Approximate proximal algorithms for generalized variational inequalities with paramonotonicity and pseudomonotonicity

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Abstract

We propose an approximate proximal algorithm for solving generalized variational inequalities in Hilbert space. Extension to Bregman-function-based approximate proximal algorithm is also discussed. Weak convergence of these two algorithms are established under the paramonotonicity and pseudomonotonicity assumptions of the operators.

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1. Introduction and preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$, respectively. Given $T : D(T) \subset H \rightarrow 2^H$ where $D(T)$ denotes the domain of $T$ and $\Omega \subset H$ be a nonempty closed and convex set, the generalized variational inequality problem for $T$ and $\Omega$, denoted by $\text{GVI}(T, \Omega)$ is the problem of finding $x^* \in D(T)$ such that

$$x^* \in \Omega, \exists u^* \in T(x^*): \langle u^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega.$$ \hspace{1cm} (1.1)

The problem $\text{GVI}(T, \Omega)$ was initially introduced in the 1970s; see, e.g. Bruck [1] and the references therein. Subsequently, Fang and Peterson [2] considered it in 1982 in the setting of finite-dimensional spaces. Since then, this problem has been extensively studied in the literature mainly on the existence of solutions of the problems. See, e.g. [3–5] and the references therein.

When $T$ is single-valued, the $\text{GVI}(T, \Omega)$ reduces to the classical variational inequalities $\text{VI}(T, \Omega)$ which have been extensively studied both in finite- and infinite-dimensional spaces. See, [6–9] and the references therein. We observe that both $\text{GVI}(T, \Omega)$ and $\text{VI}(T, \Omega)$ are closely related to optimization problems. See, e.g. [6,9,10].

In this paper we suggest and analyse the approximate proximal algorithm (Algorithm 2.1) and Bregman-function-based approximate proximal algorithm (Algorithm 3.1) for solving $\text{GVI}(T, \Omega)$, where $T$ is a paramonotone and

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Proposition 4. Let \( T : D(T) \subset H \to 2^H \) be an operator where \( D(T) \) is the domain of \( T \). Then \( T \) is said to be

(i) monotone if for all \( x, y \in \Omega, u \in T(x), \) and \( v \in T(y) \),
\[ \langle u - v, x - y \rangle \geq 0 \]

(ii) paramonotone [12] on \( \Omega \) if \( T \) is monotone and \( \langle v - u, y - z \rangle = 0 \) with \( y, z \in \Omega \), \( v \in T(y), u \in T(z) \) implies that \( u \in T(y), v \in T(z) \).

Proposition 1.1 ([12, Proposition 4]). Assume that \( T \) is paramonotone on \( \Omega \) and \( \bar{x} \) is a solution of GVI\((T, \Omega)\). Let \( x^* \in \Omega \) be such that there exists an element \( u^* \in T(x^*) \) with \( \{u^*, x^* - \bar{x}\} \leq 0 \). Then \( x^* \) also solves GVI\((T, \Omega)\).

In 2005, Burachik, Lopes and Svaiter [10] studied an outer approximation for the variational inequality problem. To prove the convergence of the method, they employed the paramonotonicity and pseudomonotonicity of multivalued operators. Let \( B \) be a reflexive Banach space and the operator \( T : D(T) \subset H \to 2^H \) be such that the domain \( D(T) \) is closed and convex. \( T \) is said to be pseudomonotone [13] if for any sequence \( \{(x_n, u_n)\} \subset G(T) \), the graph of \( T \), there holds the following:

(a) \( \{x_n\} \) converges weakly to \( x^* \in D(T) \),
(b) \( \limsup_n \langle u_n, x_n - x^* \rangle \leq 0 \),

then for every \( w \in D(T) \) there exists an element \( u^* \in T(x^*) \) such that
\[ \langle u^*, x^* - w \rangle \leq \liminf_n \langle u_n, x_n - w \rangle. \]

2. Approximate proximal algorithm for GVI\((T, \Omega)\)

Let \( \Omega \subset H \) be a nonempty closed and convex set and let \( T : D(T) \subset H \to 2^H \) be a multivalued operator with \( \Omega \cap D(T) \neq \emptyset \). Recall that the generalized variational inequality GVI\((T, \Omega)\) is the problem of finding \( x^* \in \Omega \cap D(T) \) such that there exists \( u^* \in T(x^*) \) with
\[ \langle u^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega. \tag{2.1} \]
\( S^* \) denotes the solution set of GVI\((T, \Omega)\). We fix a sequence \( \{\Omega_n\} \) of convex closed subsets of \( H \) and two sequences \( \{e_n\}, \{\lambda_n\} \subset \mathbb{R}_+ := [0, +\infty) \) satisfying the following conditions:

(A1) \( \Omega \subset \Omega_n \) for all \( n \), and there exist \( x^* \in S^* \) and \( u^* \in T(x^*) \) such that
\[ \langle u^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_n \text{ and } \forall n. \]

(A2) \( \sum_n (e_n/\lambda_n) < +\infty \) with \( \{\lambda_n\} \subset (0, M] \) for some \( M > 0 \).

Observe that there are some situations where (A1) is satisfied. For example, if \( \Omega_n \) is contained in some bounded, closed, convex subset of \( H \) for all \( n \) and the operator \( T \) is upper semicontinuous along line segments with bounded closed convex values, then (A1) is satisfied (see, e.g. [3]).

We now describe our first algorithm as follows:
Algorithm 2.1. Initialization. Take any initial value \( x_0 \in \Omega \) and \( \Omega_1 \supset \Omega \).

Iterations. For \( n = 1, 2, \ldots \), find \( x_n \in \Omega_n \cap D(T) \), a solution of the \( n \)th approximating problem, defined as follows: for given \( \Omega_n \), \( \varepsilon_n \) and \( \lambda_n \),

\[
\begin{aligned}
\text{find } x_n \in \Omega_n \cap D(T) \text{ such that there exists } u_n \in T(x_n) \text{ with }
\langle \lambda_n (x_{n-1} - x_n + e_n) - u_n, x_n - x \rangle \geq -\varepsilon_n, \quad \forall x \in \Omega_n, \\
\end{aligned}
\]

(\( AP_n \))

where \( \{\varepsilon_n\} \) is an error sequence in \( H \).

Definition 2.1. Let \( \{\Omega_n\} \), \( \{\varepsilon_n\} \) and \( \{\lambda_n\} \) be as in (A1) and (A2).

(a) A sequence \( \{x_n\} \) is called an almost-orbit if \( x_n \) solves \( (AP_n) \) for all \( n \).

(b) An almost-orbit \( \{x_n\} \) is called asymptotically feasible (AF, for short) if all weak accumulation points of \( \{x_n\} \) belong to \( \Omega \).

We remark that if \( D(T) = H \), \( \varepsilon_n = x_n - x_{n-1} \) and \( \lambda_n = 1 \) for all \( n \), then the concepts of almost-orbit and asymptotic feasibility reduce to the concepts of orbit and feasibility in [10, Definition 3.1], respectively.

Lemma 2.1 ([11, Lemma 2.1]). Let \( \{a_n\} \), \( \{b_n\} \) and \( \{c_n\} \) be nonnegative real sequences satisfying the following condition:

\[
a_{n+1} \leq (1 + b_n) a_n + c_n, \quad \forall n \geq n_0,
\]

(*)

for some integer \( n_0 \geq 1 \), where \( \sum_n b_n < +\infty \) and \( \sum_n c_n < +\infty \). Then \( \lim_n a_n \) exists.

Now, we state and prove the main result of this section.

Theorem 2.1. Suppose that the sequence \( \{x_n\} \) generated by Algorithm 2.1 is an AF almost-orbit and (A1) as well as (A2) hold. Suppose that

(i) \( T \) is paramonotone and pseudomonotone with closed domain;

(ii) \( S^* \) is nonempty.

If \( \sum_n \|e_n\| < +\infty \), then \( \{x_n\} \) is weakly convergent to a solution of GVI(\( T, \Omega \)).

