_9 (x_{13} p_1 p_3 x_{24} p_2 p_4 x_{68} p_6 p_8 x_{79} p_7 p_9)^\omega.$$ 

![FIG. 9. An interaction system for which \(\text{SIF}(\mathcal{I}) \not\subseteq \text{SPF}^- (\mathcal{I})\) and \(\text{SPF}^- (\mathcal{I}) \not\subseteq \text{SIF}^- (\mathcal{I})\).](image-url)
Observe that π₁ is singular and π₁ ≡ π₂. So π₂ ∈ SG⁺(I). However, π₂ /∈ WPF(I) because p₅ is continuously ready for an enabled interaction but p₅ never executes any interaction. So SG⁺(I) ⊈ WPF(I). Moreover, π₂ does not satisfy WPF⁺ either. So SG⁺(I) ⊈ WPF⁺(I).

On the other hand, the run

\[ \rho = p₃(p₁p₂x₁₂)^\omega \]

satisfies WPF. Moreover, all of its equivalent runs also satisfy WPF but none of them is singular. So ρ ∈ WPF⁺(I) and ρ /∈ SG⁺(I). Hence WPF⁺(I) ⊈ SG⁺(I), and WPF⁺(I) ⊈ SG⁺(I).

It can be seen that for Lemma 4.12 to hold, I must contain interactions involving more than two processes. (Recall that in the biparty case SG⁺(I) ⊈ WPF⁺(I) ⊈ WPF(I).)

**Lemma 4.13.** There exists a system I = (P, I, Mᵢ), where ∀x, y ∈ I, x ≠ y ⇒ Px ⊈ Py, such that

1. SPF(I) ⊈ WPF⁺(I) and WPF⁺(I) ⊈ SPF(I) and
2. SIF(I) ⊈ WPF⁺(I) and WPF⁺(I) ⊈ SIF(I).

**Proof.** The example presented in Lemma 4.12 can be used to establish the lemma; we omit the details.

We now summarize the results in the following theorem. A pictorial representation of the comparison is given in Fig. 7b. Recall that the stronger-than relation “→” is transitive. So two fairness notions are incomparable if there is no path connecting them.

**Theorem 4.14.** Let I = (P, I, Mᵢ) be a given multiparty interaction system such that ∀x, y ∈ I, x ≠ y ⇒ Px ⊈ Py. Then the following relations hold:

1. SIF(I) ⊆ SPF(I) ⊆ SPF⁺(I) ⊆ SG⁺(I) = SIF⁺(I) = SPF⁺(I) ⊆ WIF(I) = WPF⁺(I)
2. SIF⁺(I) ⊆ SPF⁺(I) ⊆ WPF⁺(I) ⊆ WPF(I) ⊆ WIF(I)
3. SPF⁻(I) ⊆ SPF⁻(I) ⊆ WIF(I) ⊆ WIF⁺(I) ⊆ WPF(I).

In particular, there exists an I for which all the above stronger-than relations “⊆” become strict. On the other hand, there exists some I such that the following relations hold:

1. SIF(I) is incomparable with SPF⁻(I)
2. SG⁺(I) is incomparable with WPF(I)
3. SIF(I), SPF(I), and SG⁺(I) are incomparable with WPF⁻(I).

Note that although WPF is incomparable with SG⁺, it is implementable for all interaction systems (see Part I). This, however, does not contradict Theorem 3.4 (that SG⁺ is the strongest implementable and equivalence-robust fairness notion) because WPF is not equivalence-robust. In fact, as we shall see in Section 5, there exists an implementable (but not equivalence-robust) fairness notion that is no
weaker than $SG^+$, but there is no strongest implementable fairness notion for virtually all interaction systems.

Moreover, $\text{WPF}^-$ is also incomparable with $SG^+$. Since $\text{WPF}^-$ is equivalence-robust, by Theorem 3.4, $\text{WPF}^-$ must not be strongly feasible. Indeed, $\text{WPF}^-$ does exclude some singular runs from some systems; see Lemma 4.12. Also noteworthy is that SPF becomes unimplementable in the multiparty case. By Lemma 2.6, $\text{SPF}^-$ is unimplementable too.

**Example.** Consider the Dining Philosophers problem. We can define a multiparty interaction $eating\_session_i$ involving the $i$th philosopher and its two neighboring chopsticks, where $0 \leq i \leq 4$.

