Analysing the c-minus-age strategy for life-cycle investing
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The \(c\)-minus-age strategy is a popular strategy for life-cycle investing. When applying the \(c\)-minus-age strategy, an investor first chooses an indirect preference parameter \(c\) and at age \(t\) will hold a percentage of \(c\) minus \(t\) in equity assets. In this article, we use a linear and a multiplicative mean-variance utility function to quantitatively analyse the term structure of the mean-variance tradeoffs implied by the \(c\)-minus-age strategy. We also provide an optimal procedure to determine \(c\), based on the two direct preference parameters, elicited from an investor, of a multiplicative mean-variance utility function.

I. Motivation and Background

The guideline of 100-minus-age strategy for life-cycle investing suggests that the percentage of a holding portfolio in equities for an investor at age \(t\) is equal to \(100 - t\) (e.g. Bodie and Crane, 1997). The general form for this guideline is \(c - t\), where \(c\) is a constant typically between 120 and 80. In fact, in practice many target maturity life-cycle funds implement the spirit of the \(c\)-minus-age strategy (e.g. Fidelity Freedom 2030). The portfolio managers of these type of funds adjust the asset allocation over time, with the portfolio consisting of less equities the nearer the target date. This type of fund has been popularly used as a default option in retirement plans. According to Holden and VanDerhei (2005), this default option can significantly improve the replacement rate of workers.

This popular strategy receives empirical support from the long-run predictability behaviour in equity. Fama and French (1988) find that stock returns exhibit strongly negative autocorrelation for holding periods over 1 year, implying mean-reverting behaviour of equity returns. Poterba and Summers (1988), employing variance ratio tests, and Bekaert and Hodrick (1992), Campbell (1996), Barberis (2000) and Campbell and Viceira (2005), using time-varying variance ratio statistics from vector autoregressive (VAR) models report the evidence that the variance of stock returns does not grow in proportional with the investment horizon, as predicted by the random walk model, but grows more slowly as investment horizon increases, implying negative serial correlation of stock returns in the long-run. This pattern of long-run predictability of stock returns implies the rejection of random walk hypothesis of stock returns and several rationales are provided. Asset pricing models with time-variation asset returns (e.g. Cox \textit{et al.}, 1985, among other financial economists), over-reaction hypothesis (De Bondt and Thaler, 1985) and fads hypothesis (Shiller, 1990) are among the theoretical explanations. In addition, in the portfolio construction, contrarian strategies are investigated to capture this mean-reverting behaviour of equity in the long-run (e.g. De Bondt and Thaler, 1985; Lakonishok \textit{et al.},

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determine this popular investing strategy and on how to best analysing the risk-return tradeoffs that are implied by no studies of quantitative analysis are available on (1992), which justifies that the young can invest the total wealth for the portfolio choice, it also receives a theoretical support from Bodie

Of particular implication of these findings is that the long-run predictability of equity implies that the variance of equity returns does not grow in proportion with the investment horizon and therefore, equity will appear relatively safe for young investors. By incorporating the human wealth into the total wealth for the portfolio choice, it also receives a theoretical support from Bodie et al. (1992), which justifies that the young can invest more in risky financial assets than the old. However, no studies of quantitative analysis are available on analysing the risk-return tradeoffs that are implied by this popular investing strategy and on how to best determine \( c \) for an investor based on the true preferences of that investor.

Some interesting results in portfolio choice can be obtained, both theoretically and empirically, by adopting a linear mean-variance utility function based on some restrictive assumptions on the utility forms or/and the return distributions (see, for example, Campbell and Viceira, 2002). In the presence of a risk-free asset, a linear mean-variance utility function is equal to the expected return of a holding portfolio minus a multiplication of a constant \((A/2)\) and the variance of the return on the holding portfolio, where \( A \) is the Arrow-Pratt coefficient of relative risk aversion (RRA). A linear mean-variance utility function was applied empirically by Friend and Blume (1975), Gressis et al. (1976), Siegel and Hoban (1982), Morin and Suarez (1983), Szpiro (1986) to estimate the coefficient of the RRA\((A)\). Jansen (1998) extends the linear mean-variance model with international capital market regulations and estimates the RRA and capital control effects based on the asset holdings of the German private sector. Using time-varying mean and variance estimations derived from a variant VAR model, the author finds that a reasonable degree of risk aversion and actual magnitude of return risk appear to be incompatible within the framework of mean-variance model.

