easily shown that the performance improvement is maximised when $t = 2$. Larger arrays can be dealt with by breaking the array up into pairs. By substituting 2 for t into the above formula, we calculate that the performance improvement for the canonically recoded numbers is 12/11. Thus, the common-multiplicand multiplication of n k -digit canonically recoded numbers takes $nk \times (1/3) + (12/11) = 11nk/36 \approx 0.306nk$ additions.

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Scheduling parallel tasks on hypercubes

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Indexing terms: Computer architecture, Parallel architectures, Scheduling

The authors consider the problem of non-pre-emptively scheduling independent parallel tasks with communication overhead on a d -dimensional hypercube system. To find a schedule such that the schedule length is minimised is NP-hard. Therefore, a simple heuristic algorithm is investigated and its performance bound is derived as $(2 + \ln m - 1/m)$, where $m = 2^d$.

Introduction: In the conventional scheduling problem, it is assumed that each task is processed in only one processor at a time. However in the parallel task scheduling problem [1, 2, 4, 5], each task is worked on by more than one processor at a time. Assume we have a set of n independent parallel tasks $T = \{T_1, T_2, \dots, T_n\}$ to be processed in a d -dimensional hypercube, and each task T_i has a computation requirement t_i . Then assume that each task T_i is associated with a minimum parallelism dimension Δ_i , that is, each task T_i can be processed at most on a Δ_i -dimensional subcube and this parallelism dimension, once decided for T_i , will not be altered during its processing. If a task T_i is scheduled to run on a d_i -dimensional subcube, $1 \leq d_i \leq \Delta_i \leq d$, then d_i will be called the scheduled dimension of T_i and the execution time required by T_i will be $(t_i/2^{d_i})$. For this problem type, a schedule is feasible if the scheduled dimension d_i of each task T_i is no greater than its maximum parallelism dimension Δ_i . A feasible schedule is called an optimal schedule if it has the shortest schedule length. A heuristic algorithm has a performance bound of β if $(S_H/S_O) \leq \beta$ for all problem instances, where S_H and S_O denote the heuristic schedule length and the optimal schedule length, respectively.

Under the linear speedup assumption, Wang and Cheng [4] proposed an ECT (earliest completion time) algorithm for scheduling parallel tasks and derived a performance bound of $(3 - 2/m)$ in which the number of processors required by each task and the number of processors m in systems are arbitrary. Zhu and Ahuja [5] proposed an LDF (largest dimension first) algorithm for sched-

uling parallel tasks on hypercube systems and derived a performance bound of $(2 - 1/m)$, where m is the number of processors in the hypercube system.

Fig. 1 Optimal schedule and MLDF schedule

a Optimal
b MLDF

Decision of scheduled dimension: It is obvious that while a task T is processed in parallel on a parallel machine, communication between processors is unavoidable. As a consequence, a linear speedup can never be achieved. In a hypercube system, because the average communication overhead is proportional to the dimension x of a subcube on which a task T is processed [3], the communication time required by each processor can be assumed to be Cx , where C is a given positive rational number. Thus, the total processing time $P(x, t)$ required for processing a task T is equal to $((t/2^x) + Cx)$, where t is the computational requirement of task T . By simple calculus, we then have: $\partial P(x, t)/\partial x = -(t/2^x) + C$ and $\partial^2 P(x, t)/\partial x^2 = [(t/2^x) + C]$. Because $\partial P(x, t)/\partial x = 0$ when $x = \log_2[(t/2^x) + C]$ and $\partial^2 P(x, t)/\partial x^2 > 0$, $P(x, t)$ reaches its minimum value at $x = \log_2[(t/2^x) + C]$. Based on the derived result, our strategy is to make the scheduled dimension of task T equal to $\lceil \log_2[(t/2^x) + C] \rceil$.