The philosopher processes and the chopstick processes execute the following program:

\[
\begin{align*}
\text{philosopher}_i &:: [ \text{hungry}; eating\_session_i \rightarrow \text{thinking} ] \\
\text{chopstick}_i &:: [ eating\_session_i \rightarrow \text{clean\_chopstick} \quad \square eating\_session_{i-1\mod 5} \rightarrow \text{clean\_chopstick} ]
\end{align*}
\]

The interactions have the structure shown in Fig. 11, where $p_i$ and $c_j$ represent $\text{philosopher}_i$ and $\text{chopstick}_j$, respectively. The following is a possible scenario of the processes:

all processes are ready, and then the repeat of the following forever

\[
\begin{align*}
\text{philosopher}_3, \text{chopstick}_3, \text{chopstick}_4 & \text{ establish } eating\_session_3 \\
\text{philosopher}_1, \text{chopstick}_1, \text{chopstick}_2 & \text{ ready} \\
\text{philosopher}_0, \text{chopstick}_0, \text{chopstick}_1 & \text{ establish } eating\_session_0 \\
\text{philosopher}_0, \text{chopstick}_0, \text{chopstick}_1 & \text{ ready}
\end{align*}
\]

This scenario satisfies WIF, and so is possible if the underlying implementation of the multiparty interactions guarantees only WIF. It can also be seen that the scenario does not satisfy $\text{SPF}^+$. Since by our results $\text{SPF}^+$ is implementable, we know that such a scenario can be avoided by an appropriate implementation.

On the other hand, since $\text{SPF}^+$ is the strongest implementable and equivalence-robust fairness notion, the following scenario which satisfies $\text{SPF}^+$ but not $\text{SPF}$ cannot be excluded by any implementation ensuring equivalence-robust properties:

all processes are ready, and then the repeat of the following forever

\[
\begin{align*}
\text{philosopher}_3, \text{chopstick}_3, \text{chopstick}_4 & \text{ establish } eating\_session_3 \\
\text{philosopher}_1, \text{chopstick}_1, \text{chopstick}_2 & \text{ ready} \\
\text{philosopher}_0, \text{chopstick}_0, \text{chopstick}_1 & \text{ establish } eating\_session_0 \\
\text{philosopher}_0, \text{chopstick}_0, \text{chopstick}_1 & \text{ ready}
\end{align*}
\]

It is interesting to note that in this scenario two non-neighboring philosophers (i.e., $\text{philosopher}_2$ and $\text{philosopher}_4$) are blocked from entering $eating\_session$. So for this problem no implementation of the multiparty interactions can guarantee that at most one philosopher is starving.

**FIG. 11.** The interaction structure of the Dining Philosophers problem.
4.3. When Interactions May Contain Interactions

If \( I \) contains two interactions \( x, y \) such that \( P_x \subseteq P_y \), then in the biparty case the following relation (Lemma 4.4) no longer holds:

\[
SG^+(I) = SIF^+(I) = SPF^+(I).
\]

Instead, the three fairness notions have the new relationship

\[
SG^+(I) \subseteq SIF^+(I) \subseteq SPF^+(I).
\]

To see this, observe first that in general \( SG^+(I) \subseteq SIF^+(I) \subseteq SPF^+(I) \) because every singular run satisfies SIF and SPF as well. For the proper subset relation, the run

\[
\pi = P_y \cdot (yP_yxP_x)^\omega
\]

belongs to \( SIF^+(I) - SG^+(I) \) (assuming no interaction is contained in \( x \)), while

\[
\rho = P_y \cdot (yP_y)^\omega
\]

belongs to \( SPF^+(I) - SIF^+(I) \).

Moreover, in the above example \( \pi \) also belongs to \( SIF(I) - SG^+(I) \) and \( SIF^-(I) - SG^+(I) \). So SIF and SIF\(^-\) are no stronger than \( SG^+ \). But recall that when interactions cannot contain interactions, SIF and SIF\(^-\) are stronger than \( SG^+ \). Hence, in general, \( SG^+ \) is incomparable with SIF and SIF\(^-\). Figure 12a summarizes the relationship of the fairness notions for biparty interactions.

Note that \( SG^+ \) becomes unimplementable. In fact, \( SG^+ \) is not even strongly feasible. (Recall by Lemmas 3.1 and 2.3 that if \( SG^+ \) were strongly feasible, then \( SG^+ \) would be the strongest implementable and equivalence-robust fairness notion, thus contradicting Theorem 3.8.) This is because if a system contains two interactions \( x \) and \( y \) such that \( P_x \subseteq P_y \), then every time when \( y \) is enabled, \( x \) is enabled too (assuming a program of type \( M^\delta \)). So all nonblocking schedulers for the system will inevitably generate

\[
\text{FIG. 12. The hierarchy of various fairness notions—the general case where interactions may contain interactions. The fairness notions in boldface are implementable, while the others are not.}
\]
some runs which cannot be equivalent to a singular run. Furthermore, although SIF$^+$ is the strongest implementable and equivalence-robust fairness notion shown in Fig. 12a, by Theorem 3.8 we know that there exists another implementable and equivalence-robust fairness notion which is no weaker than SIF$^+$.

When interactions are multiparty, the relationship of the fairness notions is shown in Fig. 12b. Recall that in this case SPF$^-$ (IS) $\subseteq$ SPF (IS) $\subseteq$ SPF$^+$ (IS). SG$^+$ is incomparable with SPF and SPF$^-$, but is stronger than SPF$^+$. The fact that SG$^+$ is incomparable with SPF and SPF$^-$ can be observed by the similar reason behind the incomparability between SG$^+$ versus SIF and SIF$^-$. The fact that SG$^+$ is stronger than SPF$^+$ is because SG$^+$ is stronger than SIF$^+$, which is stronger than SPF$^+$.

Moreover, SIF$^+$ is incomparable with SPF and SPF$^-$, To see this, let $x$, $y \in 1$ be two interactions such that $P_x \subseteq P_y$. Then run $(P_x, y)^w$ belongs to SPF (IS) $-$ SIF$^+$ (IS) and SPF$^-$ (IS) $-$ SIF$^+$ (IS). So SPF and SPF$^-$ are no stronger than SIF$^+$. However, we have seen instances that SPF and SPF$^-$ are stronger than SIF$^+$ when interactions cannot contain interactions. So, in general, SPF and SPF$^-$ are incomparable with SIF$^+$.

Finally, although SG$^+$ in general is classified as unimplementable when some interaction $y$ contains an interaction $x$, from Theorems 3.8 and 3.10 we know that the unimplementability holds only in the case where all other interactions contained in $y$, if any, contain $x$. Moreover, in the cases where SG$^+$ is implementable, although SG$^+$ is identical to SIF$^+$ (and SPF$^+$) when interactions can not contain interactions (see Lemma 4.4), SG$^+$ may be strictly stronger than SIF$^+$ when interactions can contain interactions. For example, let $\mathbb{S} = (P = \{p_1, p_2\}, l = \{x_1, x_2, x_{12}\}, M')$ be a system shown in Fig. 3a, where $P_{x_1} = \{p_1\}$, $P_{x_2} = \{p_2\}$, and $P_{x_{12}} = \{p_1, p_2\}$. Then the run $(p_1 x_1 p_2 x_2 p_1 p_2 x_{12})^w$ belongs to SIF$^+$ (IS) $-$ SG$^+$ (IS).

4.4. Further Remarks

In the comparison of SG$^+$ and existing fairness notions, we have divided the results into two categories: one for systems that support only biparty interactions and the other for systems that allow multiparty interactions of arbitrary arity. This classification is based on the fact that some popular languages/models (e.g., CSP, Ada, and CCS) facilitate only biparty interactions. For each category, we have further divided the results into two subcategories, depending on whether interactions may contain interactions. We have seen examples which do not need interactions to contain an interaction (see Sections 4.1 and 4.2). As shown in Theorem 3.4 and Lemma 4.4, disallowing interactions to contain interactions ensures a strongest implementable and equivalent-robust fairness notion for the system, namely, SG$^+$, which is identical to the maximal completions of SIF and SPF (regardless of biparty or multiparty interactions). Therefore, all other equivalent-robust fairness notions are either weaker than SG$^+$ or are unimplementable.

On some applications, however, one may find it useful to allow interactions to contain interactions. For example, a process $p$ may choose to establish an interaction $x$ with $q$ and $r$, or an interaction $y$ with only $q$. In the biparty case, a process $p$ may choose to interact with $q$ from a set of different interactions so as to perform different actions. If one will, one could eliminate the need for containing interactions within an interaction by defining only one interaction for the largest set of participants and let different subsets of participants establish different actions within the interaction. In the above biparty example, we may define a single interaction for $p$ and $q$. Once the interaction is established, the two processes may negotiate with each other to decide which action to perform. Similarly, in the multiparty example we may replace $x$ and $y$ with an interaction $w$ involving $p, q$, and $r$. However, $r$ would become a “dummy” participant if only $p$ and $q$ interact within $w$. Since $r$ must always be involved in $w$, the new setting is more restrictive as $p$ and $q$ may interact regardless of $r$ in the original setting.

Perhaps the need for containing interactions within an interaction becomes more evident when one wishes to allow a choice between local actions and interactions. For example, consider a variant of the Producers–Consumers problem presented in Section 4.1, where each datum computed by a producer can be overwritten by a more up-to-date one if the target consumer is not yet ready for the data, and an old data can be “recycled” by a consumer if the producer cannot generate new data in time. Assuming only one producer and one consumer, then the following is a CSP program for the problem:
\begin{verbatim}
producer :: compute(data);
   * [ consumer ! data → skip
   □ compute(data) → skip ]
consumer :: * [ producer ? data → skip
   □ digest(data) → skip ]
\end{verbatim}

In this example each process, when ready for interaction with the other process, has a choice to perform a local action. As noted in the definition of our abstract model (see Section 2.1 of Part I), such a local action can be modeled by an interaction involving solely the process. Therefore, the system has an interaction structure shown in Fig. 3a. Note that although some interaction contain interactions, by Theorem 3.10, SG\(^+\) is still the strongest equivalent-robust fairness notion that can be implemented for the system. However, observe that the run

\[(producer \hspace{1em} compute \hspace{1em} consumer \hspace{1em} digest)\]

is a possible computation of the system (where perform and digest denote the local interactions by the processes) where the two processes repeatedly execute their local interactions forever. Since the run is singular, we see that no implementable equivalent-robust fairness notion can be enforced to ensure that the two processes will ever establish an interaction. It has been argued that nonuniform choice between local actions and interactions should be avoided to prevent stuttering—repetitions of a configuration in a computation \cite{1}. From the above example we see that any attempt to impose an implementable and equivalent-robust fairness notion to prevent stuttering is doomed to fail.

Finally, although in this section we have only considered SG\(^+\), SIF, SPF, WPF, and WIF, and their completions, based on our studies other fairness notions can be included in the fairness hierarchies as well. For example, U-fairness is incomparable with SIF\(^+\) in the hierarchies of Fig. 12. The fact that SIF\(^+\)(\(\mathcal{I}\)) \(\not\subseteq\) U(\(\mathcal{I}\)) can be seen by the fact that SIF and U-fairness are incomparable, and that U-fairness is equivalent-robust \cite{4}. The other direction can be illustrated by a system consists of two processes \(p_1\) and \(p_2\) and two interactions \(x\) and \(y\), where \(P_x = P_y = \{p_1, p_2\}\), such that the run \((p_1 p_2 x p_1 \{x\} p_2 \{x\} x)^\omega\) belongs to U(\(\mathcal{I}\)) \(-\) SIF\(^+\)(\(\mathcal{I}\)). Moreover, it can also be shown that SIF\(^-\) is stronger than U-fairness, as any run that violates U-fairness must be equivalent to a run violating SIF.

5. THE IMPOSSIBILITY OF A STRONGEST IMPLEMENTABLE FAIRNESS NOTION

We now determine the possibility/impossibility of a strongest implementable (but not necessarily equivalence-robust) fairness notion for various interaction systems. For this, it is useful to recall the fairness implementability criterion and the definition of indistinguishableness relation introduced in Part I. We have shown in Theorem 3.4 that if no interaction contains an interaction, then SG\(^+\) is the strongest fairness notion for \(\mathcal{I}\) that is both implementable and equivalence-robust. However, as we shall see shortly, SG\(^+\) is not the strongest implementable fairness notion for \(\mathcal{I}\), unless \(I\) consists of only one interaction. Note that when interactions may contain interactions, SG\(^+\), in general, is not strongly feasible and so is not implementable (see Section 4.3).

Since stronger fairness notions provide more liveness properties, one would wish to define a fairness notion as strong as possible while ensuring the implementability of the notion. Recall that every nonblocking scheduler \(S\) for \(\mathcal{I}\) must be able to generate all runs in SG(\(\mathcal{I}\)). According to the fairness implementability criterion and Lemma 2.3, for every implementable fairness notion \(C\) and every \(\pi \in \text{SG}(\mathcal{I})\), \(C(\mathcal{I})\) must contain indistinct(\(\pi\)). Thus, a potential candidate for the strongest implementable fairness notion is SG\(\text{INDISTINCT}(\mathcal{I})\), defined by

\[
\text{SG\(\text{INDISTINCT}(\mathcal{I})\) = \bigcup_{\pi \in \text{SG}(\mathcal{I})} \text{indistinct}(\pi).}
\]

However, it turns out that SG\(\text{INDISTINCT}(\mathcal{I})\) is not even strongly feasible. This holds even if interactions are bipartied and cannot contain interactions. In fact, as shown in the following two theorems, for every \(\mathcal{I}\) the strongest implementable fairness notion does not exist, unless \(\mathcal{I}\) consists of only one interaction.
We argue that there exists another implementable fairness notion \( C \) such that \( \mathcal{C}(\mathcal{I}) \subseteq \mathcal{C}'(\mathcal{I}) \).

**Proof.** Let \( x, y \) be any two distinct interactions in \( I \). Let \( R_x \) and \( R_y \) be defined by

\[
R_x = \{ \pi \in \text{run}(\mathcal{I}) \mid \pi \text{ begins with the form } (P_y - P_z) \cdot P_x x \}
\]

\[
R_y = \{ \pi \in \text{run}(\mathcal{I}) \mid \pi \text{ begins with the form } (P_y \cup P_z - P_w) \cdot P_w w \text{ for some } w \neq x \text{ and } P_w \subseteq P_x \cup P_y \}
\]

Clearly, both \( R_x \) and \( R_y \) are not empty.

Assume that \( C \) is an implementable fairness notion for \( \mathcal{I} \). By the fairness implementability criterion, there exists a nonblocking scheduler \( S \) such that for every run \( \pi \) generated by \( S \), \( \text{indistinct}(\pi) \subseteq \mathcal{C}(\mathcal{I}) \).

We argue that either \( \mathcal{C}(\mathcal{I}) \cap R_x \neq \emptyset \) or \( \mathcal{C}(\mathcal{I}) \cap R_y \neq \emptyset \). To see this, consider an adversary which begins by letting the processes in \( P_x \cup P_y \) be ready. If \( S \) in response schedules \( x \) for execution, then some run in \( \text{indistinct}(r(S, A)) \) begins with the form \( (P_y - P_z) \cdot P_x x \), and so \( \text{indistinct}(r(S, A)) \cap R_x \neq \emptyset \). Since \( \text{indistinct}(r(S, A)) \subseteq \mathcal{C}(\mathcal{I}), \mathcal{C}(\mathcal{I}) \cap R_x \neq \emptyset \). Similarly, if \( S \) instead schedules a different interaction, then \( \mathcal{C}(\mathcal{I}) \cap R_y \neq \emptyset \).

Suppose that \( \mathcal{C}(\mathcal{I}) \) intersects both \( R_x \) and \( R_y \). Consider the following scheduler \( S_x \):

\( S_x \) behaves like the nonblocking scheduler for SIF presented in Part I. In particular, if \( x \) is enabled initially, then \( x \) is chosen for execution first.

We claim that for every run \( \pi \) generated by \( S_x \), \( \text{indistinct}(\pi) \cap R_x = \emptyset \). This is because if the first interaction executed in \( \pi \) is \( x \), then \( \text{indistinct}(\pi) \cap R_x = \emptyset \) (because all runs in \( \text{indistinct}(\pi) \) have \( x \) as their first interaction). If the first interaction is \( w \) for some \( w \neq x \) and \( P_w \subseteq P_x \cup P_y \), then \( x \) must not be enabled before the instance of \( w \) is to be executed, for otherwise \( S_x \) would instead choose \( x \) as the first interaction. So \( P_x - P_w \neq \emptyset \). So \( \pi \) begins with \( Q \cdot w \) for some \( Q \) such that some process (say \( p \)) in \( P_x - P_w \) does not belong to \( Q \). Then no run in \( \text{indistinct}(\pi) \) can begin with the form \( (P_x \cup P_y - P_w) \cdot P_w w \) because the relation of indistinguishableness does not allow \( p \)'s ready transition to be moved ahead of any ready transition in \( Q \). So \( \text{indistinct}(\pi) \cap R_x = \emptyset \).

Define fairness notion \( \mathcal{C}_x \) to be

\[
\mathcal{C}_x(\mathcal{I}) = \{ \psi \in \text{indistinct}(\pi) \mid \pi \text{ can be generated by } S_x \}.
\]

Then, \( \mathcal{C}_x \) is also implementable, and \( \mathcal{C}_x(\mathcal{I}) \cap R_x = \emptyset \). So \( \mathcal{C}(\mathcal{I}) \subseteq \mathcal{C}_x(\mathcal{I}) \).

Note that since \( \mathcal{C}_x \) does not intersect \( R_y \), if \( \mathcal{C}(\mathcal{I}) \) intersects only \( R_x \), then clearly \( \mathcal{C}(\mathcal{I}) \subseteq \mathcal{C}_x(\mathcal{I}) \).

Finally, if \( \mathcal{C}(\mathcal{I}) \) intersects only \( R_y \), then since the role of \( x \) and \( w \) is essentially symmetric, we can use a similar argument to show that there exists another implementable fairness notion \( \mathcal{C}_y(\mathcal{I}) \) such that \( \mathcal{C}_y(\mathcal{I}) \cap R_y = \emptyset \). So \( \mathcal{C}(\mathcal{I}) \subseteq \mathcal{C}_y(\mathcal{I}) \).

**Theorem 5.2.** Let \( \mathcal{I} = (\mathcal{P}, I, M') \). If \( |I| = 1 \), then there is only one implementable fairness notion, i.e., \( \text{run}(\mathcal{I}) \) which equals \( \text{SG}^+(\mathcal{I}) \).

**Proof.** Straightforward.

Note that Theorem 5.1 does not depend on whether interactions are biparted or multiparted, nor does it depend on whether interactions may contain interactions. It holds as long as \( I \) contains more than one interaction. Therefore, for every \( \mathcal{I} \) for which \( \text{SG}^+ \) is implementable, there exists an implementable fairness notion which is no weaker than \( \text{SG}^+ \).

6. CONCLUSIONS

We have determined the system structure for which the strongest implementable completion of a given fairness notion exists. Moreover, for systems in which interactions do not contain interactions, we have obtained a fairness notion \( \text{SG}^+ \) which is the strongest implementable and equivalence-robust fairness notion one can get for these systems. We have also presented a comprehensive comparison of
SG\(^+\) with several commonly used fairness notions and their minimal and maximal completions. The results show that SG\(^+\) is identical to the maximal completions of SPF and SIF and is stronger than WIF. Since WIF is generally accepted as the only fairness criterion by many multiparty interaction implementations, our results indicate that we could exclude more “unfair” computations from these implementations (see the examples in Sections 4.1 and 4.2). Moreover, when interactions are CSP-like biparted, SG\(^+\) is also equivalent to SPF. Therefore, SPF is the strongest equivalence-robust property one can observe from a CSP-like program executing in any asynchronous environment. Finally, we have shown that in the absence of equivalence-robustness, it is in general impossible to define a strongest implementable fairness notion, unless there is only one interaction in the system. This implies plenty of leeway in the design of fairness notions suitable for various applications.

In studying the relationships between various fairness notions and their minimal and maximal completions, we often assumed a program of type \(M^v\) when we need to distinguish two fairness notions, where \(M^v\) allows a process, whenever it is ready for interaction, to be ready for all interactions of which it is a participant. The choice of \(M^v\) also allows us to observe the structure of interactions that may distinguish two fairness notions. Based on this analysis, one may also analyze how the relationships are affected by the semantics of \(M\) for any given IS = (P, I, M), where \(M\) is not limited to type \(M^v\). In this case, the relationships are determined not only by the structure of I but also by the condition whether the semantics of \(M\) allows the interactions to be enabled as required so as to distinguish two fairness notions.

The notion of liveness enhancement is introduced in [1] as another fairness criterion. It requires a fairness notion to allow some system to gain some liveness property which the system would not have without the additional fairness requirement. Program termination is typically used to evaluate this criterion. By the example presented after Theorem 3.4, we see that SG\(^+\) is also liveness enhancing.

Although we have not explicitly presented any scheduling algorithm to implement SG\(^+\), we can easily obtain one by using the method proposed in Part I. The method transforms a nonblocking scheduler to a coordinator process running concurrently with the existing processes of the system. By communicating with the existing processes, the coordinator determines, for each ready process, when and which interaction to execute. Note that the scheduling is essentially centralized as all interactions are established by the coordinator. It is therefore worth exploring a fully distributed solution for the problem, meaning that nonconflicting interactions can be established by different coordinators. That would then indicate that SG\(^+\) is the strongest equivalence-robust fairness notion that can be distributively implemented (provided that no interaction contains an interaction).

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