Proof. Following the same proof of Theorem 2.1 in [11], we can prove the following conclusions:

(i) For \( x^* \in S^* \) as in (A1), there holds

\[
\lambda_n (x_{n-1} - x_n + e_n, x_n - x^*) \geq -\varepsilon_n.
\]

(ii) For \( x^* \in S^* \) as in (A1), there holds

\[
\|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 - \|x_n - x_{n-1}\|^2 + 2\langle \varepsilon_n, x_n - x^* \rangle + 2\|\varepsilon_n\|e_n + \lambda_n.
\]

(iii) For \( x^* \in S^* \) as in (A1), there exists an integer \( N_0 \geq 1 \) such that for all \( n \geq N_0 \)

\[
\|x_n - x^*\|^2 \leq (1 + \beta_n)\|x_{n-1} - x^*\|^2 - \frac{1}{1 - \|\varepsilon_n\|\varepsilon_n} \|x_n - x_{n-1}\|^2 + \beta_n,
\]

where \( \beta_n = \frac{\|e_n\| + 2e_n/\|\varepsilon_n\|}{1 - \|\varepsilon_n\|\varepsilon_n}, \quad \forall n \geq N_0. \)

(iv) The following statements hold:

(a) \( \lim_n \|x_n - x^*\| \) exists for \( x^* \in S^* \) as in (A1) and hence \( \{x_n\} \) is bounded;

(b) \( \lim_n \|x_n - x_{n-1}\| = 0 \).

Next, we shall prove that \( \{x_n\} \) converges weakly to a solution of GVI(\( T, \Omega \)).

Indeed, we first claim that every weak accumulation point of \( \{x_n\} \) is a solution of GVI(\( T, \Omega \)). Let \( \hat{x} \) be a weak accumulation point of \( \{x_n\} \). Then there exists a subsequence \( \{x_{n_j}\} \) weakly convergent to \( \hat{x} \). For each \( j \), \( x_{n_j} \) solves \( (AP_{n_j}) \). Thus there exists \( u_{n_j} \in T(x_{n_j}) \) such that

\[
\langle \lambda_{n_j} (x_{n_j-1} - x_{n_j} + e_{n_j}) - u_{n_j}, x_{n_j} - x \rangle \geq -\varepsilon_{n_j}, \quad \forall x \in \Omega_{n_j} \text{ and } \forall n_j.
\]
By the condition $\Omega_{n_j} \supset \Omega$, we have
\[
\langle \lambda_{n_j} (x_{n_j-1} - x_{n_j} + e_{n_j}) - u_{n_j}, x_{n_j} - x \rangle \geq -\varepsilon_{n_j}, \quad \forall x \in \Omega \text{ and } \forall n_j.
\] (2.2)

Since $\{x_n\}$ is AF, $\hat{x} \in \Omega$. Therefore
\[
\langle \lambda_{n} (x_{n-1} - x_{n} + e_{n}) - u_{n}, x_{n} - \hat{x} \rangle \geq -\varepsilon_{n}, \quad \forall n_j,
\]
which implies that
\[
\varepsilon_{n} + \lambda_{n} \langle x_{n-1} - x_{n} + e_{n}, x_{n} - \hat{x} \rangle \geq \langle u_{n}, x_{n} - \hat{x} \rangle, \quad \forall n_j.
\]

Also, utilizing (A2) we have
\[
\limsup_{j} \langle u_{n_j}, x_{n_j} - \hat{x} \rangle \leq \limsup_{j} \langle \lambda_{n_j} (x_{n_j-1} - x_{n_j} + e_{n_j}, x_{n_j} - \hat{x}) + \varepsilon_{n_j} \rangle
\]
\[
= \limsup_{j} \lambda_{n_j} \left[ \langle (x_{n_j-1} - x_{n_j} + e_{n_j}), x_{n_j} - \hat{x} \rangle + \frac{\varepsilon_{n}}{\lambda_{n_j}} \right]
\]
\[
\leq \limsup_{j} M \left[ (\|x_{n_j-1} - x_{n_j}\| + \|e_{n_j}\|) \|x_{n_j} - \hat{x}\| + \frac{\varepsilon_{n_j}}{\lambda_{n_j}} \right]
\]
\[
= 0.
\]

