Robust Nonsingularity Analysis Using Linear Fractional Transformation and Its Application to Stability Analysis

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Abstract

In this paper, a procedure is described for converting a perturbed matrix with mixed unstructured and structured uncertainties to the form of a Linear Fractional Transformation (LFT). Also, a criterion is given for keeping the nonsingularity of the perturbed matrix, provided it is nominally so. Moreover, based on the conversion method and criterion, explicit bounds on the time-varying uncertainties in continuous-time and discrete-time systems are derived for robust stability.

1. Introduction

Recently, matrix nonsingularity analysis has been utilized to analyze the robust stability problem in the presence of time-invariant/varying uncertainty [1, 2]. However, the dimensions of the matrices involved tend to be large, especially in the approaches involving Kronecker product and structured singular value (μ) analysis techniques, such as in [2]. This may cause some computation problems. Moreover, all these approaches can deal with multivariable polynomial type of uncertainties at most [1]. In this paper, we introduce a method to deal with the robust nonsingularity problem. It is shown how to convert a matrix with mixed unstructured and bounded rational type structured uncertainties to the form of an LFT. Then, for the case in which the nominal matrix is nonsingular, an explicit bound is given in terms of μ for the perturbed matrix to remain nonsingular. Compared with previous results in this respect, the current method can deal with a wider variety of uncertainties, including even the product of unstructured and structured uncertainties. In addition, the dimensions of the matrices involved now are often smaller than those from the Kronecker product approach. Furthermore, the derived result is still applicable to the robust stability analysis problems, and bounds of uncertainties for the uncertain systems to preserve stability can be obtained by using the concept of nonsingularity check. Thus a wider class of the robust stability analysis problems can be solved with the proposed method in a more efficient way.

2. Robust Nonsingularity Analysis Using LFT Representation

Let $k = [k_1 \ \dots \ k_q]^T \in \mathbf{R}^q$ be a vector of bounded uncertain parameters, $\Delta_{F_i} \in \mathbf{R}^{l_i \times l_i}$, $i = 1, \dots, p$, be bounded unstructured uncertainty matrices, $\Delta = \text{diag}\{k_1 I_{r_1}, \dots, k_q I_{r_q}, \Delta_{F_1}, \dots, \Delta_{F_p}\}$, where r_i is the multiplicity of k_i , and $\Omega_\gamma = \{\Delta : \bar{\sigma}(\Delta) < \gamma\}$, where $\bar{\sigma}(\cdot)$ is the largest singular value. Then we have

Definition 2.1 It is said that a perturbed matrix $S \in \mathbf{R}^{n \times n}$ containing $k_1, \dots, k_q$ and $\Delta_{F_1}, \dots, \Delta_{F_p}$ can be transformed to an LFT $F(M, \Delta)$ on Δ if S can be written as

$$\begin{aligned} S &= M_{22} + M_{21}(I_r - \Delta M_{11})^{-1} \Delta M_{12} \\ &\triangleq F(M, \Delta), \end{aligned} \quad (1) \quad (2)$$

where M_{11} , M_{12} , M_{21} , and M_{22} are constant matrices of suitable dimensions, M is the joint coefficient matrix defined as $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, and $r = \sum_{i=1}^q r_i + \sum_{i=1}^p l_i$.

From the definition, it is easily seen that M_{22} equals to the nominal part of S . Moreover, if $\det(I_r - \Delta M_{11}) \neq 0$ for all $\Delta \in \Omega_\gamma$, then the LFT is said to be *well-defined* over Ω_γ . Now, suppose the perturbed matrix S considered in this paper is of the following form:

$$S = S_o + S_s(k) + S_u, \quad (3)$$

where $S_o + S_s(0)$ is the nominal part of S and assumed to be nonsingular. $S_s(k) - S_s(0)$ is the structured part of the perturbation, and S_u is the unstructured part of the perturbation. Assume further $S_s(k)$ has bounded rational entries $f_{ij}(k)$ of the form

