An Adaptive Variable Structure Control for
Fast Time-Varying Unknown PlantsChiang-Ju Chien¹ and Li-Chen Fu^{1,2}¹Department of Electrical Engineering²Department of Computer Science and Information Engineering
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Abstract

In this paper, a combination of adaptive control and variable structure control methodology is proposed to solve the general fast time-varying plants with totally unstructured parameters. We show that the control scheme will work only under the assumptions of known order, relative degree, high frequency gain sign and minimum-phase property. In contrast to the conventional approaches where the variable structure design is applied mainly to the adaptation of some traditional parameters Θ by using the knowledge of unknown bounds on control parameters, a new adaptation scheme for control parameters incorporating the variable structure concept is given. Owing to this new scheme, we forsake the standard requirement of the upper bounds on some unknown parameters, which is frequently adopted in the field of robust linear MRAC using variable structure design. It is shown that without any persistent excitation, the tracking error will converge to zero in finite time for relative degree one plants and to a small residual set exponentially (whose size depends on a design parameter) for plants with any higher relative degree.

1 INTRODUCTION

A longstanding question which can be found in the literature of model reference adaptive control (MRAC) problem is how to control linear fast time-varying unknown plants adaptively. In most of the existing general studies of an adaptive control problem with time-varying plants, it is common to consider the parameter variations either as being slow but completely unknown [3, 4, 6, 10, 13, 14, 15] or as being fast but with known structure [11, 12]. The main consideration comes from the intuitive idea that a robust adaptive controller for linear time-invariant plants should also perform well in the face of slow time variation of plant parameters. The most recent and important results were given by [11, 12] where special controller structures with time-varying control parameters for both model reference control problem and indirect pole placement control problem are proposed. These controller structures will meet the control objective exactly when the linear time-varying plants are known with arbitrary fast, but finite, speed of parameter variations. The salient feature when in comparison with the existing ones which deal with the same problem is that the time-varying controllers solve the "matching" problem exactly for arbitrary fast but smooth variations of plant parameters. However, when the parameter structure of the time-varying plants is unknown, the assumption of slow time-variations is still crucial for the stability analysis of the closed-loop system there.

The only works to date that solve the model reference control problem for arbitrarily fast time-varying unknown plants are given by [1] and [9] where model reference control is combined with variable structure control design or robust control design, respectively. The crucial assumption of the slow time-variation of parameters is removed in both schemes. Either in the variable structure control scheme or in the robust control scheme, the time-varying effects are treated as some disturbances or uncertainties and then, some suitable bounding functions are generated for controller design such that the disturbances or uncertainties can be rejected. However, some bounds on the time-varying control parameters or plant parameters are needed for both schemes. In other words, the time-varying parameters of the plants are not totally unstructured.

In this paper, a successful combination of adaptive control and variable structure control methodology is proposed to solve the general fast time-varying plants with totally unstructured parameters. We show that the control scheme will work only under the assumptions of known order, relative degree, high frequency gain sign and minimum-phase property and can subject to any bounded input, output disturbances or a class of unmodeled dynamics. The main feature of the proposed adaptive variable structure scheme is summarized as follows.

- (1) In contrast to the conventional approaches where the variable structure design is applied mainly to the adaptation of some traditional parameters Θ by using the knowledge of unknown

bounds on control parameters, a new adaptation scheme for control parameters incorporating the variable structure concept is given. Owing to this new scheme, we forsake the standard requirement of the upper bounds on some unknown parameters, which is frequently adopted in the field of robust linear MRAC using variable structure design. This new feature will be shown to retain the excellent merits of the variable structure design but to require the least amount of conditions on the plant to be controlled.

- (2) Without any persistent excitation, the tracking error will converge to zero in finite time for relative degree one plants and to a small residual set exponentially (whose size depends on a design parameter) for plants with any higher relative degree.
- (3) With a suitable choice of control parameters in the very beginning, the transient behavior of the tracking errors is improved. From the computer simulations, the desired results are observed which are hardly achievable by traditional MRAC schemes, especially for time-varying plants.

In the subsequent discussions, the following notations will be used.

- (1) $G(s)[u](t)$: denotes the filtered version of $u(t)$ with any proper or strictly proper transfer function (matrix) $G(s)$.
- (2) $|\cdot|$: denotes the absolute value of any scalar or the norm of any vector or matrix.
- (3) $\|(\cdot)\|_\infty = \sup_{\tau \leq t} |(\cdot)(\tau)|$: denotes the truncated L_∞ norm of the argument function or vector.
- (4) $\|G(s)\|_\infty$: denotes the H_∞ norm of the transfer function $G(s)$.

2 SYSTEM DESCRIPTION AND NEW MATCHING
CONDITION

Consider a single-input single-output (SISO) linear time-varying plant which can be described by the following form of differential equations:

$$\begin{aligned} D_p(s, t)[x_p](t) &= u_p(t); & X_p(0) &= X_{p0} \\ y_p(t) &= k_p(t)N_p(s, t)[x_p](t) \end{aligned} \quad (2.1)$$

where $x_p \in R^n$ is an internal state; $u_p, y_p \in R$ are input and output of the plant, respectively; $D_p(s, t), N_p(s, t)$ are monic linear time-varying PDO's (see [11, 12]); $k_p(t) \neq 0, \forall t \geq 0$, is the high frequency gain and $X_p(0) = [x_p(0), \dots, x_p^{(n-1)}(0)]^T$ is the initial condition. Several assumptions on the plant are made as follows:

- (S1) $D_p(s, t), N_p(s, t)$ are strongly right coprime PDO's with known degrees n and m , respectively, and with unknown time-varying coefficients. Furthermore, all the plant parameters (the coefficients of the PDO's) are assumed to be uniformly bounded smooth functions of t .
- (S2) The sign of $k_p(t)$ is constant with respect to t and is known *a priori*. Without loss of generality, we will assume that $k_p(t) > 0, \forall t \geq 0$, and the range of $k_p(t)$ is a subset of a closed interval on the real axis that does not contain 0.
- (S3) $N_p^{-1}(s, t)$ is an exponentially stable PIO.

The control objective is to design a control law $u_p(t)$ such that the output $y_p(t)$ of the plant tracks the output $y_m(t)$ of a linear time-invariant reference model

$$y_m(t) = M(s)[r_m](t) = k_m D_m^{-1}(s)[r_m](t)$$

where $r_m(t)$ is a uniformly bounded reference signal and the reference model satisfies the following assumptions :

(S4) $D_m(s)$ is a monic Hurwitz polynomial of degree $n - m$ and the sign of k_m is positive.

For solving the MRAC problem with time-invariant plants, the standard model reference control (SMRC) scheme [7] is usually preferred with auxiliary signals w_i being generated as

$$\dot{w}_1(t) = \Lambda w_1(t) + b u_p(t), \quad \dot{w}_2(t) = \Lambda w_2(t) + b y_p(t) \quad (2.2)$$

where (Λ, b) is a controllable pair with $\Lambda \in \mathbb{R}^{n-1 \times n-1}$ being a stable matrix. The input for SMRC scheme is designed as $u_p(t) = \Theta^T(t)w(t)$ with $w(t) = [r_m(t), w_1^T(t), w_2^T(t), y_p(t)]^T$ and $\Theta(t) = [\theta_0(t), \theta_1^T(t), \theta_2^T(t), \theta_3(t)]^T$. It has been shown that there exist unique parameters $\Theta^* \in \mathbb{R}^{2n}$ such that the closed-loop system matches the given reference model when plant is time-invariant [7] and the fact fails when plant becomes time-varying [10, 11]. However, a new model reference control (NSMRC) scheme [11] is proposed by composing the control input from the outputs of observable filters instead of the controllable one and then, obtaining an exact match between the closed-loop system and the reference model. Later, in our work [1], the NSMRC scheme, which outgrows from the SMRC scheme, is also proposed to provide a condition for an exact match. The following lemma summarizes the condition.

Lemma 2.1 : There exists a vector $\Theta^*(t)$ such that if $u_p(t)$ is designed as :

$$\begin{aligned} u_p &= \left\{ \theta_0^*(t)r_m(t) + \theta_1^{*T}(t)w_1(t) + \theta_2^{*T}(t)w_2(t) + \theta_3^*(t)y_p(t) \right\} \\ &+ \left\{ d^{-1}(s)X_1^*(s,t)d^{-1}(s)[u_p](t) + d^{-1}(s)X_2^*(s,t)d^{-1}(s)[y_p](t) \right\} \\ &= \Theta^{*T}(t)w(t) + \eta_1(t) + \eta_2(t) \end{aligned} \quad (2.3)$$

where $d(s) = \det(sI - \Lambda)$ and $X_i^*(s, t)$ is a PDO whose coefficients depend on the derivatives of $\theta_i^*(t)$, for $i = 1, 2$, the I/O operator of the closed-loop plant $G_c(s, t) : r_m \rightarrow y_p$ is equal to the time-invariant reference model. $\square$

Proof : See [2] for detail.

