Abstract—This paper discusses the application of a new mechanical element, called Inerter, to building suspension control. The inerter was proposed as a real two-terminal mechanical element, which is a substitute for the mass element, with the applied force proportional to the relative acceleration across two terminals. To investigate the performance benefits of building suspension with inerters, three building models were utilized to analyze the performance using two proposed performance indices. From the simulation results, inerters were deemed effective in reducing vibrations from earthquakes and traffic.

I. INTRODUCTION

The vibration control of buildings can be mainly divided into two categories, namely the resistant structure and the isolated structure. The former utilizes the elastic properties of buildings to sustain or absorb vibration energy, such as earthquakes [1]. The latter applies isolators at the bases of buildings in order to suppress the input energy by specifying the resonance frequencies [2]. Two passive elements, namely rubber isolators and steel springs, are frequently adopted in the current applications. Alder and Fuller [3] verified the elastic properties of rubber isolators, and discussed their design and implementation. Waller [4] investigated steel springs to cope with traffic vibration. Kelly [5] reviewed the proposed structures which decoupled the damaging effects of ground movements. Clough and Penzien [6] utilized Dynamic-Stiffness matrices to explain the resonance phenomena. Chua et al. [7] and Thornely-Taylor [8] applied finite-element and finite-difference methods, respectively, to analyze two-dimensional beam structures and discussed the influence of element numbers. Talbot and Hunt [9] proposed Insertion Gain and Power Flow Insertion Gain to evaluate the performance of building isolation. In this paper, we propose two performance indices to discuss the performance benefits of building suspension employing inerters. Three building models, namely the rigid-body, flexible column and portable frame models, are utilized to discuss the parameter optimization and corresponding responses.

The inerter was recently proposed as an ideal two-terminal mechanical element to substitute for the mass element, with the applied force proportional to the relative acceleration across two terminals. Ideal inerters have been applied to vehicle, motorcycle and train suspension control [10, 11, 12], where significant performance improvement was noticed. In [13], the Inerter nonlinearities and their impact on system performance were investigated. In this paper, we explore the application of inerters to building suspension control. It is arranged as follows: in section II, a rigid-body building model is considered for performance analysis. We propose two measures to discuss the performance improvement by suspension struts with inerters. In section III, the ideas are extended by considering a flexible-column model of buildings. In section IV, a portal frame building model is established to analyze the system performance. Finally, we draw some conclusions in section V.

II. THE RIGID-BODY MODEL

In modeling the building as a rigid body with two degrees of freedom (DOF), as shown in Figure 1 [9], where \( Q \) represents the suspension arrangements, the dynamics of the system can be represented as

\[
\begin{bmatrix}
M s^2 + 2Q_s & 0 & 0 \\
0 & M s^2 + 2Q_s & -2HQ_s \\
0 & -2HQ_s & 2H^2 s^2 + H'Q_s + W'Q_s
\end{bmatrix}
\begin{bmatrix}
y_s \\
y_s \\
\theta
\end{bmatrix} = \begin{bmatrix}
2Q_b & 0 & 0 \\
0 & 2Q_v & 0 \\
0 & 0 & -2HQ_c
\end{bmatrix}
\begin{bmatrix}
x_s \\
x_v \\
\theta
\end{bmatrix},
\]  

where \( M \) and \( I \) are the mass and inertia of the building, while \( x_b \) and \( x_v \) are the horizontal and vertical displacements of the building base, and \( y_b \), \( y_v \) and \( \theta \) represent the horizontal, vertical and rotational displacements of the building.

Three suspension arrangements, as illustrated in Table 1, are considered to discuss the performance benefits of inerters. It is

Figure 1. A two-DOF rigid-body model. [9]
noted that S2 is a basic parallel arrangement and S3 is a basic serial arrangement.