One could generalize a linear mean-variance utility function into an additive mean-variance utility function, where an additive mean-variance utility function is a nonnegatively weighed average of the expected return (mean) utility and the variance utility. For example, Van Eaton and Conover (1998) investigate a special form for an investigation of time diversification with the mean utility being equal to the mean and the variance utility being equal to the SD raised to the power of a positive number. Several works are available (e.g., Selvanathan, 1987; Clement et al., 1997; Selvanathan and Selvanathan, 2005) on the procedures for testing the adequacy of an additive mean-variance utility function.

However, for most investors there are matching effects between mean and variance, and an additive mean-variance utility function fails to accommodate these effects. For an illustration of the matching effects, consider the example with two expected return levels, \( r_1 \) and \( r_2 \), and two variances, \( v_1 \) and \( v_2 \). The occurring of \( r_1 \) and \( r_2 \) has half and half chance and the occurring of \( v_1 \) and \( v_2 \) has also half and half chance. There are matching effects between mean and variance if and only if the lottery with a half chance of \((r_1, v_1)\) and a half chance of \((r_2, v_2)\) is not indifferent with the lottery with a half chance of \((r_1, v_2)\) and a half chance of \((r_2, v_1)\). This motivates us to generalize an additive mean-variance utility function into a multiplicative one, which can accommodate the matching effects.

A multiplicative mean-variance utility function is a nonlinear function of the mean utility and the variance utility consisting of a mean utility term, a variance utility term and a nonlinear term with the product of the mean utility and the variance utility. Such a mean-variance utility function is not only able to accommodate the matching effects but is
also simple enough for practical use. It should be noted that there is one preference parameter \( A \) associated with a linear mean-variance utility function, one preference parameter is associated with an additive mean-variance utility function and two preference parameters are associated with a multiplicative mean-variance utility function. In the case of all risky assets, the preference parameter \( k \) associated with a linear mean-variance utility function is an approximation to \( A \), the coefficient of the RRA of an investor. The preference parameter \( k_1 \), \( 0 \leq k_1 \leq 1 \), associated with an additive mean-variance utility function indicates the relative weight \( k_1 \) of the mean utility to the total utility and the relative weight \( 1 - k_1 \) of the variance utility to the total utility. Each \( k_i, 0 \leq k_i \leq 1, i = 1, 2, \) of the two preference parameters associated with a multiplicative mean-variance utility function, respectively, indicates the relative weight \( k_1 \) of the mean utility to the total utility and the relative weight \( k_2 \) of the variance utility to the total utility. Readers who are interested in a theoretical explanation of a general multiplicative utility function may refer to the work of Keeney and Raiffa (1976). The multiplicative utility functions have been applied to a wide variety of decision contexts. For example, Levy et al. (2003) used a multiplicative utility function to show that the optimal saving may be positive even at a negative rate of return, where the nonlinear term is the product of the utility of current consumption and the utility of the consumption one period ahead. Guerrero and Herrero (2005) provided conditions for modelling individual preferences for lotteries on health profiles by using a multiplicative utility function. 

In this article, we use a linear and a multiplicative mean-variance utility function to quantitatively analyse the term structure of the mean-variance tradeoffs implied by the c-minus-age strategy. We also provide an optimal procedure to determine \( c \) for the c-minus-age strategy based on the two preference parameters, \( k_1 \) and \( k_2 \), expressed by an investor, of a multiplicative mean-variance utility function.

II. Theoretical Model

To analyse the mean-variance tradeoffs implied by the c-minus-age strategy, we express the mean-variance efficient portfolio weight functions and efficient variance functions in terms of expected return levels. In addition, we express the expected return levels in terms of \( c \) and \( t \). These functions are all linked to preference parameters in a linear and a multiplicative utility functions.

Efficient portfolio weight functions and efficient variance function

Suppose that there are a set \( \{ X_i, i = 1, \ldots, n \} \) of \( n \) risky asset classes available for investing. For a given investment horizon, let \( r = (r_1, \ldots, r_n) \) denote the expected (mean) return vector, where \( r_i = E(X_i), i = 1, \ldots, n \) and let \( V = [v_{ij}] \) denote the \( n \times n \) symmetric covariance matrix of the \( n \) asset classes, where \( v_{ij} = \text{cov}(X_i, X_j) \) for \( i, j = 1, \ldots, n \) during this horizon. Without loss of generality, we assume that \( r_1 \leq \cdots \leq r_n \). Based on the classic portfolio model of Markowitz (1952), we can express the efficient portfolio weight function \( w^*_i (r) \) of each asset class \( i, i = 1, \ldots, n \) as a linear function of \( r \) and express the efficient variance function \( f(r) \) as a convexly quadratic function of \( r \), where we let

\[
w^*_i (r) = d_{1i} r + d_{0i}
\]

and

\[f(r) = a_2 r^2 + a_1 r + a_0\]

The formulas for computing the \( n \) pair of coefficients \((d_{0i}, d_{00})\), \( i = 1, \ldots, n \) and the three coefficients of \( a_2, a_1 \) and \( a_0 \) in terms of \( r = (r_1, \ldots, r_n) \) and \( V = [v_{ij}] \) can be written as follows. Let:

\[
\begin{align*}
(a) & \quad Y_i = X_i - X_1 - u_i (X_n - X_1), \quad i = 2, \ldots, n - 1 \\
& \quad Y_0 = X_1 + u_i (X_n - X_1); \\
(b) & \quad y_{ij} = \text{var}(Y_i), \quad y_{00} = \text{cov}(Y_0, Y_i), \quad i = 2, \ldots, n - 1, \\
& \quad y_{0j} = \text{cov}(Y_0, Y_j), \quad i < j, \quad i, j = 2, \ldots, n - 1 \text{ and } y_{00} = \text{var}(Y_0); \\
(c) & \quad b = (y_{20}, \ldots, y_{(n-1)0}) = b_r + b_0 \quad \text{(since } b \text{ is a linear function } r), \quad V = [v_{ij}], \quad i, j = 2, \ldots, n - 1, \\
& \quad d_i = -V^{-1} b_1 \quad \text{and} \quad d_0 = -V^{-1} b_0. 
\end{align*}
\]

We have that

\[
\begin{align*}
d_{1i} &= (d_{12}, d_{13}, \ldots, d_{1(n-1)}), \\
d_0 &= (d_{02}, d_{03}, \ldots, d_{0(n-1)}), \\
d_{11} &= -\sum_{i=2}^{n-1} [(r_{n} - r_{i})/(r_{n} - r_{i}) - \sum_{j=2}^{n-1} (r_{n} - r_{j})/(r_{n} - r_{j}) - \sum_{k=2}^{n-1} (r_{i} - r_{k})/(r_{i} - r_{k})] \\
d_{10} &= (r_{n} - r_{i})/(r_{n} - r_{i}) - \sum_{j=2}^{n-1} (r_{n} - r_{j})/(r_{n} - r_{j}) - \sum_{k=2}^{n-1} (r_{i} - r_{k})/(r_{i} - r_{k})] \\
d_{00} &= (r_{n} - r_{i})/(r_{n} - r_{i}) - \sum_{j=2}^{n-1} (r_{n} - r_{j})/(r_{n} - r_{j}) - \sum_{k=2}^{n-1} (r_{i} - r_{k})/(r_{i} - r_{k})] \\
since f(r) = (d_r + d_0)^t V(d_r + d_0)’, \quad \text{we have that} \quad a_2 = d_1 V d_1’, \quad a_1 = 2d_1 V d_0 \quad \text{and} \quad a_0 = d_0 V d_0, \quad \text{where superscript } t \text{ denotes transpose.}
\end{align*}
\]

Multiplicative mean-variance utility function

A multiplicative mean-variance utility function is as follows:

\[U(x, y) = k_1 U_1(x) + k_2 U_2(y) + (1 - k_1 - k_2) U_1(x) U_2(y)\]
where
\[ U_1(x) \text{ is the expected return utility function with } x \text{ being scaled to be in } [0, 1], \]
\[ U_2(y) \text{ is the variance utility function with } y \text{ being scaled to be in } [0, 1]. \]

By letting \[ x = (r - \zeta)/(\bar{r} - \zeta) \text{ and } y = (f(\bar{r}) - f(r))/(f(\tilde{r}) - f(\bar{r})), \] where \( \zeta \) is the return level associated with the global minimum variance of \( f(r) \), i.e. \( \zeta = -a_1/2a_2 \), and \( \tilde{r} \) is the best return level judiciously specified by a financial analyst or investor, we can scale \( x \) and \( y \) such that, \( U_1(0) = U_2(0) = U(0, 0) = 0 \). \( U_1(1) = U_2(1) = U(1, 1) = 1 \), \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \). In addition, it is shown in the Appendix that \( y = 1 - x^2 \). The two preference parameters \( k_1 \) and \( k_2 \) of an investor can be elicited as follows. Let the ‘paradise portfolio’ denote the portfolio with both its expected return and variance attaining the best level and the ‘evil portfolio’ with both its expected return and variance attaining the worst level. Also, let ‘paradise-evil portfolio’ denote the portfolio with its expected return attaining the best level and its variance attaining the worst level and let ‘evil-paradise portfolio’ denote the portfolio with its expected return attaining the worst level and its variance attaining the best level. We let alternative A0 denote a two-outcome lottery with the probability of the ‘paradise portfolio’ being \( p \) and the probability of the ‘evil portfolio’ being \( 1 - p \); let alternative A1 denote a sure outcome of the ‘paradise-evil portfolio’; let alternative A2 denote a sure outcome of the ‘evil-paradise portfolio’.

The preference probability \( p \) of an investor making alternatives A0 and A1 indifferent is his/her \( k_1 \) and the preference probability \( p \) making alternatives A0 and A2 indifferent is \( k_2 \). Intuitively, \( k_1 \) is the relative weight of the whole mean utility to the total utility, and \( k_2 \) is the relative weight of the whole variance utility to the total utility. We note that if \( k_1 + k_2 = 1 \), then the multiplicative mean-variance utility function reduces to an additive one.

**Determination of the specified expected return level by the c-minus-age strategy**

The \( c \)-minus-age strategy implies that, at a given age \( t \), the total percentage of all the efficient portfolio weights in equities is equal to \( c - t \). The \( n \) risky asset classes are partitioned into two categories, the category of equity asset class (denoted by \( S \)) and the category of fixed income asset class. Let \( D_1 = \sum_{i \in S} d_{1i} \) and \( D_0 = \sum_{i \in S} d_{0i} \) for every asset class \( i \in S \). Since \( D_1t + D_0 = c - t \) by the strategy, we have that \( r = (c - t - D_0)/D_1 \) and thus have the following relation on the scaled expected return level \( x \) (i.e. \( x = (r - \zeta)/(\bar{r} - \zeta) \)) determined by the \( c \)-minus-age strategy:

\[ x = \frac{(c - t - D_0)}{(\bar{r} - \zeta)D_1} \frac{r}{\bar{r} - \zeta} \]  

**III. Procedures for Determining the \( c \) Value**

We present an optimal procedure to determine \( c \) value in the \( c \)-minus-age strategy. Suppose that the beginning age and the ending age of the

Consider a linear mean-variance utility function \( U(r) = r - (k/2)f(r) \), where \( r \) is the expected return level, \( f(r) \) is the efficient variance function, i.e. \( f(r) = a_2r^2 + a_1r + a_0 \), and \( k \) is an approximation of the Arrow-Pratt RRA coefficient. We have, by the first-order-condition of \( U(r) \) with respect to \( r \) and by \( r = (c - t - D_0)/D_1 \), the relationship:

\[ \frac{1}{k} = \frac{a_2(c - t - D_0) + a_1}{D_1} \]  

We now consider a multiplicative mean-variance utility function \( U(x) \) with \( U(x) = k_1U_1(x) + k_2U_2(y) + (1 - k_1 - k_2)U_1(x)U_2(y) \), where \( y = 1 - x^2 \).

Given a scaled expected return level \( x \), we can obtain, by the first-order-condition of \( U(x) \) with respect to \( x \), the set of the preference parameters \( k_1 \) and \( k_2 \), which makes the maximum of \( U(x) \) be achieved at the given expected return level of \( x \), as follows:

\[ k_2 = e_1(x)k_1 + e_0(x) \]  

where

\[ U_{01} = \frac{\partial(U_1U_2)}{\partial x}, \quad U_{11} = \frac{\partial U_1}{\partial x}, \quad U_{21} = \frac{\partial U_2}{\partial x}, \]

\[ e_1(x) = \frac{U_{11}(x) - U_{01}(x)}{U_{01}(x) - U_{21}(x)} \]

\[ e_0(x) = \frac{U_{01}(x)}{U_{01}(x) - U_{21}(x)}. \]

In the case of a linear mean utility and a linear variance utility, we have that \( U_1(x) = x \) and \( U_2(y) = 1 - x^2 \). We now have that \( U(x) = k_1x + k_2(1 - x^2) + (1 - k_1 - k_2)x(1 - x^2) \), or alternatively, \( U(x) = (k_1 + k_2 - 1)x^3 - k_2x^2 + (1 - k_2)x + k_2 \), and have that \( k_2 = e_1(x)k_1 + e_0(x) \), where \( e_1(x) = (3x^2)/(-3x^2 + 2x + 1) \) and \( e_1(x) = (1 - 3x^2)/(-3x^2 + 2x + 1) \).
investment horizon for an investor is \( t_1 \) (say 30) and \( t_2 \) (say 65), respectively. The optimal procedure for determining the \( c \) value can be described as follows:

**Step 1:** elicit from the investor the preference parameters of \( k_1 \) and \( k_2 \) for each age \( t \) from \( t_1 \) to \( t_2 \).

**Step 2:** for each investment horizon facing at the age of \( t \), where \( t \in [t_1, t_2] \), we use (Equation 1) to estimate the expected return level specified by the \( c \)-minus-age strategy.

**Step 3:** for each investment horizon facing at the age of \( t \), where \( t \in [t_1, t_2] \), plug the corresponding \( k_1 \) elicited in Step 1 and the expected return level obtained in Step 2 into (Equation 3) to obtain an estimated value of \( k_2 \).

**Step 4:** the \( c \) value is determined by minimizing the total of the squared error of the elicited \( k_2 \) and the estimated \( k_2 \) summing across all the periods facing at the age of \( t \), where \( t \in [t_1, t_2] \).

**IV. Data, Numerical Example and Results**

In this article, all the numerical results are based on the following dataset. The indices for asset classes\(^1\) based on Fabozzi et al. (2002) come from Stocks, Bonds, Bills and Inflation published by Ibbotson Associates. Data include real total rates of return on US 30-day T-Bills for the asset class of US cash, Lehman Brothers Aggregate Bonds for US bonds, S&P 500 for US large-cap equity, Russell 2000 for US small-cap equity, MSCI EAFE for Europe/Japan equity and MSCI Pacific for emerging markets equity.\(^2\) The annual data from 1979 to 2004 is adopted in this article since the \( c \)-minus-age strategy for life-cycle investing implies a modification of the portfolio decision each year. Based on historical asset return data, the bootstrapping method is used for generating the returns of the future. For each resampling there are 500 times of sequential random selections with replacement for each of the investment horizons \( n \) year long, \( n = 1, 2, \ldots, 35 \), where each random selection is a selection of the year in the time span of the historical return data. Once a year is selected, the returns of all the indices associated with the year are sampled.

Figure 1 plots the \( k \) values of (Equation 2) of the linear mean-variance utility function which serves as an approximation of the coefficients of the RRA over the years-to-retirement from 1 to 35 for the \( c \)-minus-age strategy with \( c = 80, 90, 100, 110 \) and 120. From Fig. 1, we know that the \( k \) values decrease but their slopes increase as the years-to-retirement increase for each curve of \( c \). For a user of a \( c \)-minus-age strategy, the implications are: (1) his/her RRA decreases but the rates increase across the years-to-retirement; (2) the coefficient of the RRA for \( c = 80 \) is, on average, 1.37 times that of \( c = 90 \), 1.73 times that of \( c = 100 \), 2.10 times that of \( c = 110 \) and 2.46 times that of \( c = 120 \).

Figure 2(a) and (b) show the iso-utility curve of (Equation 3) of the two preference parameters \((k_1 \text{ and } k_2)\) associated with a multiplicative mean-variance utility function corresponding to each of the four age groups (ages 30–34, 40–44, 50–54 and 60–64) for \( c = 100 \) and 120 (plots for \( c = 80, 90 \) and 110 indicate similar results and are available upon request). Since an additive mean-variance utility function is a special case when \( k_1 + k_2 = 1 \), a user of an additive mean-variance utility function refers to the line segment with negative 45 degrees. The plots indicate that for any level of \( c \): (1) the iso-utility curve corresponding to an older age group is above the iso-utility curve corresponding to a younger age group; (2) the intercept of the iso-utility curve corresponding to an older age group is larger than that corresponding to a younger age group; (3) the slope of the iso-utility

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1. Fabozzi et al. (2002) presented the commonly used indexes for asset classes and justified these indexes not only with longer histories, but also with more accuracy.

2. Our results may be sensitive to the number of funds (asset classes) in the analysis. Fortunately, Cromwell et al. (2000) found that approximately 70% of the reduction in the portfolio risk can be obtained with a four-fund portfolio.
curve corresponding to an older age group is smaller than that corresponding to a younger age group; (4) all slopes corresponding to all age groups are nonnegative; (5) each of all the iso-utility curves is a line segment. For a user of a $c$-minus-age strategy, the implications are: (1) the maximum amount of risk (variance) willing to trade for one unit of the expected return is nonnegative and is decreasing as one gets older; (2) if the weight of the expected return utility for one investor remains constant across age, then his/her weight of the risk (variance) utility increases as s/he gets older. In addition, many financial advisors suggest taking 120 minus age instead of 100 minus age, so as to increase equity holdings for supporting the longer life expectancies in retirement.

We now consider a numerical example for the optimal procedure to determine the $c$ value. We suppose that the investor is now 30 years old and plan to retire at the age of 65. We also assume the growth rates of $k_1$ and $k_2$ for an investor over his/her life cycle, where the decreasing rate of $k_1$ is set at the level of 1.5% and the increasing rate of $k_2$ is set at the level of 1%.

Other figures mapping out the region of $c$ with $c = 70, 80, 90, 100, 110$ and 120 for different combinations of the growth rates of $k_1$ and $k_2$ are available upon request.

V. Conclusions

We use a linear and a multiplicative mean-variance utility function to quantitatively analyse the term structure of the mean-variance tradeoffs implied by the $c$-minus-age strategy. We also provide an optimal procedure to determine $c$ based on the two preference parameters, elicited from an investor, of a multiplicative mean-variance utility function.

When a linear mean-variance utility function is used, our investigation of the term structure of the mean-variance tradeoffs implied by the $c$-minus-age strategy reveals that the $k$ values decreases as the years-to-retirement increases for each curve of $c$. For a user of the $c$-minus-age strategy, the implications...
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When a multiplicative mean-variance utility function is used, our investigation of the term structure of the mean-variance tradeoffs implied by the \( c \)-minus-age strategy reveals that for any level of \( c \): (1) the iso-utility curve corresponding to an older age group is above the iso-utility curve corresponding to a younger age group; (2) the intercept of the iso-utility curve corresponding to an older age group is larger than that corresponding to a younger age group; (3) the slope of the iso-utility curve corresponding to an older age group is smaller than that corresponding to a younger age group; (4) all slopes corresponding to all age groups are nonnegative; (5) each of all the iso-utility curves is a line segment. For a user of the \( c \)-minus-age strategy, the implications are: (1) the maximum amount of risk (variance) willing to trade for one unit of the expected return is nonnegative and is decreasing as one gets older; (2) if the weight of the expected return utility for one investor remains constant across age, then his/her weight of the risk (variance) utility increases as s/he gets older. Figure 3 provides a numerical example on choosing a suitable \( c \) value according to one’s preference parameters.

References


Appendix

Proof of $y(x) = 1 - x^2$ in the multiplicative mean-variance utility function

$$y(x) = \frac{(r + a_1/2a_2)}{\tilde{r} + a_1/2a_2}$$

Recall that $f(r) = a_2r^2 + a_1r + a_0$ and $\xi = -a_1/2a_2$.

Since

$$x = \frac{r - \tilde{r}}{\tilde{r} - \xi}$$

$$= \frac{(r + a_1/2a_2)}{\tilde{r}} + \frac{a_1}{2a_2}$$

$$= \frac{2a_2r + a_1}{2a_2\tilde{r} + a_1}$$

we have that

$$x = \frac{2a_2r + a_1}{2a_2\tilde{r} + a_1}$$

We note that

$$1 - x = 1 - \frac{r - \tilde{r}}{\tilde{r} - \xi} = \frac{\tilde{r} - r}{\tilde{r} - \xi}$$

and

$$1 + x = 1 + \frac{2a_2\tilde{r} + a_1}{2a_2\tilde{r} + a_1}$$

$$= \frac{a_2\tilde{r} + a_1}{a_2\tilde{r} + (a_1/2)}$$

$$= \frac{a_2(\tilde{r} + r) + a_1}{a_2(\tilde{r} + \tilde{r}) + a_1}$$

Since

$$y(r) = \frac{f(\tilde{r}) - f(r)}{f(\tilde{r}) - f(\tilde{r})} = \frac{a_2\tilde{r}^2 + a_1\tilde{r} - a_2r^2 - a_1r}{a_2\tilde{r}^2 + a_1\tilde{r} - a_2\tilde{r}^2 - a_1\tilde{r}}$$

$$= \frac{a_2(\tilde{r}^2 - r^2) + a_1(\tilde{r} - r)}{a_2(\tilde{r}^2 - \tilde{r}^2) + a_1(\tilde{r} - \tilde{r})} = \frac{\tilde{r} - r}{\tilde{r} - \xi} \times \frac{a_2(\tilde{r} + r) + a_1}{a_2(\tilde{r} + \tilde{r}) + a_1}$$

$$= (1 - x)(1 + x)$$

$$= 1 - x^2,$$

we have that $y(x) = 1 - x^2$. 

References