Take any $\bar{x} \in S^*$. From the pseudomonotonicity of $T$, we conclude that there exists $\hat{u} \in T(\hat{x})$ such that
\[
\liminf_{j} \langle u_{n_j}, x_{n_j} - \bar{x} \rangle \geq \langle \hat{u}, \hat{x} - \bar{x} \rangle.
\]

Since $\bar{x}$ lies in $\Omega$, from (2.2), we have
\[
\liminf_{j} \langle u_{n_j}, x_{n_j} - \bar{x} \rangle \leq \liminf_{j} \langle \lambda_{n_j} (x_{n_j-1} - x_{n_j} + e_{n_j}, x_{n_j} - \hat{x}) + \varepsilon_{n_j} \rangle
\]
\[
\leq \limsup_{j} \lambda_{n_j} \left[ \langle (x_{n_j-1} - x_{n_j} + e_{n_j}), x_{n_j} - \hat{x} \rangle + \frac{\varepsilon_{n_j}}{\lambda_{n_j}} \right]
\]
\[
\leq \limsup_{j} M \left[ (\|x_{n_j-1} - x_{n_j}\| + \|e_{n_j}\|) \|x_{n_j} - \hat{x}\| + \frac{\varepsilon_{n_j}}{\lambda_{n_j}} \right]
\]
\[
= 0.
\]

Combining the last two inequalities we infer that
\[
\langle \hat{u}, \hat{x} - \bar{x} \rangle \leq 0.
\]

Now taking into account the paramonotonicity of $T$ and Iusem [12, Proposition 4], we deduce that $\hat{x}$ is a solution of the GVI($T, \Omega$).

On the other hand, suppose that $\hat{x}$ and $\bar{x}$ are any two weak accumulation points of $\{x_n\}$ and that two subsequences $\{x_{n_j}\}$ and $\{x_{m_j}\}$ of $\{x_n\}$ weakly converge to $\hat{x}$ and $\bar{x}$, respectively. Then both $\hat{x}$ and $\bar{x}$ belong to $S^*$. Thus, by conclusion (iv) (a), we know that both $\lim_n \|x_n - \hat{x}\|$ and $\lim_n \|x_n - \bar{x}\|$ exist. Now, observe that
\[
\lim_{n} \|x_n - \hat{x}\|^2 = \lim_{i} \|x_{n_i} - \hat{x}\|^2 = \lim_{i} \|x_{n_i} - \hat{x} + \hat{x} - \bar{x}\|^2
\]
\[
= \lim_{i} \|x_{n_i} - \hat{x}\|^2 + 2 \langle x_{n_i} - \hat{x}, \hat{x} - \bar{x} \rangle + \|\hat{x} - \bar{x}\|^2
\]
\[
= \lim_{i} \|x_{n_i} - \hat{x}\|^2 + \|\hat{x} - \bar{x}\|^2
\]
\[
= \lim_{n} \|x_n - \hat{x}\|^2 + \|\hat{x} - \bar{x}\|^2.
\] (2.3)

Replacing the role of $\hat{x}$ by $\bar{x}$, we similarly derive
\[
\lim_{n} \|x_n - \bar{x}\|^2 = \lim_{n} \|x_n - \bar{x}\|^2 + \|\bar{x} - \hat{x}\|^2.
\] (2.4)
Adding up (2.3) and (2.4) we immediately get $\hat{x} = \bar{x}$. Therefore, $\{x_n\}$ is weakly convergent to a solution of $\text{GVI}(T, \Omega)$.

\[\square\]

3. Extension to Bregman function-based approximate proximal algorithm

Let $\Lambda$ be a convex open subset in $H$ and $h : \overline{\Lambda} \to H$ be a Bregman function where $\overline{\Lambda}$ denotes the closure of the set $\Lambda$. We refer Definition 2.1 in [14] for the definition of Bregman functions. We observe that although [14, Definition 2.1] is in finite-dimensional setting, it is not difficult to see that it can be extended to Hilbert space. The Bregman distance between $x$ and $y$ is defined via the “$D$-function”

$$D_h(x, y) = h(x) - h(y) - \langle \nabla h(y), x - y \rangle,$$

where $x \in \overline{\Lambda}$ and $y \subseteq \Lambda$. From the strict convexity of $h$, one can prove that $D_h(x, y) \geq 0$, and $D_h(x, y) = 0$ if and only if $x = y$. If $h(x) = \frac{1}{2}\|x\|^2$, then $D_h(x, y) = \frac{1}{2}\|x - y\|^2$. In the following, we will use a class of functions that is presented as

$$h(x) = h_0(x) + \frac{1}{2}\|x\|^2,$$

where $h_0$ is a Bregman function. It is easy to see that $h$ is also a Bregman function. Thus for all $x \in \overline{\Lambda}$ and $y \in \Lambda$, we have as in [11]

$$D_h(x, y) \geq \frac{1}{2}\|x - y\|^2.$$  \hfill (3.2)