<table>
<thead>
<tr>
<th>Table 1. Three suspension models.</th>
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<tbody>
<tr>
<td>S1</td>
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<tr>
<td>$Q_i(\omega) = j\omega + k$</td>
</tr>
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</table>

To discuss the performance, we consider the Insertion Gain (IG) of the system as [14]:

$$IG_i = \frac{\gamma_i}{x_i},$$  \hspace{1cm} (2)

where $i = h$ (or $i = v$) represents the frequency responses and performance in the horizontal (or vertical) directions. It is noted from (1) that the dynamics in the horizontal and rotational directions are coupled. Therefore, we can evaluate the system performance in the vertical and horizontal directions. Two performance indices $J_\infty$ and $J_2$ are proposed as follows:

$$J_{\infty, i} = \sup_{\omega \in [\omega_1, \omega_2]} |IG_i(\omega)|,$$ \hspace{1cm} (3)

$$J_{2, i} = \sqrt{\int_{\omega_1}^{\omega_2} |IG_i(\omega)| \, d\omega},$$ \hspace{1cm} (4)

in which $\omega \in [\omega_1, \omega_2]$ are the concerned frequency range. The indices are so chosen since the $H_{\infty}$ norm of the system equals to the supreme ratio of the output 2-norm to the input 2-norm, while the $H_2$ norm of the system indicates the maximum ratio of the output $\infty$-norm to the input 2-norm [15]. That is, they indicate the supreme (worst) outputs in terms of energy or the absolute magnitude corresponded to a fixed input. Considering the disturbance sources, the frequencies are set as $\omega \in [0.5, 10]$ Hz for the primary earthquake and $\omega \in [5, 25]$ Hz for the traffic factor (wheel-passing) [16].

The following parameters are used for simulations: $H=50m$, $W=20m$, $M=10^6$ kg. The suspension stiffness is set as $k=10^5 - 10^8$ N/m and the resonance frequency of the building is set between 0.05 Hz and 1.59 Hz [17]. To optimize the performance measures, the damping rates ($c's$) and the inertance ($b's$) are tuned to achieve the smallest values of $J_2$ and $J_\infty$ at each fixed stiffness settings ($k's$).

### A. Vertical Direction

The optimization of $J_{\infty, i}$ is illustrated in Figure 2, where S2 achieves significant performance improvement for the traffic factor, while S3 is slightly better than S2 for earthquakes. Considering specific stiffness settings $k=10^8$ N/m for the traffic factor and $k=1.1\times10^7$ N/m for the earthquake, the optimal parameter settings are illustrated in Table 2, where the S2 (S3) layout achieves 55% (58%) for the traffic factor (earthquake). The corresponding Bode plots of these models are shown in Figure 3, where the notches are noted to be beneficial in improving the performance.

$$10 \log_{10} [IG(dB)]$$

- Traffic factor 5-25Hz (S2, S3)
- Earthquake 0.5-10Hz (S2, S3)

(a) For traffic factor.
(b) For earthquake.

Figure 2. Optimization of $J_{\infty, i}$.

<table>
<thead>
<tr>
<th>Table 2. $J_{\infty, i}$ optimization on the vertical direction.</th>
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<tbody>
<tr>
<td>layout</td>
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<td>--------</td>
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<td></td>
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<tr>
<td>S1</td>
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</table>
| S2     | $J_{\infty} = 0.0485$  
$b=55047$, $c = 0$ | $J_{\infty} = 0.4494$  
$b = 8.19\times10^5$, $c = 0$,  
57% Improvement |
| S3     | $J_{\infty} = 0.1127$  
$b = 11034$, $c = 0$ | $J_{\infty} = 0.4219$  
$b = 8.98\times10^5$, $c = 1.39\times10^7$,  
58% Improvement |

(a) For traffic factor.
(b) For earthquake.

Figure 3. Bode plots with the optimal settings for $J_{\infty, i}$.

The optimization of $J_{2, i}$ is illustrated in Figure 4, where S2 achieves about 29% performance improvement for the traffic factor at all stiffness settings, while S3 is a better choice for the earthquake. Setting the stiffness $k=10^8$ N/m for the traffic effect $(k = 1.1\times10^7$ N/m for earthquake), the optimal Bode plots are shown in Figure 5, where the S2 (S3) achieves 30% (37%) improvement.
From the discussion in IIA and IIB, it is noted that applying inerters to the suspension design introduces notches on the Bode plots, which is beneficial in reducing the performance indices. Besides, we also observed that adding dampers tends to have little effect on the performance optimizations. Therefore, in this section multi-layer suspensions are proposed to the building base to discuss the performance improvements.

A multi-layer building suspension is shown in Figure 8(a). Suppose the masses between each layers are \( m_1 \cdots m_N \), the IG of the system on the vertical direction can be expressed as

\[
IG_N(\frac{y}{x}) = \frac{Q_1 \cdots Q_{Nt}}{(M_1 s^2 + Q_1)(M_2 s^2 + Q_2) \cdots (M_N s^2 + Q_{Nt})}.
\]

Suppose the suspension \( Q_i \) is a parallel combination of spring \( k_i \) and inverter \( b_i \), (5) can be further simplified by \( Q_N = k_N + b_N s^2 \). For example, suppose \( N=2 \), the suspension arrangement is illustrated in Figure 8(b). Given \( M = 10^6 \text{ kg}, k_1 = k_2 = k_3 = 10^6 \text{ N/m}, m_1 = 0.05M, m_2 = 0.05m_1, b_1 = 0.01m_1, b_2 = 0.01m_1, b_3 = 0.01m_1 \), the Bode plots are shown in Figure 8(c), where the multi-layer design can potentially reduce the responses.

III. FLEXIBLE COLUMN MODEL

The aforementioned rigid-body model provides some intuitions on the suspension design of buildings. However, the elastic effects of buildings are ignored by the simplification. Therefore, in this section we consider the flexible column model, and discuss the suspension benefits of inerters.

A flexible column model of building is illustrated in Figure 9 [17], in which \( x_i \) and \( x_i \) are the horizontal and vertical disturbance inputs, while \( y_a \) and \( y_a \) are the horizontal and vertical displacements on the base of the building, and \( y_a \) and \( y_a \) are the horizontal and vertical displacements on the top of the building. The base isolator and side isolator are represented by \( Q_b \) and \( Q_s \). The dynamic equations of the model are as follows [18]:
Similarly, the optimization of $J_2$ is shown in Figure 11, where the S3 layout is always the best.

![Figure 11. Optimization of $J_2$ for the flexible column model.](image)

### IV. Portal Frame Model

![Figure 12. A portal frame model.](image)

![Figure 13. Free-body diagram of beam $i$.](image)

The portal-frame model, as shown in Figure 12 [17], is frequently used for the analyses of modern buildings, where the floors and walls are represented by elastic beams. Isolators are located at the base of the building. Each beam has two joints, which are connected to other beams. Each joint has three DOF, in the translational, lateral and rotational directions. Therefore, there are twenty-four DOF for the
model of Figure 12. To discuss the coupling effects of the structures, we firstly analyze the free-body diagram of $beam_{34}$, as illustrated in Figure 13. Suppose the translation, lateral and rotational displacements of joint 3 (joint 4) are labeled as $u_3$, $v_3$ and $\theta_3$ ($u_4$, $v_4$ and $\theta_4$), and the forces/moments on these directions are labeled as $f_3$, $s_3$ and $q_3$ ($f_4$, $s_4$ and $q_4$), then the dynamics of $beam_{34}$ can be represented as follows [17]:

$$
K = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

in which

$$
K_{ii} = \begin{bmatrix}
-k_{ii} & -k_{ii} & 0 & 0 & 0 & 0 \\
-k_{ii} & -k_{ii} & 0 & 0 & 0 & 0 \\
0 & 0 & -k_{ii} & -k_{ii} & 0 & 0 \\
0 & 0 & -k_{ii} & -k_{ii} & 0 & 0 \\
0 & 0 & 0 & 0 & -k_{ii} & -k_{ii} \\
0 & 0 & 0 & 0 & -k_{ii} & -k_{ii} \\
\end{bmatrix}
$$

and

$$
\beta = \left( \frac{\rho A a^2}{EI} \right), \quad \alpha = \sqrt{\frac{\beta}{\omega}}.
$$

Considering the rotation of other beams, the complete dynamic equations of the system can be represented as:

$$
\Sigma F_{24x1} = [K]_{24x24} \begin{bmatrix} U \end{bmatrix}_{24x1}.
$$

Suppose the road inputs on joints 1 and 8 are $z_1$ and $z_8$ with suspension $Q_1$ and $Q_2$, the forces acting on the joints are:

$$
F_b = \begin{bmatrix} F_{b,1} \\
F_{b,2} \end{bmatrix} = \begin{bmatrix} Q_1 (U_1 - Z_1) \\
Q_2 (U_2 - Z_2) \end{bmatrix}
$$

To simplify the expression of (12), we define $H = K^{-1}$ and partition (12) conformally:

$$
\begin{bmatrix} U_b \end{bmatrix}_{24x1} = \begin{bmatrix} U_{b,1} \\
U_{b,2} \end{bmatrix}_{24x1} = \begin{bmatrix} H_{11} & H_{12} \\
H_{21} & H_{22} \end{bmatrix}_{24x24} \begin{bmatrix} F_b \end{bmatrix}_{24x1}.
$$

If we ignore the weights of joints, the resultant forces acting on each joint of structures equal to zero, i.e. $\Sigma F_i = 0$ and (14) is simplified as:

$$
\begin{bmatrix} U \end{bmatrix}_{24x1} = \begin{bmatrix} H_{11} \\
H_{12} \\
H_{21} \\
H_{22} \end{bmatrix}_{24x24} \begin{bmatrix} F_b \end{bmatrix}_{24x1}.
$$

Combining (13) with (15), the displacements of all joints can be evaluated from the inputs $z_1$ and $z_8$:

$$
\begin{bmatrix} U \end{bmatrix}_{24x1} = \left( \begin{bmatrix} I \end{bmatrix}_{24x24} - \begin{bmatrix} H_1 \end{bmatrix}_{24x24} \right)^{-1} \begin{bmatrix} H_1 \end{bmatrix}_{24x24} \begin{bmatrix} P \end{bmatrix}_{24x1} \begin{bmatrix} Z \end{bmatrix}_{24x1}.
$$

Since the model of (16) is much more complicated than the previous rigid-body or flexible beam models, in order to analyze the performance, we consider the Power Flow Insertion Gain (PFIG) defined as follows [19]:

$$
PFIG = \frac{P_{total}}{P_{ambient}}.
$$

where $P_{ambient}$ and $P_{total}$ represent the instant power of the structure without and with isolators, respectively, which are defined as:

$$
P = \frac{1}{2} \Re \left[ \omega (u' + v_0') + \theta q' \right].
$$

We then defined the performance indices $J_\omega$ and $J_2$ of the system as follows:

$$
J_\omega = \sup_{\tilde{\omega} \in [\omega_1, \omega_2]} \left| PFIG(\tilde{\omega}) \right|,
$$

$$
J_2 = \int_{\omega_1}^{\omega_2} \left| PFIG(\omega) \right|^2 d\omega,
$$

in which $\omega \in [\omega_1, \omega_2]$ are the concerned frequency range. We carried out the performance analyses using the following parameters [17]: Young’s modulus of the elements $E=10 \times 10^8$ N/m$^2$, cross-sectional area $A=0.25 \pi m^2$, density of elements $\rho = 2400$ kg/m$^3$, damping loss factor $\eta = 0.1$ and length of elements $L=10$ m.
The optimization of $J_*$ is illustrated in Figure 14. For the traffic factor, S2 can improve 100% of the performance for nearly all stiffness settings, as illustrated in Figure 14(a). As for earthquakes, both S2 and S3 achieve about 100% improvement for all stiffness settings (except when $k = 10^3 ~ 10^5$ N/m), as shown in Figure 14(b). Setting the stiffness $k = 5\times 10^7$ N/m, the corresponding Bode plots using optimal parameters are illustrated in Figure 14(c)(d).

![Figure 14. Optimization of $J_*$ using the portal frame model.](image)

The optimization of $J_2$ is shown in Figure 15. For the traffic factor, using inerter is not very beneficial in improving performance, as seen in Figure 15(a). On the other hand, the performance improvement for earthquake is mainly at low stiffness settings, as illustrated in Figure 15(b). Setting the stiffness $k = 1.1\times 10^7$ for the traffic factor and $k = 5.1\times 10^8$ N/m for earthquakes, the corresponding Bode plots using optimal parameter settings are illustrated in Figure 15(c)(d).

![Figure 15. Optimization of $J_2$ using the portal frame model.](image)

V. CONCLUSION

This paper has discussed the performance benefits of building suspension design employing inerter. Three building models, namely the rigid-body, flexible beam and portal frame models, were considered for the performance analyses. Three basic suspension layouts were introduced for numerical optimization. From the simulation results, inerter were deemed effective in suppressing vibrations from both traffic and earthquake vibrations. Further studies can be carried out by considering complicated suspension layouts or LMI approaches, as illustrated in [20].

REFERENCES