Dimension-decision procedure: {Assign $\min\{\Delta_i, \lceil \log_2[(t_i/2^{d_i}) + C] \rceil\}$ to the scheduled dimension d_i of task T_i , for $i = 1, 2, \dots, n$ }

Modified largest dimension first (MLDF) scheduling algorithm: Using the scheduled dimension d_i determined for each task T_i by the above dimension-decision procedure, a given set of n independent parallel tasks $T = \{T_1, T_2, \dots, T_n\}$ can be decomposed into $(\rho + 1)$ subsets of tasks: $T^0 = \{T_1^0, T_2^0, \dots, T_{n_0}^0\}$, $T^1 = \{T_1^1, T_2^1, \dots, T_{n_1}^1\}$, $T^2 = \{T_1^2, T_2^2, \dots, T_{n_2}^2\}$, $T^3 = \{T_1^3, T_2^3, \dots, T_{n_3}^3\}$, $T^4 = \{T_1^4, T_2^4, \dots, T_{n_4}^4\}$, $T^5 = \{T_1^5, T_2^5, \dots, T_{n_5}^5\}$, $T^6 = \{T_1^6, T_2^6, \dots, T_{n_6}^6\}$, $T^7 = \{T_1^7, T_2^7, \dots, T_{n_7}^7\}$, $T^8 = \{T_1^8, T_2^8, \dots, T_{n_8}^8\}$, $T^9 = \{T_1^9, T_2^9, \dots, T_{n_9}^9\}$, $T^{10} = \{T_1^{10}, T_2^{10}, \dots, T_{n_{10}}^{10}\}$, $T^{11} = \{T_1^{11}, T_2^{11}, \dots, T_{n_{11}}^{11}\}$, $T^{12} = \{T_1^{12}, T_2^{12}, \dots, T_{n_{12}}^{12}\}$, $T^{13} = \{T_1^{13}, T_2^{13}, \dots, T_{n_{13}}^{13}\}$, $T^{14} = \{T_1^{14}, 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Proof of theorem 1: Assume that task T_w^q is finished at time S_H . By lemma 2,

$$S_H \leq \frac{\sum_{j=1}^p \sum_{i=1}^{n_j} \left[2^j \left(\frac{t_i^j}{2^j} + C_j \right) \right] + (m - 2^q) S_w^q}{m}$$

where t_i^j is the computation requirement of task T_i^j . From the result of the dimension-decision procedure, we know that $C \leq (\ln 2/2)t_i^j$; then,

$$S_H \leq \frac{\sum_{j=1}^p \sum_{i=1}^{n_j} [1 + \ln m] t_i^j}{m} + \left(\frac{m - 2^q}{m} \right) S_w^q$$

It is obvious that

$$\frac{\sum_{j=1}^p \sum_{i=1}^{n_j} t_i^j}{m} \leq S_O$$

and

$$S_w^q \leq S_O$$

where S_O is the optimal schedule length of task set T . Thus

$$\frac{S_H}{S_O} \leq \left(2 + \ln m - \frac{1}{m} \right)$$

Example: Let $t_i = t$ and $\Delta_i = \log_2 m$ for $i = 1, 2, \dots, m-1$, $t_m = t$ and $\Delta_m = 1$, and $C = (\ln 2)t/m$. Fig. 1a and b show that the lengths of the optimal algorithm and the MLDF algorithm are t and $[2 + \ln m - (1/m)(1 + \ln m)]t$, respectively. Thus, $(S_H/S_O) \approx (2 + \ln m - 1/m)$.

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Singularity of Green dyadic for bi-isotropic media

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Indexing terms: Chiral materials, Green function method

The singularity of the Green dyadic for the general bi-isotropic medium is constructed through decomposition of the bi-isotropic problem in two simple isotropic problems. It is seen that, in contrast to the isotropic medium case, the singularity cannot be written in terms of a simple delta function term but it involves a second term containing the derivative of the delta function as well.

Introduction: The singularity of the Green dyadic corresponding to time-harmonic electromagnetic fields in isotropic media has been under investigation for the last 20 years because it plays a key role when numerical methods are applied to integral equation formulations of problems. Van Bladel recognised that, when computing

fields inside the source region, the field integral must be treated in a special way because of the inherent singularity of the Green function [1]. The result can be written in terms of a Green dyadic consisting of a principal-value part and a delta-function term. The principal value part gives the field from the source of which a contracting volume around the field point is extracted. The delta-function term gives the contribution from the source in the contracting volume. In the original formulation by Van Bladel the volume was spherical, which leads to the representation

$$\overline{G}(\mathbf{r} - \mathbf{r}') = PV(\overline{I} + \frac{1}{k^2} \nabla \nabla) G(\mathbf{r} - \mathbf{r}') - \frac{1}{3k^2} \overline{I} \delta(\mathbf{r} - \mathbf{r}') \quad (1)$$

with

$$G(\mathbf{r} - \mathbf{r}') = \frac{e^{-jkD}}{4\pi D} \quad D = \sqrt{(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')} \quad (2)$$