In this section we still consider the $\text{GVI}(T, \Omega)$ defined by (2.1). We still fix a sequence $\{\Omega_n\}$ of convex closed subsets of $H$ and two sequences $\{\epsilon_n\}, \{\lambda_n\} \subset \mathcal{R}_+: = [0, +\infty)$ satisfying the assumptions (A1) and (A2) in Section 2. In addition, assume also that

(A3) $\nabla h(\cdot)$ is uniformly continuous on any closed bounded subsets of $H$.

These sequences and $h$ define new approximating problems which form a general Bregman function-based approximate proximal point scheme.

**Algorithm 3.1.** Initialization. Take any initial value $x_0 \in \Omega$ and $\Omega_1 \supset \Omega$.

Iterations. For $n = 1, 2, \ldots$, find $x_n \in \Omega_n \cap D(T) \cap \Lambda$, a solution of the $n$th approximating problem, defined as follows: for given $\Omega_n, \epsilon_n$ and $\lambda_n$,

$$\left\{ \begin{array}{l}
\text{find } x_n \in \Omega_n \cap D(T) \cap \Lambda \text{ such that there exists } u_n \in T(x_n) \text{ with } \\
(\lambda_n \nabla h(x_{n-1}) - \nabla h(x_n) + \epsilon_n) - u_n, x_n - x \geq -\epsilon_n, \forall x \in \Omega_n, \\
\end{array} \right. \quad (\text{BAP}_n)$$

where $\{\epsilon_n\}$ is an error sequence in $H$.

**Definition 3.1.** Let $\{\Omega_n\}, \{\epsilon_n\}$ and $\{\lambda_n\}$ be as in (A1) and (A2).

(a) A sequence $\{x_n\}$ is called an $h$-almost-orbit if $x_n$ solves (BAP$_n$) for all $n$.

(b) An $h$-almost-orbit $\{x_n\}$ is called asymptotically feasible (AF, for short) if all weak accumulation points of $\{x_n\}$ belong to $\Omega$.

Next we discuss the convergence of **Algorithm 3.1** under the assumptions of paramonotonicity and pseudomonotonicity imposed on $T$. To prove the convergence of **Algorithm 3.1**, we need additionally the following condition:

(A4) $\nabla h(\cdot)$ is sequentially continuous from the weak topology of $H$ to the weak topology of $H$.

**Theorem 3.1.** Suppose that the assumptions (A1)–(A4) hold and that the sequence $\{x_n\}$ generated by **Algorithm 3.1** is an AF $h$-almost-orbit. Suppose that

(i) $T$ is paramonotone and pseudomonotone with closed domain;

(ii) $S^*$ is nonempty.
If $\sum_n \|e_n\| < +\infty$, then $\{x_n\}$ is weakly convergent to a solution of GVI$(T, \Omega)$.

**Proof.** From the same proof of Theorem 3.1 in [11], we can prove the following conclusions:

(i) For $x^* \in S^*$ as in (A1), there holds

$$
\lambda_n (\nabla h(x_{n-1}) - \nabla h(x_n) + e_n, x_n - x^*) \geq -\varepsilon_n, \quad \forall n.
$$

(ii) For $x^* \in S^*$ as in (A1), there holds

$$
D_h(x^*, x_n) \leq D_h(x^*, x_{n-1}) - D_h(x_n, x_{n-1}) + \langle e_n, x_n - x^* \rangle + \frac{\varepsilon_n}{\lambda_n}, \quad \forall n.
$$

(iii) For $x^* \in S^*$ as in (A1), there exists an integer $N_0 \geq 1$ such that for all $n \geq N_0$

$$
D_h(x^*, x_n) \leq (1 + \beta_n)D_h(x^*, x_{n-1}) - \frac{1}{1 - \|e_n\|}D_h(x_n, x_{n-1}) + \beta_n,
$$

where $\beta_n = \|e_n\| + \varepsilon_n/\lambda_n$, $\forall n \geq N_0$.

(iv) The following statements hold:

(a) $\lim_n D_h(x^*, x_n)$ exists for $x^* \in S^*$ as in (A1) and hence $\{x_n\}$ is bounded;

(b) $\lim_n D_h(x_n, x_{n-1}) = 0$ and hence $\lim_n \|x_n - x_{n-1}\| = 0$.

Next, we shall prove that $\{x_n\}$ is weakly convergent to a solution of GVI$(T, \Omega)$.

Indeed, we first claim that every weak accumulation point of $\{x_n\}$ is a solution of GVI$(T, \Omega)$. Let $\hat{x}$ be a weak accumulation point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_j}\}$ weakly convergent to $\hat{x}$. For each $j$, $x_{n_j}$ solves (BAP$_{\lambda_{n_j}}$). Thus there exists $u_{n_j} \in T(x_{n_j})$ such that

$$
\langle \lambda_{n_j} (\nabla h(x_{n_j-1}) - \nabla h(x_{n_j}) + e_{n_j}) - u_{n_j}, x_{n_j} - x \rangle \geq -\varepsilon_{n_j}, \quad \forall x \in \Omega_{n_j} \text{ and } \forall n_j.
$$

By the condition $\Omega_{n_j} \supset \Omega$, we have

$$
\langle \lambda_{n_j} (\nabla h(x_{n_j-1}) - \nabla h(x_{n_j}) + e_{n_j}) - u_{n_j}, x_{n_j} - x \rangle \geq -\varepsilon_{n_j}, \quad \forall x \in \Omega \text{ and } \forall n_j. \quad (3.3)
$$

Since $\{x_n\}$ is AF and $\hat{x} \in \Omega$, we have

$$
\langle \lambda_{n_j} (\nabla h(x_{n_j-1}) - \nabla h(x_{n_j}) + e_{n_j}) - u_{n_j}, x_{n_j} - \hat{x} \rangle \geq -\varepsilon_{n_j}, \quad \forall n_j.
$$

This implies that

$$
\varepsilon_{n_j} + \lambda_{n_j} \langle \nabla h(x_{n_j-1}) - \nabla h(x_{n_j}) + e_{n_j}, x_{n_j} - \hat{x} \rangle \geq \langle u_{n_j}, x_{n_j} - \hat{x} \rangle, \quad \forall n_j.
$$

Note that $\lim_n \|x_n - x_{n-1}\| = 0$, and $\{x_n\}$ is bounded. Thus we derive $\lim_n \|\nabla h(x_n) - \nabla h(x_{n-1})\| = 0$ by virtue of (A3). Now utilizing (A2), we have

$$
\limsup_j \langle u_{n_j}, x_{n_j} - \hat{x} \rangle \leq \limsup_j [\lambda_{n_j} (\nabla h(x_{n_j-1}) - \nabla h(x_{n_j}) + e_{n_j}, x_{n_j} - \hat{x} + \varepsilon_{n_j})]
$$

$$
= \limsup_j \lambda_{n_j} \left[ (\nabla h(x_{n_j-1}) - \nabla h(x_{n_j}) + e_{n_j}, x_{n_j} - \hat{x} + \varepsilon_{n_j}) \right] \leq \limsup_j M \left[ \|\nabla h(x_{n_j-1}) - \nabla h(x_{n_j}) + e_{n_j} \| \|x_{n_j} - \hat{x} + \varepsilon_{n_j} \| \right] \leq \limsup_j M \left[ (\|\nabla h(x_{n_j-1}) - \nabla h(x_{n_j})\| + \|e_{n_j}\|) \|x_{n_j} - \hat{x} + \varepsilon_{n_j} \| \right] = 0.
$$

Take $\tilde{x} \in S^*$. By pseudomonotonicity of $T$, we conclude that there exists $\hat{u} \in T(\tilde{x})$ such that

$$
\liminf_j \langle u_{n_j}, x_{n_j} - \tilde{x} \rangle \geq \langle \hat{u}, \tilde{x} - \hat{x} \rangle.
$$
Since $\tilde{x}$ lies in $\Omega$ and from (3.3), we conclude that
\[
\liminf_j (\mu_{n_j}, x_{n_j} - \tilde{x}) \leq \liminf_j [\lambda_{n_j} (\nabla h(x_{n_j}) - \nabla h(x_{n_j}) + e_{n_j}, x_{n_j} - \tilde{x}) + \epsilon_{n_j}]
\]
\[
\leq \limsup_j [\lambda_{n_j} (\nabla h(x_{n_j}) - \nabla h(x_{n_j}) + e_{n_j}, x_{n_j} - \tilde{x}) + \epsilon_{n_j}]
\]
\[
= \limsup_j \lambda_{n_j} \left[ (\nabla h(x_{n_j}) - \nabla h(x_{n_j}) + e_{n_j}, x_{n_j} - \tilde{x}) + \frac{\epsilon_{n_j}}{\lambda_{n_j}} \right]
\]
\[
\leq \limsup_j M \left[ (\nabla h(x_{n_j}) - \nabla h(x_{n_j})) \| x_{n_j} - \tilde{x} \| + \frac{\epsilon_{n_j}}{\lambda_{n_j}} \right] = 0.
\]
Combining the last two inequalities we infer that
\[
\langle \tilde{u}, \tilde{x} - \tilde{x} \rangle \leq 0.
\]
Again taking into account the paramonotonicity of $T$ and Iusem [12, Proposition 4], we deduce that $\tilde{x}$ is a solution of the GVI($T, \Omega$).

On the other hand, suppose that $\hat{x}$ and $\tilde{x}$ are any two weak accumulation points of $\{x_n\}$ and that two subsequences $\{x_{n_k}\}$ and $\{x_{n'_k}\}$ of $\{x_n\}$ are weakly convergent to $\hat{x}$ and $\tilde{x}$, respectively. Then both $\hat{x}$ and $\tilde{x}$ belong to $S^*$. Thus, by conclusion (iv) (a) we know that both $\lim_n D_h(\hat{x}, x_n)$ and $\lim_n D_h(\tilde{x}, x_n)$ exist, that is, there exist $\tilde{l}, \tilde{l} \in R_+$ such that
\[
\lim_n D_h(\hat{x}, x_n) = \tilde{l} \quad \text{and} \quad \lim_n D_h(\tilde{x}, x_n) = \tilde{l}.
\]
According to Theorem 3.1,
\[
D_h(\hat{x}, x_n) = D_h(\tilde{x}, x_n) + (\nabla h(x_n) - \nabla h(\tilde{x}), \tilde{x} - \hat{x}) + D_h(\hat{x}, \tilde{x}).
\]
From (3.4), we have
\[
\lim_n (\nabla h(x_n) - \nabla h(\tilde{x}), \tilde{x} - \hat{x}) = -\tilde{l} - D_h(\hat{x}, \tilde{x}).
\]
The left-hand side of (3.5) vanishes since $\tilde{x}$ is a weak cluster point of $\{x_n\}$, and since $\nabla h(\cdot)$ is sequentially continuous from the weak topology of $X$ to the weak topology of $X$ by (A4). So we have
\[
\tilde{l} - \tilde{l} = D_h(\hat{x}, \tilde{x}).
\]
Reversing the roles of $\hat{x}$ and $\tilde{x}$, a similar reasoning leads to $\tilde{l} - \tilde{l} = D_h(\hat{x}, \tilde{x})$, which, combined with (3.6), yields $D_h(\hat{x}, \tilde{x}) + D_h(\hat{x}, \tilde{x}) = 0$, i.e., $D_h(\hat{x}, \tilde{x}) = D_h(\hat{x}, \tilde{x}) = 0$, and hence $\hat{x} = \tilde{x}$, establishing the uniqueness of the weak cluster point of $\{x_n\}$. It follows that $\{x_n\}$ is weakly convergent to a solution of GVI($T, \Omega$).

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