$$f_{ij}(k) = \frac{n_{ij}(k)}{1 + d_{ij}(k)}, \quad (4)$$

where $n_{ij}(k)$ and $d_{ij}(k)$ are multivariable polynomials in the elements of k , and $1 + d_{ij}(k) \neq 0 \ \forall k \in \Omega_\gamma$. Moreover, S_u is of the form

$$S_u = D_1 \Delta_F E_1 G_\Delta, \quad (5)$$

where D_1 and E_1 are constant matrices, $\Delta_F = \text{diag}\{\Delta_{F_1}, \dots, \Delta_{F_p}\}$, and G_Δ has the meaning explained as follows. If S_u is a linear function of Δ_F only, then $G_\Delta = I$, i.e., $S_u = D_1 \Delta_F E_1$. If S_u is the quadratically-coupled unstructured uncertainty, then we can set $G_\Delta = \Delta_F' E_2$, where E_2 is a constant matrix, and $\Delta_F' = \text{diag}\{\Delta_{F_{p+1}}, \dots, \Delta_{F_p}\}$. Finally, if S_u is the product of structured and unstructured uncertainties, then we can set $G_\Delta = G(k)$, where $G(k)$ is a matrix of the bounded rational entries, just the same as $S_s(k)$.

With the techniques proposed in [6], it is known that $f_{ij}(k)$ can always be converted to an LFT. As a result, by simply adding the (i, j) th entry of S_o to $f_{ij}(0)$, the (i, j) th entry of $S_o + S_s(k)$ can be converted to an LFT $F(M^{ij}, \Delta^{ij}(k))$, where $M^{ij} = \begin{bmatrix} M_{11}^{ij} & M_{12}^{ij} \\ M_{21}^{ij} & M_{22}^{ij} \end{bmatrix}$, and $\Delta^{ij}(k)$ is a diagonal matrix with elements of k repeated on the diagonal. After transforming the entries of $S_o + S_s(k)$ to LFTs individually, we have

$$S_o + S_s(k) = \bar{M}_{22} + \bar{M}_{21}[\bar{I}_d - \bar{\Delta}(k) \bar{M}_{11}]^{-1} \bar{\Delta}(k) \bar{M}_{12} \quad (6)$$

where

$$\begin{aligned} \bar{M}_{11} &= \text{diag}\{M_{11}^{11}, \dots, M_{11}^{1n}, \dots, M_{11}^{n1}, \dots, M_{11}^{nn}\}, \\ \bar{M}_{12} &= \begin{bmatrix} \text{diag}\{M_{12}^{11}, \dots, M_{12}^{1n}\} \\ \vdots \\ \text{diag}\{M_{12}^{n1}, \dots, M_{12}^{nn}\} \end{bmatrix}, \\ \bar{M}_{21} &= \text{diag}\{M_{21}^{(1)}, \dots, M_{21}^{(n)}\} \end{aligned}$$

with $M_{21}^{(j)} = [M_{21}^{j1} M_{21}^{j2} \dots M_{21}^{jn}]$, $\bar{M}_{22}$ is a block matrix with the (i, j) th block M_{22}^{ij} , $\bar{\Delta}(k) = \text{diag}\{\Delta^{11}(k), \dots, \Delta^{1n}(k), \dots, \Delta^{n1}(k), \dots, \Delta^{nn}(k)\}$, and I_d is the identity matrix with the same dimension as that of $\bar{\Delta}(k)$. After interchanging rows and columns of $\bar{\Delta}(k)$ so that the elements of k appear in order, the procedure discussed in [6] can be used to reduce the size of the uncertainty matrix.

Next we turn to the unstructured part. If G_Δ falls in any of the above-mentioned categories of uncertainty, it is obviously an LFT, or can be transformed to one. Thus, S_Δ can be converted to an LFT because the product of two LFTs can always be converted to another LFT. Therefore, since the sum of LFTs can be converted to still another LFT, we reach at the conclusion that S can be expressed as an LFT. In fact, in addition to the aforementioned categories of G_Δ , at this point it is easy to see that the same conclusion holds for any G_Δ that is an LFT itself, or one that is obtained by using the procedure recursively. Without loss of generality, we assume that S can be written as

$$S = M_{22} + M_{21}(I_r - \Delta M_{11})^{-1} \Delta M_{12}, \quad \Delta \in \Omega_\gamma, \quad (7)$$

where $\det M_{22} \neq 0$. Thus, the remaining question is how large γ can be to keep S (in form of an LFT) nonsingular.

Theorem 1 ([3, 4]) *The LFT of S is well-defined and S keeps nonsingularity over Ω_γ if and only if*

$$\gamma \leq 1 / \max\{\mu(M_{11}), \mu(M_{11} - M_{12}M_{22}^{-1}M_{21})\} \quad (8)$$

where $\mu(\cdot)$ is the structured singular value [5] with respect to the set of real matrices of the form Δ .

From the above theorem and the conversion techniques introduced, it is clear that the robust nonsingularity problems which can be dealt with by the method of this paper is more general than those in [1, 2]. It is also noted that the sizes of resulting matrices are often smaller than that obtained by using Kronecker product (if possible) from numerical experiences.

3. Robust Stability Analysis

Consider linear uncertain systems of order n with the state-space model

$$\begin{aligned} \delta x(t) &= A(k(t))x(t) \\ &= [A_0 + \sum_{i=1}^m f_i(k(t))A_i]x(t), \end{aligned} \quad (9)$$

where δ denotes the differentiation operator for continuous-time systems, or the shift operator for discrete-time systems, $x(t) \in \mathbb{R}^n$ is the state vector, $k(t)$ is the vector of q time-varying uncertain parameters, f_i 's are bounded rational functions similar to that in eq. (4) but $f_i(0) = 0$, and A_i is the structural influence matrix through which f_i affects $A(k(t))$. Moreover, it is assumed that A_0 is stable. The argument t will be omitted if no ambiguities arise in the following discussion.

Define the Lyapunov function candidate by $V(x) = x^T P x$, where P satisfies

$$A_0^T P + P A_0 = -Q \quad (10)$$

for the continuous-time case and

$$A_0^T P A_0 + P = -Q \quad (11)$$

for discrete-time case, and Q is positive definite. Thus, the Lyapunov derivative (difference) $\mathcal{L}(V(x))$ becomes

$$\begin{aligned} \mathcal{L}(V(x)) &= x^T L(k)x \\ &= x^T F(M, \Delta(k))x \end{aligned} \quad (12)$$

where $L(k) = -Q + \sum_{i=1}^m f_i(k)(A_i^T P + P A_i)$ for continuous-time systems and $L(k) = -Q + \sum_{i=1}^m f_i(k)(A_0^T P A_i + A_i^T P A_0) + \sum_{i,j=1}^m f_i(k)f_j(k)A_i^T P A_j$ for discrete-time systems. Note that for either systems $L(k)$ has bounded rational entries, and hence we can rewrite $L(k)$ as $F(M, \Delta(k))$. Since $M_{22} = -Q$, by using the concept of robust nonsingularity [2], the following robust stability theorem can be obtained.

Theorem 2 *System (9) is asymptotically stable if*

$$\|k\|_\infty < \left\{ \max\{\mu(M_{11}), \mu(M_{11} - M_{12}M_{22}^{-1}M_{21})\} \right\}^{-1}. \quad (13)$$

From the above theorem, it is clear that the proposed method can deal with a wider class of uncertainties than those in [1, 2, 7, 8]. Furthermore, for a given Q , the derived bound is maximal to ensure the negativeness of $\mathcal{L}(V(x))$ in the sense of l_∞ norm. However, it is noted that different choices of Q may affect the bound found.

4. Conclusion

In this paper, we develop an LFT approach to solve the robust nonsingularity problem via μ -analysis. It is shown that the proposed method can deal with a wider variety of uncertainties, such as the mixed unstructured and/or rational type structured uncertainties, and the obtained uncertainty matrix is often of smaller size. Also, an application in robust stability analysis is given to show the usefulness of the method.

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