In the 1970s there was much interest in the case when the contracting volume has a form other than spherical. In 1980, Yaghjian [2] and Lee *et al.* [3] (for another derivation see [4]) showed that eqn. 1 must be generalised to

$$\overline{G}(\mathbf{r} - \mathbf{r}') = PV(\overline{I} + \frac{1}{k^2} \nabla \nabla) G(\mathbf{r} - \mathbf{r}') - \frac{1}{k^2} \overline{L} \delta(\mathbf{r} - \mathbf{r}') \quad (3)$$

The dyadic $\overline{L}$ depends on the form of the contracting volume and can be defined as

$$\overline{L} = \frac{1}{4\pi} \int_{4\pi} \mathbf{n} \mathbf{u}_r d\Omega \quad (4)$$

where $\mathbf{n}$ is the unit normal to the surface of the volume and $\mathbf{u}_r$ the radial unit vector directed to the point of the normal vector $\mathbf{n}$. The integration in eqn. 4 is over the whole spherical angle 4π and the origin corresponds to the field point in question. For a sphere and any other surface with three-fold symmetry, we have $\overline{L} = \overline{I}/3$.

In this Letter, we further generalise this theory to bi-isotropic media making it possible to apply integral equation methods to problems in such media.

Wavefield decomposition: The method of solution applies the well known decomposition of the electromagnetic fields in two wavefields (also called self-dual fields or Bohren decomposition). The wavefields see the original bi-isotropic medium as effective simple isotropic media. Thus, we have two problems involving isotropic media and they can be readily solved. The final solution is obtained by combining the wavefield solutions. It is not necessary to describe the decomposition because it is well documented in [5]. Instead, we define the notation and briefly give the result.

If the bi-isotropic medium is defined through the constitutive equations

$$\mathbf{D} = \epsilon \mathbf{E} + (\sin \vartheta - j\kappa_r) \sqrt{\mu\epsilon} \mathbf{H} \quad (5)$$

$$\mathbf{B} = \mu \mathbf{H} + (\sin \vartheta + j\kappa_r) \sqrt{\mu\epsilon} \mathbf{E} \quad (6)$$

with

$$\eta = \sqrt{\mu/\epsilon} \quad k = \omega \sqrt{\mu\epsilon} \quad (7)$$

the electric field $\mathbf{E}$ due to an electric current source $\mathbf{J}$ in such a medium is

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \int_V \overline{G}_e(\mathbf{r} - \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') dV' \quad (8)$$

where $\overline{G}_e$ defines the electric Green dyadic.

If the source is split in two parts $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$, each of them sees an isotropic medium of its own, defined by the parameters

$$\epsilon_{\pm} = \epsilon(\cos \vartheta \pm \kappa_r) e^{\pm j\vartheta} \quad \mu_{\pm} = \mu(\cos \vartheta \pm \kappa_r) e^{\mp j\vartheta} \quad (9)$$

$$k_{\pm} = \omega \sqrt{\mu_{\pm} \epsilon_{\pm}} = k(\cos \vartheta \pm \kappa_r) \quad \eta_{\pm} = \sqrt{\mu_{\pm} / \epsilon_{\pm}} = \eta e^{\mp j\vartheta} \quad (10)$$

The Green dyadic can then be split in two parts

$$\overline{G}_e(\mathbf{r} - \mathbf{r}') = \overline{G}_{e+}(\mathbf{r} - \mathbf{r}') + \overline{G}_{e-}(\mathbf{r} - \mathbf{r}') \quad (11)$$

with

$$\overline{G}_{e\pm}(\mathbf{r}) = \pm \frac{1}{k_{\pm} + k_{\mp}} (\nabla \times \overline{I} \pm k_{\pm} \overline{I}) \cdot \overline{G}_{\pm}(\mathbf{r}) \quad (12)$$

where $\overline{G}_{\pm}(\mathbf{r})$ are the isotropic Green dyadics corresponding to the two respective media: