Bequest division and population growth: A lineal extinction probability approach

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Abstract. This study uses a *lineal-extinction-probability-minimization approach* proposed by Chu (1991) to study the relation between bequest division and population growth. Using a Markov branching process and “first-degree stochastic dominance” technique, our model’s results imply that any specific bequest division, primogeniture or equal divisions, does not necessarily increase or reduce the population growth by itself. Whether the population does grow, or by how much, will be determined by the concurrent social-economic conditions. We find that if all family heads faced an increasing MRTS technology and/or a relatively unfair market chance, then they would *tend* to choose an unequal division as their optimal bequest policy, and population growth would increase. However, if an equal division was adopted involuntarily by family heads or was enforced by laws, then the population growth *might* decline.

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1. Introduction

Most demographic economists agree with the notion that the rules of inheritance have certain impacts on population growth, though they do not agree

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with how. More than four decades ago, Habakkuk (1955) seemed to suggest that the equal division system would lead to a positive population growth and the single-heir system to a negative one. However he pointed out that historically both equal and unequal division rules of inheritance have been associated with population growth and decline.¹ Goody, Thirsk and Thompson (1979) also came to a conclusion similar to Habakkuk's.

Habakkuk's argument concerning bequest division and population growth for Napoleonic France placed emphasis on the *formation of a family*, and assumed that family heads have total control of births. Habakkuk argued that there are two opposing effects each inheritance rule could have on population growth. On the one hand, equal division of family property enables *all* the children to have better opportunities to marry and establish families encouraging population growth. On the other hand, a family head (maybe a peasant) may try to keep his property intact by limiting the number of children born lessening population growth. Single-heir inheritance makes it easier for the heir to prosper, and to raise many children, while making it difficult for non-heir children to do likewise. To measure the net impact that an inheritance rule has on population growth, we must be able to identify these two opposing effects.

We think Habakkuk's argument might not be an appropriate way to describe reality at that time for the following reasons. First, birth control techniques were not popularly available prior to the nineteenth century, and the parents' ability to control the number of births as a means to maintain the integrity of their property is questionable. Second, the number of children should not only be taken as a denominator for the parents' property. It might also be considered a product of the altruism that parents extend to children.² These inappropriate assumptions might have compromised some of Habakkuk's predictions. To counter such problems emerging from the "family-formation approach," this study uses a *minimization of lineal-extinction-probability approach* proposed by Chu (1991) to reexamine the relationship between inheritance rules and population growth.³

Several historical observations motivate us to use Chu's (1991) approach. First of all, in a pre-transitional society with a relatively high mortality rate, family heads might have to be concerned not only with their own well being, but also the certainty of their lineal succession when choosing means of bequest divisions. For instance, Adam Smith ([1776] 1937) proposed that primogeniture, in which one child received most of the bequest, was "introduced to preserve a certain lineal succession" (p. 362) of the inherited land. Cole and Wolf (1974, p. 176) also argued that primogeniture might have fulfilled the goal to "perpetuate one's name against the future of his children." Freedman (1966) and Nakane (1967) also pointed out that family line succession was viewed as a duty of family heads in traditional societies of both China and Japan.

Secondly, the minimization of the lineal-extinction-probability approach also has the characteristics featured in a "Malthusian" model where the survival probability increases with parental income. It is reasonable to assume that the head of a richer family is more capable of preserving his own prosperity, and providing his children with better chances to stay prosperous. Thus, the level of parental income might be thought to be positively related to the probability of lineal succession.⁴ Hence, the minimization of lineal extinction probability may serve as one of the objectives of bequest decisions in an earlier time or pre-industrial revolution economies.

In Chu's (1991) approach, an individual's inheritance and market luck would determine his income level and the income level would then in turn affect his lineal extinction probability. Chu defined two sources of uncertainty involved in the process of minimizing the lineal extinction probability. One is the *life risk*, which denoted the uncertainty regarding whether a child could successfully grow up as an adult. The other is the *income risk*, which denotes the income uncertainty after a child grew up. Given these two uncertainties, the family head could only *speculate* concerning the optimal bequest when minimizing the lineal extinction probability. Therefore, in Chu's model, the family head was required to be aware of the stochastic processes of the two uncertainties prior to the bequest decision being made.

Our model is designed to find the conditions needed for a bequest division to achieve the highest population growth, by using *first-degree stochastic dominance* as proposed by Hardar and Russell (1969). In our model we allow several income groups for each generation. We also specify intergenerational income mobility as a Markov branching process similar to that used by Chu (1990) and Chu and Koo (1995). Though Chu (1991) provided a projection regarding the relation between bequest division and population growth similar to this study's results, he did not prove it mathematically. Only a simulation was provided. In the current study we not only provide an analytical framework, but also include environmental factors in the model. Thus, we can achieve more specific predictions on the relationship between bequest division and population growth.

The results of our model show that if there were exogenous restrictions placed on inheritance divisions, then the population growth rate might be lower than that implied by an optimal bequest division. We also show that any specific bequest division, primogeniture or equal division, does not necessarily increase or reduce the population growth. Whether the population grows and by how much, would be determined by concurrent social-economic conditions, the shape of the income production function, the distribution of market luck, and the probabilities of raising children successfully to adulthood. Based on these findings, we are able to determine what division rules may lead to a higher population growth under these specific environmental conditions.

This study is organized as follows. In Sect. 2 we first specify a lineal extinction function and make some necessary assumptions. We then derive the optimal bequest division choice for a family head. Section 3 uses a stationary intergenerational income mobility system to derive a general relationship between the bequest division rules and the population growth in a steady state. In Sect. 4, by adding social-economic conditions to the model, we reduce our general model to a more specific one. Doing so allows us to focus on how a bequest division influences population growth. Finally, we present our concluding remarks in Sect. 5.

2. The basic model

We follow Chu's (1991) setting, and consider a one-sex overlapping generation model in which an individual lives either one or two periods. The first period is denoted as childhood and the second as adulthood. People who die in the first period cannot produce any children, and those who survive to the

second period are able to produce N children.⁵ How many of the N children survive to adulthood depends on the degree of parental care or the prior genetic endowment. We ignore the inherited genetic factors, and assume a positive relationship between the degree of parental care and the earning power of parents. Thus, for $0 \leq n \leq N$, let $p(n|y)$ be the probability that n out of N children survive to adulthood in a family with income y ,⁶ where $\underline{y} < y < \bar{y}$. Terms $\underline{y}$ and $\bar{y}$ denote respectively the lowest and the highest family income in the society. Clearly, $\sum_{n=0}^N p(n|y) = 1$.

The income of a family head ($\tilde{y}$) is determined by a production function of the bequest transfer (b) and market luck (α). The market luck α is a random variable which may represent an individual's fortune in operating his business. We assume that the production function of the bequest transfer and the probability distribution of α do not vary over time t . Furthermore, the distribution of market luck is independent of bequest transfer b .⁷ The marginal products of the bequest transfer and market luck are both positive. And the marginal product of market luck can increase the marginal product of the bequest transfer. That is, the relationships among income $\tilde{y}$, bequest transfer b , and market luck α can be described as follows:

$$\tilde{y} = f(b, \alpha), \quad f_b, f_\alpha > 0, \quad f_{b\alpha} \geq 0,$$

$$\text{where } f_b \equiv \frac{\partial f}{\partial b}, \quad f_\alpha \equiv \frac{\partial f}{\partial \alpha}, \quad f_{b\alpha} \equiv \frac{\partial^2 f}{\partial b \partial \alpha}.$$

The family income is defined as the sum of total family expenses E and bequests B .⁸ We let E be the sum of a family head's own consumption and his child-rearing expense, which is a function of income y and the number of surviving children n , namely, $E(y, n)$. Therefore,

$$y - E(y, n) = B(y, n),$$

where $B(y, n)$ represents the total bequests that an income- y parent gives to his n adult children.

Assume that family heads intend to minimize lineal extinction probability. Define the present as time 0, and let $q_t(y)$ be the probability that a family head with income y at time 0 has no surviving offspring at time t . We can construct a similar iterative equation to Chu's (1991) as follows:

$$q_{t+1}(y) = \left[1 - \sum_{n=1}^N p(n|y) \right] + \sum_{n=1}^N p(n|y) \cdot \min_{\mathbf{b} \in \Omega(y, n)} \left(\prod_{i=1}^n E_{\tilde{y}_i | b_i(y, n)} q_t(\tilde{y}_i) \right), \quad t \geq 1, \quad (1)$$

$$\text{and } q_1(y) = 1 - \sum_{n=1}^N p(n|y), \quad t = 0,$$

where $\tilde{y}_i = f(b_i(y, n), \alpha)$, $\Omega(y, n) = \{\mathbf{b} \in \mathbf{R}_+^n \mid \sum_{i=1}^n b_i(y, n) = B(y, n)\}$, where $b_i(y, n)$ denotes the bequest transfer of the i th child among n surviving children

whose parent has income y . Denote $H(\tilde{y}_i | b_i(y, n))$ as the conditional cumulative probability distribution of $\tilde{y}_i$ given $b_i(y, n)$. Let $E_{\tilde{y}_i | b_i(y, n)} q_t(\tilde{y}_i)$ be the i th child's conditional expected probability of having no surviving offspring t periods later given that i receives $b_i(y, n)$. Then,

$$E_{\tilde{y}_i | b_i(y, n)} q_t(\tilde{y}_i) = \int q_t(\tilde{y}_i) dH(\tilde{y}_i | b_i(y, n)).$$

Suppose that an income y family head has n adult (surviving) children. He would choose a bequest division rule to minimize the probability that all the n succession lines of his children would come to an end t periods later. Assuming the n adult children's lineal extinction probabilities are independent, then the term in the last parentheses of (1) represents the probability that all n succession lines vanish at time $t + 1$. Hence, we can divide the lineal extinction probability $q_{t+1}(y)$ into two terms. The first term $1 - \sum_{n=1}^N p(n|y)$ denotes the probability that the family head fails to bring up any children, i.e. $p(0|y)$. The second term denotes the probability that the family head has a number of children at period 1, but the children fail to have any surviving offspring t periods later.

Since the bequest transfer $b_i(y, n)$ is unrelated to the distribution of market luck α by our definition, $H(\cdot | b_i(y, n))$ would be affected by the distribution of α and the shape of the production function. We define $\psi(\cdot)$ and $\Psi(\cdot)$ as the p.d.f. and c.d.f. of α respectively, and denote the inverse function of f with respect to α as $f^{-1}(b, y)$; that is, $f(b, f^{-1}(b, y)) = y$. Since

$$H(x | b_i(y, n)) \equiv \Pr(\tilde{y}_i \leq x | b_i(y, n)) = \Pr(\alpha \leq f^{-1}(b_i(y, n), x)),$$

where $\underline{y} < x < \bar{y}$,

then the relationship between $H(\cdot | \cdot)$ and $\Psi(\cdot)$ would be

$$H(x | b_i(y, n)) = \Psi(f^{-1}(b_i(y, n), x)).$$

Before formally getting into our model, we make several important assumptions regarding the bequest division as follows:

A1. The total amount of inheritance remains the same, regardless of the bequest decision.

A2. The higher a family head's income is, the more the total bequest is; i.e. $B(y', n) > B(y, n)$, $\forall y' > y$.

A3. The higher a family head's income is, the more children there are that survive to adulthood. That is,

$$\sum_{n=0}^k p(n|y) \geq \sum_{n=0}^k p(n|y'), \quad \text{for } y' > y \text{ and } 0 \leq k \leq N.^9$$

A4. The more the bequest transfer is, the greater (upper cumulative) the probability is that a child earns a higher income; i.e. $H(x|b') \leq H(x|b)$, $\forall b' > b$.

Theorem 1. *If A1, A2, and A3 are satisfied, then $q_t(y)$ will be a decreasing function of y , regardless of the division rule adopted. That is, $\partial q_t(y)/\partial y < 0$, $\forall t$.*

Proof. See Appendix A.

From Theorem 1, it is more likely for family heads with higher income to bring up all their children to adulthood. Moreover, wealthy children would receive more inheritances than do those from poorer backgrounds; they would thus have more opportunities to be financially successful, and have more chances to *preserve the lineal succession*. Suppose everyone faced the same market uncertainty. Then the richer family heads' expected extinction probabilities would be lower than those of the poorer ones. Therefore, for any given time t , the lineal extinction probability would be negatively related to the family head's income.

We use the superscripts e and u to denote “equal” and “unequal” bequest divisions, respectively.¹⁰ Let $g_i^e(x, y; n)$ and $g_i^u(x, y; n)$ be the probabilities that the i th child among n surviving children of an income- y parent will earn income x , under the “equal” and “unequal” bequest divisions, respectively. Hence, we can define $g^j(x, y; n)$ as the *average mobility rate* at which the n surviving children with income- y parents can move to income- x group, under the j th division rule. That is,

$$\begin{aligned} g_i^e(x, y; n) &\equiv h\left(x \mid \sum_{i=1}^n \frac{1}{n} b_i(y, n)\right) = g^e(x, y; n), \\ g_i^u(x, y; n) &\equiv h(x \mid b_i(y, n)), \\ g^j(x, y; n) &\equiv \sum_{i=1}^n \frac{1}{n} g_i^j(x, y; n), \quad j = e, u. \end{aligned} \quad (2)$$

Similarly, we denote $q_t^e(y)$ and $q_t^u(y)$ as the lineal extinction probabilities under the “equal” and “unequal” bequest divisions at time t , respectively.

Definition. Let $f(\cdot)$, $g(\cdot)$ be two different probability density functions, and $F(\cdot)$, $G(\cdot)$ be their corresponding cumulative density functions. If $G(x) \leq F(x)$ for any x , then $g(x)$ first-degree stochastically dominates (F.S.D. hereafter) $f(x)$.

Lemma 1. (Hadar and Russell 1969) $\int h(x)g(x)dx \geq \int h(x)f(x)dx$ and $h'(x) > 0 \quad \forall x$ if and only if $g(x)$ F.S.D. $f(x)$. Otherwise, $\int h(x)g(x)dx \leq \int h(x)f(x)dx$ and $h'(x) < 0 \quad \forall x$ if and only if $g(x)$ F.S.D. $f(x)$.

Theorem 2. *If A1, A2, A3, and A4 were all satisfied, then $g^u(x, y; n)$ would F.S.D. $g^e(x, y; n) \forall n$, and $q_t^u(y) < q_t^e(y) \forall y$.¹¹*

Proof. See Appendix B.

Suppose that the lineal extinction probabilities will converge to steady-state $q^j(y)$, $j = e, u$. It is obvious that Theorem 2 would hold in a steady state, regardless of whether the bequest division is optimal or not.

Proposition 1. *If $g^u(x, y; n)$ F.S.D. $g^e(x, y; n)$ is satisfied for all n surviving children, then an income- y family head would choose the unequal bequest division as his optimal policy.*

From Proposition 1, if a family head's bequest-division decision is determined by the minimization of lineal extinction probability, then under the sufficient conditions of stochastic dominance, $g^u(x, y; n)$ would F.S.D. $g^e(x, y; n) \forall y, n$. All family heads in a society would adopt an unequal inheritance rule, primogeniture for example.¹² In such a case, there is a higher probability that the children would move to a higher income group. This would also decrease the expected extinction probability for the society as a whole.

3. Bequest division and population growth

Define $k^j(x, y)$ as the stationary transition rate for an average number of income- x children that an income- y parent has, under the j th division rule. That is,

$$k^j(x, y) = \sum_{n=1}^N p(n|y) \sum_{i=1}^n g_i^j(x, y; n), \quad j = e, u.$$

Letting $n(y)$ be the expected net reproduction number of an income- y parent,¹³ then

$$n(y) = \sum_{n=1}^N n \cdot p(n|y).$$

Given A3 and Lemma 1, a family head with a higher income would have a greater of expected net reproduction, that is $n(y') \geq n(y)$, $\forall y' > y$. We can denote $m^j(x, y)$ as the mobility rate of an income- y parent with income- x adult children. By definition, the sum of all probabilities over what the next generation would earn must be one, that is,

$$k^j(x, y) = n(y) \cdot m^j(x, y),$$

$$\int_{\underline{y}}^{\bar{y}} m^j(x, y) dx = 1, \quad j = e, u.$$

We denote $P_t(y)$ as the population with income y at time t . The inter-generational income stochastic transition can be viewed as a Markov branching process described as follows:

$$P_{t+1}(x) = \int k^j(x, y) \cdot P_t(y) dy. \quad (3)$$

Letting $P_t \equiv \int P_t(x) dx$, and dividing (3) by P_t , we obtain

$$\rho_{t+1} \cdot \mu_{t+1}(x) = \int k^j(x, y) \cdot \mu_t(y) dy, \quad (4)$$

where the population growth rate $\rho_{t+1} \equiv P_{t+1}/P_t$, and the proportion of income- y people in the total population $\mu_t(y) \equiv P_t(y)/P_t$. According to the Perron-Frobenius theory and the limit theorem,¹⁴ the population growth rate, distributions of income and reproductive value will converge to a dominant eigenvalue ρ , and the sequences $\mu(y)$ and $v(y)$, respectively, in a steady state.¹⁵

In a steady state, we therefore have

$$\rho^e \mu^e(x) = \int k^e(x, y) \mu^e(y) dy = \int n(y) m^e(x, y) \mu^e(y) dy, \quad (5)$$

$$\text{and } \rho^u \mu^u(x) = \int k^u(x, y) \mu^u(y) dy = \int n(y) m^u(x, y) \mu^u(y) dy. \quad (6)$$

Subtracting (5) from (6), we have

$$\begin{aligned} & (\rho^u - \rho^e) \mu^e(x) + \rho^u (\mu^u(x) - \mu^e(x)) \\ &= \int \mu^u(y) \cdot n(y) (m^u(x, y) - m^e(x, y)) dy \\ &+ \int (\mu^u(y) - \mu^e(y)) n(y) \cdot m^e(x, y) dy. \end{aligned} \quad (7)$$

Multiplying both sides of Eq. (7) by $v^e(x)$ and integrating it with respect to x , we would obtain a simplified Eq. (8) expressed as follows.

$$\begin{aligned} \rho^u - \rho^e &= \frac{\int \mu^u(y) n(y) \left[\int v^e(x) (m^u(x, y) - m^e(x, y)) dx \right] dy}{\int \mu^u(y) v^e(y) dy} \\ &= \frac{\int \mu^u(y) \sum_{n=1}^N np(n|y) \left[\int v^e(x) (g^u(x, y; n) - g^e(x, y; n)) dx \right] dy}{\int \mu^u(y) v^e(y) dy}. \end{aligned} \quad (8)$$

Since $\mu^u(y)$, $np(n|y)$ and $\int \mu^u(y) v^e(y) dy$ are all positive in (8), and $v^e(y)$ is intuitively an increasing function of y ,¹⁶ then whether ρ^u is greater or smaller than ρ^e would depend on the stochastic processes of $g^u(x, y; n)$ and $g^e(x, y; n)$. Following Lemma 1, Theorem 3 is deduced directly as follows.

Theorem 3. *From A1 through A4, if $g^u(x, y; n)$ F.S.D. $g^e(x, y; n) \forall y$ and $\forall n$, then $\rho^u > \rho^e$. If $g^e(x, y; n)$ F.S.D. $g^u(x, y; n) \forall y$ and $\forall n$, then $\rho^e > \rho^u$.*

Combining the arguments from Proposition 1 and Theorem 3, we could get Proposition 2.

Proposition 2. *If $g^u(x, y; n)$ F.S.D. $g^e(x, y; n)$ for all y and n , and all family heads have chosen the unequal bequest division, then there would be an increase in population growth and all family heads would choose the unequal division rules voluntarily as their optimal bequest division policy. Contrarily, if $g^e(x, y; n)$ F.S.D. $g^u(x, y; n)$ for all y and n , and all family heads have chosen the equal bequest division, then there would be also an increase in population growth. However, here we cannot mathematically prove whether all family heads would adopt the equal bequest divisions as their optimal policy.*

In Proposition 2 we show that the intergenerational income mobilities under equal and unequal divisions are linked to specific environmental conditions, and the optimal bequest division can be viewed as the one with the greatest high-income mobility rate. That is, the population would grow at a faster rate if a specific division increased the probability that people would move to higher-income groups. Hence, we cannot conclude whether equal or unequal division would lead to population growth without first referring to environmental variables. The path through which an inheritance system influences population growth is through the environmental variables which affect the mobility rate and the net reproduction function.¹⁷

Given that $g^u(x, y; n)$ F.S.D. $g^e(x, y; n)$ for all y and n , and all family heads adopted the unequal division rules, then the population growth would increase as stated in Proposition 2. If family heads adopted the equal division rule involuntarily, or when the equal division is enforced by laws, then the population growth would be retarded. We can draw a similar conclusion when $g^e(x, y; n)$ F.S.D. $g^u(x, y; n)$ for any y and n . Since every family head may face a different set of environmental variables, the optimal bequest division for each family should not be the same. Any specific bequest division enforced by laws or customs would not be optimal for *all* families, and it may retard the population growth. It is clearly shown that this effect is not induced by the rule itself, but by the differing exogenous variables each family faces.

4. The environmental factors, bequest division and population growth

In this section we will further analyze the relationship between each stochastic dominance and environmental conditions and provide some implications. The “exogenous environmental variables” considered in our work include the shapes of production function $f(b, \alpha)$ and the probability distribution of market luck $\psi(\alpha)$. Both depend on the conditions of the concurrent social system, customs and market structure. We first want to find out when $g^u(x, y; n)$ would F.S.D. $g^e(x, y; n)$ according to Propositions 1 and 2.

Lemma 2. *If $H(\cdot|b)$ is a concave function of b , then $g^u(x, y; n)$ would F.S.D. $g^e(x, y; n)$ according to (2). If $H(\cdot|b)$ is a convex function of b , then $g^e(x, y; n)$ would F.S.D. $g^u(x, y; n)$.*

Proof. See Appendix C.

According to Lemma 2, whether $g^u(x, y; n)$ would F.S.D. $g^e(x, y; n)$ or $g^e(x, y; n)$ would F.S.D. $g^u(x, y; n)$ depends on the concavity or convexity of $H(\cdot|b)$ with respect to b . Letting $\Phi(b) \equiv \Psi(f^{-1}(b, \cdot)) = H(\cdot|b)$, and $\text{MRTS}_{b\alpha}$ be the marginal rate of technological substitution of b for α , we can obtain the first- and second-order differentials of $\Phi(b)$ with respect to b as follows.¹⁸

$$\frac{\partial \Phi(b)}{\partial b} = -\psi \cdot \text{MRTS}_{b\alpha} = -\psi \cdot \left(\frac{f_b}{f_\alpha} \right), \quad (9)$$

$$\frac{\partial^2 \Phi(b)}{\partial b^2} = \psi \cdot \text{MRTS}_{b\alpha}^2 \left[-\frac{\partial \text{MRTS}_{b\alpha}}{\partial b} \cdot \frac{1}{\text{MRTS}_{b\alpha}^2} + \frac{\partial \psi}{\partial \alpha} \cdot \frac{1}{\psi} \right]. \quad (10)$$

Since both f_x and f_b are positive, $\partial\Phi(b)/\partial b$ is clearly negative. The cumulative density would decline if a decrease in α came with an increase in b .

For simplicity, we would characterize our concept of unequal division as only *marginally unequal* versus equal division, and so we must check the local convexity or concavity of $\Phi(b)$ at $\bar{b}$ ($\equiv \sum_{i=1}^n b_i/n$).¹⁹ If $\Phi(b)$ is a locally convex function of b , then equal division would provide the family head a better opportunity to preserve the lineal succession. On the other hand, the concave property of $\Phi(b)$ will make relatively unequal division better for the perpetuation of the family line.

From Eq. (10), we find that $-(\partial\text{MRTS}_{bx}/\partial b) \cdot (1/\text{MRTS}_{bx}^2)$ and $(\partial\psi/\partial\alpha) \cdot (1/\psi)$ are two key factors in deciding the local concavity or convexity of $\Phi(b)$. If MRTS_{bx} was a decreasing function of b , then the first term on the right-hand side of (10) would be positive, and the isoquant would be convex to the origin, and vice versa. If MRTS_{bx} increases in b ,²⁰ then other things being equal, $\Phi(b)$ would tend to be concave (or less convex); it is more likely that $g^u(x, y; n)$ would F.S.D. $g^e(x, y; n)$. This would encourage a family head to adopt an unequal division, to increase his children's average income level, and to reduce the lineal extinction probability, hence the population would grow. However, if some family heads who faced quasi-convex production technologies were forced to choose the equal division, then the population growth might decline. On the other hand, if all family heads faced quasi-concave income production functions, then it is more likely that $g^e(x, y; n)$ would F.S.D. $g^u(x, y; n)$, and an adoption of equal division would lead to a population growth, though we cannot conclude that the equal division is optimal.

We assume that a production function is separable and additive with respect to bequest transfer b and market luck α , for instance, $f(b, \alpha) = f(b) + \alpha$. If the production function was subject to increasing marginal returns in bequest transfer b , i.e. $f_{bb} > 0$, then a family head would tend to adopt an unequal division as his optimal policy. These implications are similar to those of Chu (1991) and Chu and Koo (1995).²¹

The value of $(\partial\psi/\partial\alpha) \cdot (1/\psi)$ depends on the market structure or the fairness of the social systems. Given a non-negative income assumption, market luck is specified as being distributed over $(0, \infty)$, so $\psi(\alpha)$ is a skewed-to-right distribution.²² Therefore, we can assume $\psi(\cdot)$ is a *normalized Gamma distribution*, and²³

$$\psi(z) = \frac{z^{\gamma-1} \cdot e^{-z}}{\Gamma(\gamma)},$$

$$\text{where } \Gamma(\gamma) = \int_0^{\infty} t^{\gamma-1} e^{-t} dt, \quad \gamma > 0, \quad z \geq 0.$$

According to the properties of moments, we can derive the skewness coefficient α_3 and the kurtosis coefficient α_4 as follows:

$$\alpha_3 = 2\gamma^{-1/2}, \quad \alpha_4 = 3 + 6\gamma^{-1}.$$

When γ approaches infinity, the Gamma distribution would approach a normal distribution asymptotically.²⁴ If a normal distribution of market luck was

characterized as a sign of a fair market, then the market chances to earn would become fairer with a larger γ . Moreover, we can define

$$\frac{\partial \psi}{\partial \alpha} \cdot \frac{1}{\psi} = \left(\frac{\gamma - 1 - \alpha}{\alpha} \right). \quad (11)$$

From (11), we find that γ has a positive marginal effect on $(\partial \psi / \partial \alpha) \cdot (1/\psi)$, $\forall \alpha > 0$. That is, in a society with a fairer market chance, the average marginal density of market luck would be higher.

Here we would like to use our model to examine some of the arguments regarding bequest division and population growth made by previous researchers. Ho (1962) once proposed a possible link between primogeniture and positive population growth. He found that China had experienced a period of almost continuous population growth from the late fourteenth century until 1850, which coincided with imperial China having a system of civil service examination. Ho argued that the civil service examination opened up opportunities for children of low- and middle-income families to move up socially. Thus, the system contributed to the popularity of unequal distribution of parental bequests, which made family lines prosperous, and sustained the continuous population growth.

In our model the civil service examination would increase γ (market chances become fairer), as shown in (11). Given any market luck α , $(\partial \psi / \partial \alpha) \cdot (1/\psi)$ would also increase. Substituting (11) into (10), other things being equal, the sign of $\partial^2 \Phi(b) / \partial b^2$ would tend to be positive. If $\Phi(b)$ is a convex function of b , according to Propositions 1 and 2, then the condition for $g''(x, y; n)$ to F.S.D. $g^e(x, y; n)$ would be less likely to hold. If the unequal division had been a popular choice, then population growth might have declined, contrary to historical observation. Thus, the civil service examination per se might not be enough to lead to an increasing population growth. In our opinion, the reason that an unequal bequest division (primogeniture) led to an increase in population growth might be due to other environmental variables, such as the shapes of the production function. The civil service examination system might have created a fairer opportunity to move up socially, and in turn it encouraged the society's productivity. This might lead to an increase in the child-rearing ability for each family, and eventually increase population growth.

According to Proposition 2 and our previous analysis, we realize that if the production function is quasi-concave and/or the market opportunity is fair, then it is more likely that population growth would increase when an equal division is adopted. However, if laws enforce unequal division (e.g. primogeniture) instead, then population growth may be retarded. On the other hand, if the production function is quasi-convex and/or the market opportunity is unfair, then the population would increase when an unequal division was chosen, and the population would decline when an equal division was enforced.

Not all families would face the same set of environmental conditions. That is, some families may have $g^e(x, y; n)$ F.S.D. $g''(x, y; n)$, while others have $g''(x, y; n)$ F.S.D. $g^e(x, y; n)$. It is likely that population growth would decline if any specific bequest division was enforced. Hence, we cannot conclude for certain that any bequest division, equal or unequal, would have led to an increase in population growth. We can summarize the previous results as follows in Table 1.

Table 1. Environmental factors, the optimal bequest division and population growth

Environmental conditions	Tendency of stochastic dominance	Optimal division	Population growth
MRTS _{bz} decreases in b	$g^e(x, y; n)$	Uncertain	$\rho^e > \rho^u$
MRTS _{bz} increases in b	$g^u(x, y; n)$	Unequal	$\rho^e < \rho^u$
Market chance is fair (γ is larger)	$g^e(x, y; n)$	Uncertain	$\rho^e > \rho^u$
Market is unfair (γ is smaller)	$g^u(x, y; n)$	Unequal	$\rho^e < \rho^u$

5. Concluding remarks

The purpose of this study is to show that it is not the form of the bequest divisions, but rather whether the division is optimal for the concurrent environment which influences the population growth of a society. A specific bequest division, primogeniture or equal division, may either increase or reduce population growth. The population growth would be further determined by the concurrent environmental variables such as production functions and market luck. When all family heads face an increasing MRTS technology and/or a relatively unfair market chance, they would *tend* to choose an unequal division as their optimal policy, and the population growth would increase. However, if an equal division was adopted involuntarily by family heads or was enforced by laws, then the population growth *might* decline.

Concurrent social-economic and exogenous factors may have different impacts on different families when choosing the bequest system.²⁵ We believe that not all income-group parents would choose the same bequest division to be their optimal inheritance policy even in the same society. If families faced different environmental conditions,²⁶ then a specific type of division limited by laws may retard the population growth. Only when families are allowed to choose their optimal bequest divisions can an increasing population growth be assured.

In this study, for convenience, we ignore the fertility decision of a family head, and assume that those who have a higher income would have a higher expected net reproduction rate. In fact, there could be either a positive or negative relation between the number of births and the income level of a family. Though most people believe that the income level has a positive relationship with the family head's ability to raise children to adulthood, we recognize that the expected number of children successfully raised to adulthood may not be positively related to family income in the real world.²⁷ Next, we do not further extend the bequest transfer to some more generalized inter-generational transfers, for example, parental human capital investment on children. Moreover, this study does not look into the variety of the "unequal" bequest divisions. More specifically defined unequal bequest divisions, such as bequest division rules based on the children's age, children's sex, and children's abilities, for example, may have different impacts on population, and these will be left for future researches.²⁸

Appendix A

Proof of Theorem 1. From A2, we know that $B(y', n) > B(y, n)$, $\forall y' > y$. Suppose that the income- y' family head disposes his inheritance freely, then he

at least adopts the same division as the income- y family head does and gives away the remaining inheritance $B(y'n) - B(y, n)$ without any cost. Hence, the feasible bequest decision set of an income- y' family head should be greater than and include the set of an income- y family head. That is, $\Omega(y, n) \subseteq \Omega(y', n)$, $\forall n \in [0, N]$. Furthermore, we can apply A3, and obtain the following inequality:

$$\begin{aligned}
 q_t(y') &= p(0|y') + \sum_{n=1}^N p(n|y') \cdot \min_{\mathbf{b} \in \Omega(y', n)} \left(\prod_{i=1}^n E_{\tilde{y}_i | b_i(y', n)} q_t(\tilde{y}_i) \right) \\
 &\leq p(0|y') + \sum_{n=1}^N p(n|y') \cdot \min_{\mathbf{b} \in \Omega(y, n)} \left(\prod_{i=1}^n E_{\tilde{y}_i | b_i(y, n)} q_t(\tilde{y}_i) \right) \\
 &= Q(y) \cdot \sum_{n=0}^N p(n|y') + (1 - Q(y)) \cdot p(0|y') \\
 &\leq Q(y) + (1 - Q(y)) \cdot p(0|y) \\
 &= q_t(y),
 \end{aligned}$$

$$\text{where } Q(y) \equiv \min_{\mathbf{b} \in \Omega(y, n)} \left(\prod_{i=1}^n E_{\tilde{y}_i | b_i(y, n)} q_t(\tilde{y}_i) \right) < 1. \quad \square$$

Appendix B

Proof of Theorem 2. Using (1), the lineal extinction probabilities under equal and unequal division rules can be written as $q_t^e(y)$ and $q_t^u(y)$ and defined as follows.

$$\begin{aligned}
 q_t^e(y) &= \left[1 - \sum_{n=1}^N p(n|y) \right] + \sum_{n=1}^N p(n|y) \prod_{i=1}^n \int q_{t-1}^e(x) g^e(x, y; n) dx, \\
 q_t^u(y) &= \left[1 - \sum_{n=1}^N p(n|y) \right] + \sum_{n=1}^N p(n|y) \prod_{i=1}^n \int q_{t-1}^u(x) g_i^u(x, y; n) dx.
 \end{aligned}$$

Using mathematical induction, we first assume that $q_{t-1}^e(y) > q_{t-1}^u(y)$, $\forall y$. From the interpretation of (1), we find that: if an unequal division were more beneficial to the preservation of lineal succession, then for any n surviving children (i.e. $\forall p(n|y)$) it would satisfy

$$\prod_{i=1}^n \int q_{t-1}^e(x) g^e(x, y; n) dx > \prod_{i=1}^n \int q_{t-1}^u(x) g_i^u(x, y; n) dx. \quad (B.1)$$

Given that $q_{t-1}^u(y) < q_{t-1}^e(y)$, $\forall y$, and the assumption that $g^u(x, y; n)$ F.S.D.

$g^e(x, y; n)$, the extinction probability would decrease as the income level x increases, regardless of the equal or unequal division rules, so $\partial q_{t-1}^e(x)/\partial x < 0$ must be satisfied. According to Lemma 1, we apply the familiar inequality between the arithmetic and geometric means of n and obtain the following relation:

$$\begin{aligned} \prod_{i=1}^n \int q_{t-1}^u(x) g_i^u(x, y; n) dx &< \prod_{i=1}^n \int q_{t-1}^e(x) g_i^u(x, y; n) dx \\ &< \left(\sum_{i=1}^n \frac{1}{n} \int q_{t-1}^e(x) g_i^u(x, y; n) dx \right)^n \\ &= \prod_{i=1}^n \int q_{t-1}^e(x) g^u(x, y; n) dx \\ &< \prod_{i=1}^n \int q_{t-1}^e(x) g^e(x, y; n) dx. \quad \square \end{aligned}$$

Appendix C

Proof of Lemma 2. By the definitions of $g^e(x, y; n)$ and $g^u(x, y; n)$ from (2), $G^u(x, y; n) \leq (\geq) G^e(x, y; n)$, $\forall x$ must satisfy

$$\sum_{i=1}^n \frac{1}{n} H(x | b_i(y, n)) \leq (\geq) H\left(x | \sum_{i=1}^n \frac{1}{n} b_i(y, n)\right).$$

That is, $G^u(x, y; n) \leq (\geq) G^e(x, y; n)$ must depend on the concavity of $H(\cdot | b)$ with respect to b . $\square$

Endnotes

¹ In Habakkuk's (1955) paper, he stated that "French demographers generally argued that the provision of equal division in the Napoleonic Code tended to retard population. Friedrich List suggested otherwise that under the same application of the Napoleonic Code, the marriage and birth rate grew in Rhineland. Furthermore, in eighteenth-century Bavaria a division of properties was advocated specifically as a means of encouraging population growth, and most German writers lean towards the view that the single-heir system tended to restrain the birth rate."

² See Becker (1991). Becker proposed that parents might have altruistic incentives in raising their children.

³ Commenting on the relation between the bequest division and population growth under the rule of the Napoleonic Code, Chu (1991) provided an interpretation that was different from Habakkuk's. He proposed that the equal division provision retarded France's population growth, because it was not an optimal policy *chosen* by family heads, but was *imposed* by the Napoleonic Code. Since family heads had little discretion over how the heritage should be divided, they were unable to make optimal intergenerational transfers and fertility decisions, and this led to a population decline.

⁴ However, the income level might have a negative impact on fertility in certain developed countries or high-growth economies. Galor and Weil (1999) showed that the Malthusian rela-

tionship between income and population growth existed in Western Europe only from the fall of the Roman Empire until the industrial revolution, and disappeared after the middle of nineteenth century. Hence the Malthusian hypothesis might not hold in a more recent times.

- ⁵ For simplicity, we do not consider the fertility decision here. Hence, a family head would have the same number of births N regardless of the income level, and the income level would only affect the probabilities that children survive to adulthood.
- ⁶ We can also define y as “wealth” without altering the model’s merit.
- ⁷ We ignore the relationship between the distribution of market luck and bequest transfers for convenience. Loury (1981), Chu and Koo (1990) and Chu (1991) specified the variable of market luck as α_b , which implies that the distribution of market luck may be affected by b . This often occurs when the capital market is imperfect, especially for less-developed economies. Furthermore, we also exclude the cases of negative market luck α , that is, the probability density is defined as zero when $\alpha < 0$.
- ⁸ The bequests can be all forms of assets, for example, cash, securities, real estates, and other physical assets.
- ⁹ This assumption employs a similar concept to the first degree of stochastic dominance, which will be formally introduced in Lemma 1.
- ¹⁰ “Equal division” equally divides the total bequest among all children. On the other hand, the extreme case of “unequal division” is *primogeniture*, which is characterized as “one child receives most of the bequest.” For convenience, in this model we consider the unequal case as *marginally unequal*, rather than a “corner solution” like primogeniture.
- ¹¹ However, we cannot apply the same properties to prove mathematically the case in which $q'_t(y) < q''_t(y)$ as in Theorem 2. Hence, Theorem 2 can only be applied when the optimal division is unequal.
- ¹² If $H(x|b)$ is a strictly concave function of b , then primogeniture would be the (globally) optimal unequal division. We will discuss this in Sect. 5.
- ¹³ The “expected net reproduction number” takes into account both the effects of birth and death rates. Since we assume the same birth size N for all income groups, the expected net reproduction number would only depend on the distribution of $p(n|y)$, i.e. the survival functions.
- ¹⁴ According to the Perron-Frobenius theory, if the transition matrix $\mathbf{k} \equiv [k(x, y)]$ is *positively regular*, then there must be a dominant eigenvalue ρ in which $\rho > |\rho'|$, where ρ' is any other eigenvalue of $\mathbf{k}$. Corresponding to ρ , there are left and right eigenfunctions $v \gg 0$ and $\mu \gg 0$ satisfying the conditions:

$$\rho\mu(x) = \int k(x, y)\mu(y) dy,$$

$$\rho v(y) = \int k(x, y)v(x) dx,$$

$$\int \mu(x)v(x) dx = 1.$$

From the limit theorem, when $\rho > 1$, the sequence $\{\mu_t(x)\}_{t=1}^{\infty}$ will converge almost surely to $\{\mu(x)\}$. More details can be found in Mode (1971).

- ¹⁵ Fisher (1930) defined $v(y)$ as the “reproductive value,” which measures *the value of an individual as a seed for future population growth*.
- ¹⁶ Since $v^e(y)$ is defined as the *reproductive value*, from assumptions A2 through A4, then the richer a parent gets, the more surviving children he would have. Without considering the fertility decision, the reproduction rate would increase with the income level of the family heads. One can request a more detailed mathematical proofs of this footnote from the authors.
- ¹⁷ In this study the birth cohort size of any income-group parent is given as N , and the expected net reproduction rate should be positively related to income. On the other hand, if the number of births had a negative relation with the family’s income, then the conclusion of Proposition 2 would be reversed.
- ¹⁸ Assuming that (b, α) are the only two inputs of the production function, then given an income level x (equivalently, the isoquant phase), we could obtain

$$\text{MRTS}_{bx} = -\frac{\partial \alpha}{\partial b} = \frac{f_b}{f_x},$$

$$\frac{\partial \text{MRTS}_{bx}}{\partial b} = \frac{f_{bb}f_x^2 - 2f_{bx}f_bf_x + f_{xx}f_b^2}{f_x^3}.$$

Next, we substitute this into the second-order condition and get (10).

- ¹⁹ If $\Phi(b)$ is globally concave on b , then primogeniture would be the optimal bequest division.
- ²⁰ We understand that an increasing MRTS in b means isoquants are concave to the origin, and a corner solution is implied by this. The reason we include the case of increasing MRTS into our consideration is to make our model more complete though we understand the chance of it is remote. Our intention is to facilitate a model in which we are able to get an unequal bequest choice, such as primogeniture, which happens to be a corner solution characterized by a production function with an increasing MRTS.
- ²¹ Chu (1991) and Chu and Koo (1995) also specified market luck as an additive factor to the bequest, such as $f(b, \alpha) = f(b) + \alpha$. Therefore, we obtain $\text{MRTS}_{bx} = f'(b)$, $\partial \text{MRTS}_{bx} / \partial b = f''(b)$, and implicitly $f_{bx} = 0$. We only need to know whether the marginal return to bequest transfer versus income is increasing, decreasing, or constant before determining $\Phi(b)$ (see Lemma 2). However, the additive setting can only be seen as a special case rather than a general one. According to our observations, it is possible that $f_{bx} \neq 0$. For example, an increase in the market luck of an individual who receives a higher amount of bequest may yield higher income on the margin, that is, $f_{bx} > 0$. Therefore, we here use MRTS to expand our theory, instead of marginal return of bequest transfer, and this would be a more general case.
- ²² With an upper limit being infinity and a lower limit being a definite value, it is unlikely to see a left-skewed or symmetric distribution.
- ²³ See Chpt. 17 of Johnson and Kotz (1970). The generalized Gamma distribution can be written as:

$$\psi(z) = \frac{(z - \lambda)^{\gamma-1} \exp[-(z - \lambda)/\beta]}{\beta^\gamma \Gamma(\gamma)}, \quad \gamma, \beta > 0, \quad z > \lambda,$$

where $\beta = 1$ and $\lambda = 0$, under the normalized Gamma distribution.

- ²⁴ For a normal distribution, $\alpha_3 = 0$ and $\alpha_4 = 3$.
- ²⁵ For example, there may exist more than one critical income level satisfying Eq. (1), such that parents with income greater than one critical level would have a different optimal bequest division from the one with income below this critical level. Hence, the income level could be an important factor that causes families to choose different bequest division.
- ²⁶ For example, when the income production function and distribution of market luck are related to bequest transfer or parental income level, the condition $g^e(x, y; n)$ F.S.D. $g^u(x, y; n)$ may not hold for $\forall y$ and $\forall n$.
- ²⁷ Becker (1991) suggested that the relationship between fertility and income is not affirmatively negative or positive. He believed that the interaction between the quantity and quality of children is the most important reason why the effective price of children rearing rises with income. Alternatively, Ahuwalia (1976) argued that families with lower income would have a higher natural fertility rate in developing countries.
- ²⁸ Menchik (1980), Tomes (1981), Sheshinski and Weiss (1982), and Becker (1991) provided some explanations for the unequal divisions of bequests among children.

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