Remark 2.2 : It is clear from (2.3) that compensators used in the NSMRC scheme differ from those in the SMRC structure mainly in the extra terms $\eta_i(t)$, $i = 1, 2$, which accounts for the time-varying effect of the plant parameters. This fact reveals that it is impossible to get an exact matching condition by the SMRC scheme when plant is time-varying, which has been proved in [10]. Actually, this can be attributed to the fact that the SMRC scheme is insufficient by the additional signals $\eta_i(t)$, $i = 1, 2$. For slowly time-varying case, it usually assumes that $|\frac{d}{dt}\theta_i^*(t)| \leq \mu^*$, $i = 0, 1, 2, 3$, for some small constant $\mu^* > 0$, which implies that $X_i^*(s, t)$ in (2.3) is approximate to a zero operator. However, under our adaptive variable structure design, the requirement of slowly time-varying rate is not needed. In other words, the perturbations $\eta_1(t)$ and $\eta_2(t)$ from the effect of time-varying parameters can be arbitrarily large. $\square$

Now, the matching condition in Lemma 2.1 will be used to derive a suitable error model if the time-varying parameters of the plant are unknown. It is noted that under the matching condition (2.3), the plant output can always be expressed, from an operator point of view, as :

$$y_p(t) = M(s)\left[\theta_0^{*-1}(u_p - \Theta^{*T}w - \eta_1 - \eta_2) + r_m\right](t)$$

Hence, define the output tracking error e_o by $e_o = y_p - y_m$, then we can get the output error equation as :

$$e_o(t) = M(s)\left[\theta_0^{*-1}(u_p - \Theta^{*T}w - \eta_1 - \eta_2)\right](t) \quad (2.4)$$

Let (A_m, b_m, c_m) be any minimal realization of the reference model $M(s)$, the state space representation of (2.4) will become :

$$\begin{aligned} \dot{e}(t) &= A_m e(t) + b_m \theta_0^{*-1}(t)(u_p(t) - \Theta^{*T}(t)w(t) - \eta_1(t) - \eta_2(t)) \\ e_o(t) &= c_m^T e(t) \end{aligned} \quad (2.5)$$

Remark 2.3 : In the error equation (2.5), $\Theta^{*T}(t)w(t)$ is the uncertainty due to unknown parameters of the plant and $\eta_i(t)$, $i = 1, 2$,

is the uncertainty resulted from time-variation effect. In traditional MRAC systems, one usually uses controller (2.2) with a suitable parameter update law, e.g., integral update laws with σ switching modification [5], to overcome these uncertainties when they are small enough. Here, we will focus on the design of an adaptive variable structure control u_p instead such that only some constant upper bounds on unknown system parameters, but not the time-varying control parameters themselves, are to be updated. $\square$

3 ADAPTIVE VARIABLE STRUCTURE CONTROLLER FOR TIME-VARYING PLANTS

The adaptive variable structure controller is designed, respectively, for fast time-varying unknown plants with relative degree one or greater than one. The main difference between these two cases can be described as follows. When the plant is relative degree one, the reference model $M(s)$ can be chosen to be strictly positive real (SPR) [8]. Moreover, the control structure and its subsequent analysis of global stability and tracking performance are much simpler for relative degree one systems. On the contrary, when the relative degree is greater than one, the reference model $M(s)$ is no longer SPR so that the controller design becomes more complicated. But based on the design for relative degree one systems, the general case can then be achieved in a more straightforward manner. In this section, we will first introduce the adaptive variable structure controller and give the analysis of this control scheme in the next section.

• The Adaptive Variable Structure Controller for Relative Degree One Plants

For fast time-varying unknown systems with relative degree one, the reference model $M(s)$ is chosen to be SPR and the adaptive variable structure controller u_p is designed as follows :

$$u_p(t) = -sgn(e_o)(\beta_1(t)|w(t)| + \beta_2(t)m(t)) \quad (3.1)$$

$$sgn(e_o) = \begin{cases} 1 & \text{if } e_o > 0 \\ 0 & \text{if } e_o = 0 \\ -1 & \text{if } e_o < 0 \end{cases}$$

where the control parameters $\beta_1(t), \beta_2(t)$ are updated as

$$\begin{aligned} \dot{\beta}_1(t) &= |e_o(t)||w(t)| \\ \dot{\beta}_2(t) &= |e_o(t)|m(t) \end{aligned} \quad (3.2)$$

with $\beta_i(0) > 0$ and $m(t)$ being the normalization signal defined as the state of the following system :

$$\dot{m}(t) = -\delta_0 m(t) + \delta_1(|u_p(t)| + |y_p(t)| + 1), \quad m(0) > \frac{\delta_1}{\delta_0} \quad (3.3)$$

where $\delta_0, \delta_1 > 0$ and $\delta_0 + \delta_2 < \lambda$ for some $\delta_2 > 0$. The parameter $\lambda > 0$ is such that the roots of $d(s - \lambda)$ are located at the left hand side of complex s -plane. Since $d(s) = \det(sI - \Lambda)$, design of the normalization signal (3.3) can always be achieved without any *a priori* knowledge about the time-varying plants. This is quite different from the use of normalization signal in the context of robust adaptive control schemes where some prior knowledge about stability margins of unmodeled dynamics and plant's zero dynamics are usually needed.

• The Adaptive Variable Structure Controller for Arbitrary Relative Degree Plants

Systematic Design Procedure

- (1) Choose an operator $L_1(s) = \ell_1(s) \cdots \ell_{p-1}(s) = (s + \alpha_1) \cdots (s + \alpha_{p-1})$ such that $M(s)L_1(s)$ is SPR and denote $L_i(s) = \ell_i(s) \cdots \ell_{p-1}(s)$, $L_p(s) = 1$.
- (2) Define the augmented signal

$$y_a(t) = M(s)L_1(s)\left[-v_1 + \frac{1}{L_1(s)}[u_p]\right](t) \quad (3.4)$$

and auxiliary errors

$$\begin{aligned} e_{a1}(t) &= e_o(t) - y_a(t) \\ e_{a2}(t) &= -[v_1]_{av} + \frac{1}{\ell_1(s)}[v_2](t) \\ &\vdots \\ e_{ap}(t) &= -[v_{p-1}]_{av} + \frac{1}{\ell_{p-1}(s)}[v_p](t) \end{aligned} \quad (3.5)$$

where $[v_i]_{av}$ is the average control of u_i which is given as $[v_i]_{av} = \frac{1}{F(\tau)}[v_i] = \frac{1}{(\tau+1)^2}[v_i]$ with τ being small enough.

(3) Design the control signals $u_p, v_i, i = 1, \dots, \rho$, as follows :

$$\begin{aligned} v_i(t) &= -sgn(e_{ai}(t))(\beta_{i1}(t)\xi_i(t) + \beta_{i2}(t)m(t) + \beta_{i3}(t)) \\ u_p(t) &= v_p(t) \end{aligned} \quad (3.6)$$

where

$$\begin{aligned} \xi_1(t) &= \frac{1}{\ell_1(s)} \cdots \frac{1}{\ell_{\rho-1}(s)} I[|w|](t) = \frac{1}{L_1(s)}[|w|](t) \\ \xi_i(t) &= \frac{1}{F^{i-1}(\tau s)} \frac{1}{\ell_1(s)} \cdots \frac{1}{\ell_{\rho-1}(s)} I[|w|](t) \\ &= \frac{1}{F^{i-1}(\tau s)} \frac{1}{L_i(s)}[|w|](t), \quad i = 2, \dots, \rho \\ \dot{m}(t) &= -\delta_3 m(t) + \delta_4(|u_p(t)| + |y_p(t)| + 1), \quad m(0) > \frac{\delta_4}{\delta_3} \end{aligned} \quad (3.7)$$

with $\delta_3, \delta_4 > 0$ and $\delta_3 + \delta_5 < \min(\lambda, \alpha_1, \dots, \alpha_{\rho-1}, \frac{1}{\tau})$ for some $\delta_5 > 0$.

(4) Finally, the adaptation law for the control parameters $\beta_{i1}, \beta_{i2}, \beta_{i3}, i = 1, \dots, \rho$, are given as follows :

$$\begin{aligned} \dot{\beta}_{i1}(t) &= |e_{ai}(t)|\xi_i(t) \\ \dot{\beta}_{i2}(t) &= |e_{ai}(t)|m(t) \\ \dot{\beta}_{i3}(t) &= |e_{ai}(t)| \end{aligned} \quad (3.8)$$

with $\beta_{i1}(0) > 0, \beta_{i2}(0) > 0, \beta_{i3}(0) > 0$.

The reasons for construction of the regressor signals $\xi_1, \dots, \xi_\rho$ will be clear in the analysis later. Now, in order to analyze the tracking performance and the closed-loop stability of the proposed adaptive variable structure controller, we first give the following key lemma which is very important in the subsequent analysis.

Lemma 3.1 : Under the controller design as in (3.1), (3.3) for relative-degree one case or in (3.4) - (3.7), in the general relative-degree case, if the control parameters $\beta_i(t), \beta_{ij}(t), i = 1, 2, 3, j = 1, \dots, \rho$, are uniformly bounded, then the plant input u_p will be bounded by the plant output y_p , i.e., there exists constant $\kappa > 0$ such that

$$\|(u_p)_t\|_\infty \leq \kappa \|(y_p)_t\|_\infty + \kappa \quad (3.9)$$

□

Proof : See [2] for detail.

4 ANALYSIS OF GLOBAL STABILITY AND TRACKING PERFORMANCE

In this section, a complete analysis of global stability and tracking performance for the proposed adaptive variable structure controller will be given.

4.1 RELATIVE DEGREE ONE CASE

Theorem 4.1 : Consider the system (2.1) satisfying (S1) - (S3) with $n - m = 1$. If the control input is designed as in (3.1) (3.3) with adaptation law (3.2), then, for any fast but fixed time-varying rate of plant parameters, the output tracking error will converge to zero at finite time while all signals inside the closed-loop system remain uniformly bounded. □

Proof : The proof consists of two parts.

Part I : Prove the boundedness of all signals inside the closed-loop system.

Consider the realization of the error model in (2.5). Since $\deg(X_i^*(s, t)) \leq (n-1) + (n-1) - 1 = 2n-3$ which along with the fact that $\deg(d(s)) = n-1$ implies $d^{-1}(s)X_i^*(s, t)d^{-1}(s)$ is a strictly proper stable operator. Hence, we can show that

$$|\eta_i(t)| \leq k_i^* m(t), \quad i = 1, 2$$

for some positive constants $k_i^*, i = 1, 2$. Furthermore, it is noted that

$\theta_0^*(t) > 0, \theta_0^{*-1}(t) > 0, \forall t \geq 0$ and

$$\sup_t |\theta_0^{*-1}(t)| \triangleq k_3^*, \quad \inf_t |\theta_0^{*-1}(t)| \triangleq k_4^*$$

exist for some $k_3^*, k_4^* > 0$ according to the assumption (S2). Hence, if we define

$$\beta_1^* = \frac{k_3^*}{k_4^*} \sup_t |\Theta^*(t)|, \quad \beta_2^* = \frac{k_3^*}{k_4^*} (k_1^* + k_2^*) \quad (4.1)$$

and construct a Lyapunov function,

$$V_a = \frac{1}{2} e^T P_m e + \sum_{i=1}^2 \frac{1}{2} k_4^* (\beta_i - \beta_i^*)^2$$

where

$$A_m^T P_m + P_m A_m = -Q_m, \quad P_m b_m = c_m \quad (4.2)$$

due to the SPR property of reference model $M(s)$, then the time derivative of V_a along the trajectories of (2.5), (3.2) subject to u_p in (3.1), (3.3) can be computed as :

$$\begin{aligned} \dot{V}_a &= -e^T Q_m e + e^T P_m b_m \theta_0^{*-1} (u_p - \Theta^* w - \eta_1 - \eta_2) \\ &\quad + \sum_{i=1}^2 k_4^* (\beta_i - \beta_i^*) \dot{\beta}_i \end{aligned} \quad (4.3)$$

which implies that

$$\begin{aligned} \dot{V}_a &\leq -q_m |e|^2 - |e_o| \theta_0^{*-1} (\beta_1 |w| + \beta_2 m - |\Theta^*| |w| - |\eta_1| - |\eta_2|) \\ &\quad + \sum_{i=1}^2 k_4^* (\beta_i - \beta_i^*) \dot{\beta}_i \\ &\leq -q_m |e|^2 - |e_o| k_4^* (\beta_1 |w| + \beta_2 m) \\ &\quad + |e_o| k_3^* \left(\sup_t |\Theta^*(t)| |w| + k_1^* m + k_2^* m \right) \\ &\quad + \sum_{i=1}^2 k_4^* (\beta_i - \beta_i^*) \dot{\beta}_i \\ &\leq -q_m |e|^2 - |e_o| k_4^* \left(\beta_1 |w| - \frac{k_3^*}{k_4^*} \sup_t |\Theta^*(t)| |w| \right. \\ &\quad \left. + \beta_2 m - \frac{k_3^*}{k_4^*} (k_1^* + k_2^*) m \right) \\ &\quad + \sum_{i=1}^2 k_4^* (\beta_i - \beta_i^*) \dot{\beta}_i \\ &\leq -q_m |e|^2 \end{aligned} \quad (4.4)$$

for some positive constant q_m . This implies that $e \in L_2 \cap L_\infty$ and $\beta_1, \beta_2, y_p, e_o \in L_\infty$ and hence, all signals inside the closed-loop system are uniformly bounded owing to Lemma 3.1. On the other hand, we can conclude that $\dot{e} \in L_\infty$ by (2.5) and hence, e and e_o will at least converge to zero asymptotically by Barbalat's lemma [8].

Part II : Investigate the tracking performance of output errors e_o .

Supposing the output error doesn't reach zero in some finite time and remain zero afterwards, the control parameters $\beta_i(t), i = 1, 2$ will increase according to the adaptation law (3.2). This implies that there exists a time constant T_1 such that $\beta_i(t) \geq \frac{\beta_i^* k_4^*}{k_3^*}, i = 1, 2, \forall t \geq T_1$. At this moment, consider another Lyapunov function $V_b(e) = \frac{1}{2} e^T P_m e$ for the time interval $[T_1, \infty)$. The time derivative of V_b becomes

$$\begin{aligned} \dot{V}_b &= -e^T Q_m e - |e_o| \theta_0^{*-1} \left(\beta_1 |w| + \beta_2 m - \sup_t |\Theta^*(t)| |w| \right. \\ &\quad \left. - (k_1^* + k_2^*) m \right) \\ &\leq -e^T Q_m e \\ &\leq -k_5 V_b \end{aligned}$$

for some $k_5 > 0$, since $\beta_i(t) \geq \frac{\beta_i^* k_4^*}{k_3^*}, \forall t \geq T_1$. This implies that e approaches zero at least exponentially fast when $t \geq T_1$. Furthermore, by the fact that

$$\begin{aligned} e_o \dot{e}_o &= e_o c_m \dot{e} \\ &\leq |e_o| \left\{ |c_m A_m| |e| - c_m b_m \theta_o^{*-1} \right. \\ &\quad \left. \left[\left(\beta_1 - \sup_t |\Theta^*(t)| \right) |w| + (\beta_2 - k_1^* - k_2^*) m \right] \right\} \end{aligned}$$

there exists a finite time $T_2 > T_1 > 0$ such that $e_o \dot{e}_o \leq -k_3 |e_o| < 0$ for some $k_3 > 0$ for all $t \geq T_2$ and $|e_o| \neq 0$. This implies that the sliding surface $e_o = 0$ is guaranteed to be reached in finite time.

4.2 ARBITRARY RELATIVE DEGREE CASE

In order to analyze the closed-loop stability and the tracking performance, the error equation (2.4) is first rewritten as :

$$e_o(t) = M(s) \left[u_p - \theta_o^{*-1} \Theta^* w - \theta_o^{*-1} \eta_1 - \theta_o^{*-1} \eta_2 - (1 - \theta_o^{*-1}) u_p \right] (t) \quad (4.5)$$

Then, by the design of the above auxiliary errors (3.5), we can find that $e_{a1}(t)$ will satisfy

$$\begin{aligned} e_{a1}(t) &= M(s) L_1(s) \left[v_1 - \frac{1}{L_1(s)} \left[\theta_o^{*-1} \Theta^* w \right] \right. \\ &\quad \left. - \frac{1}{L_1(s)} \theta_o^{*-1} \left[\eta_1 + \eta_2 + (\theta_o^* - 1) u_p \right] \right] (t) \quad (4.6) \end{aligned}$$

and hence,

$$\begin{aligned} e_{a1}(t) &= \frac{1}{L_{i-1}(s)} \left[v_i - \frac{1}{F^{i-1}(\tau s)} \frac{1}{L_i(s)} \left[\theta_o^{*-1} \Theta^* w \right] \right. \\ &\quad \left. - \frac{1}{F^{i-1}(\tau s)} \frac{1}{L_i(s)} \theta_o^{*-1} \left[\eta_1 + \eta_2 + (\theta_o^* - 1) u_p \right] - \varepsilon_{i-1} \right] \quad (4.7) \end{aligned}$$

where

$$\begin{aligned} \varepsilon_1 &= \frac{1}{F(\tau s)} \frac{1}{L_2(s)} M^{-1}(s) [e_{a1}] \\ \varepsilon_{i-1} &= \frac{1}{F(\tau s)} \varepsilon_{i-1}(s) [\varepsilon_{i-2}] + \frac{1}{F(\tau s)} \varepsilon_{i-1}(s) \varepsilon_{i-2}(s) [e_{a_{i-1}}] \end{aligned}$$

for $i = 3, \dots, \rho$. According to the above observations, the following lemma tells us that $\xi_i, i = 1, \dots, \rho$, designed in the adaptive controller are suitable candidates as bounding functions.

Lemma 4.2 : There exist some positive constants $\beta_{i1}^*, \beta_{i2}^*, i = 1, \dots, \rho$ such that

$$\begin{aligned} (1) \quad \left| \frac{1}{F^{i-1}(\tau s)} \frac{1}{L_i(s)} \left[\theta_o^{*-1} \Theta^* w \right] \right| &\leq \beta_{i1}^* \frac{1}{F^{i-1}(\tau s)} \frac{1}{L_i(s)} [|w|] \\ &= \beta_{i1}^* \xi_i \quad (4.8) \end{aligned}$$

$$(2) \quad \left| \frac{1}{F^{i-1}(\tau s)} \frac{1}{L_i(s)} \theta_o^{*-1} \left[\eta_1 + \eta_2 + (\theta_o^* - 1) u_p \right] \right| \leq \beta_{i2}^* m \quad (4.9)$$

for $i = 1, \dots, \rho$. $\square$

Proof : See [2] for detail.

We are now ready to state the main result of the systematically designed adaptive variable structure controller for linear fast time-varying unknown plants with arbitrary relative degree.

Theorem 4.3 : Consider the system (2.1) satisfying (S1) - (S3) with $n - m > 1$. If the control input is designed as in (3.6) (3.7) and the adaptation law is chosen as in (3.8), then for any fast, but finite, time-varying rate of plant parameters, there exists a $\tau^* > 0$ such that for all $\tau \in (0, \tau^*]$, the following facts will hold,

- (1) all signals inside the closed-loop system remain uniformly bounded,
- (2) the auxiliary errors $e_{ai}, i = 1, \dots, \rho$, converge to zero in finite time,
- (3) the output tracking errors e_o will converge to a residual set exponentially whose size is a class K function of τ . $\square$

Proof : The proof consists of three parts.

Part I : Prove the boundedness of e_{ai} and $\beta_{ij}, i = 1, \dots, \rho, j = 1, 2, 3$.

Step 1 : First, consider the auxiliary error e_{a1} which satisfies (4.6). Since $M(s)L_1(s)$ is SPR, we have the following realization of (4.6)

$$\begin{aligned} \dot{e}_1 &= A_1 e_1 + b_1 \left(v_1 - \frac{1}{L_1(s)} \left[\theta_o^{*-1} \Theta^* w \right] \right. \\ &\quad \left. - \frac{1}{L_1(s)} \theta_o^{*-1} \left[\eta_1 + \eta_2 + (\theta_o^* - 1) u_p \right] \right) \\ e_{a1} &= c_1 e_1 \end{aligned}$$

where $A_1^T P_1 + P_1 A_1 = -Q_1$, $P_1 b_1 = c_1^T$ for some $P_1 = P_1^T > 0$ and $Q_1 > 0$. If we choose a Lyapunov function as :

$$V_1 = \frac{1}{2} e_1^T P_1 e_1 + \frac{1}{2} (\beta_{11} - \beta_{11}^*)^2 + \frac{1}{2} (\beta_{12} - \beta_{12}^*)^2 + \frac{1}{2} \beta_{13}^2$$

and apply the controller (3.6) and adaptation law (3.8), then

$$\begin{aligned} \dot{V}_1 &= -e_1^T Q_1 e_1 + e_{a1}^T \left(v_1 - \frac{1}{L_1(s)} \left[\theta_o^{*-1} \Theta^* w \right] \right. \\ &\quad \left. - \frac{1}{L_1(s)} \theta_o^{*-1} \left[\eta_1 + \eta_2 + (\theta_o^* - 1) u_p \right] \right) \\ &\quad + (\beta_{11} - \beta_{11}^*) \dot{\beta}_{11} + (\beta_{12} - \beta_{12}^*) \dot{\beta}_{12} + \beta_{13} \dot{\beta}_{13} \\ &\leq -q_1 |e_1|^2 - |e_{a1}| \left((\beta_{11} - \beta_{11}^*) \xi_1 + (\beta_{12} - \beta_{12}^*) m + \beta_{13} \right) \\ &\quad + (\beta_{11} - \beta_{11}^*) \dot{\beta}_{11} + (\beta_{12} - \beta_{12}^*) \dot{\beta}_{12} + \beta_{13} \dot{\beta}_{13} \\ &\leq -q_1 |e_1|^2 \end{aligned}$$

This implies that $e_1, \beta_{11}, \beta_{12}, \beta_{13} \in L_\infty$ and $e_{a1} \in L_2 \cap L_\infty$.

Step 2 : From (4.7), e_{a2} satisfies

$$\begin{aligned} \dot{e}_{a2} &= -\alpha_1 e_{a2} + \left(v_2 - \frac{1}{F(\tau s)} \frac{1}{L_2(s)} \left[\theta_o^{*-1} \Theta^* w \right] \right. \\ &\quad \left. - \frac{1}{F(\tau s)} \frac{1}{L_2(s)} \theta_o^{*-1} \left[\eta_1 + \eta_2 + (\theta_o^* - 1) u_p \right] - \varepsilon_1 \right) \end{aligned}$$

Note that $|\varepsilon_1| \leq \bar{\varepsilon}^*$ for some $\bar{\varepsilon}^* > 0$ since e_{a1} is bounded by the result in Step 1. Now choose the Lyapunov function

$$V_2 = \frac{1}{2} e_{a2}^2 + (\beta_{21} - \beta_{21}^*)^2 + (\beta_{22} - \beta_{22}^*)^2 + (\beta_{23} - \bar{\varepsilon}^*)^2$$

so that we can again show that $\dot{V}_2 \leq -e_{a2}^2 \leq 0$ if the controller in (3.6) and the update law in (3.8) are applied. This concludes that $e_{a2} \in L_2 \cap L_\infty$ and $\beta_{21}, \beta_{22}, \beta_{23} \in L_\infty$.

Step i : Since for $3 \leq i \leq \rho$

$$\begin{aligned} \dot{e}_{ai} &= -\alpha_{i-1} e_{ai} + \left(v_i - \frac{1}{F^{i-1}(\tau s)} \frac{1}{L_i(s)} \left[\theta_o^{*-1} \Theta^* w \right] \right. \\ &\quad \left. - \frac{1}{F^{i-1}(\tau s)} \frac{1}{L_i(s)} \theta_o^{*-1} \left[\eta_1 + \eta_2 + (\theta_o^* - 1) u_p \right] - \varepsilon_{i-1} \right) \end{aligned}$$

and $|\varepsilon_{i-1}| \leq \bar{\varepsilon}^*$ (with some abuse of notation), the Lyapunov function

$$V_i = \frac{1}{2} e_{ai}^2 + (\beta_{i1} - \beta_{i1}^*)^2 + (\beta_{i2} - \beta_{i2}^*)^2 + (\beta_{i3} - \bar{\varepsilon}^*)^2$$

can be used to guarantee $\dot{V}_i \leq -e_{ai}^2 \leq 0$ if the controller in (3.6) and the update law in (3.8) are applied. Consequently, $e_{ai} \in L_2 \cap L_\infty$ and $\beta_{i1}, \beta_{i2}, \beta_{i3} \in L_\infty$.

Part II : Prove the boundedness of all signals inside the closed-loop system.

Define $\bar{e}_{ai} = M(s)L_{i-1}(s)[e_{ai}], i = 2, \dots, \rho$ and $E_a = e_{a1} + \bar{e}_{a2} + \dots + \bar{e}_{a\rho}$ which is uniformly bounded due to the boundedness of e_{ai} . Then, from (3.5), it can be derived that

$$\begin{aligned} E_a &= e_o + M(s)L_1(s) \left[v_1 - \frac{1}{L_1(s)} [u_p] \right] \\ &\quad + M(s)L_1(s) \left[-[v_1]_{av} + \frac{1}{L_1(s)} [v_1] \right] \\ &\quad + M(s)L_2(s) \left[-[v_2]_{av} + \frac{1}{L_2(s)} [v_2] \right] \\ &\quad \vdots \\ &\quad + M(s)L_{\rho-1}(s) \left[-[v_{\rho-1}]_{av} + \frac{1}{L_{\rho-1}(s)} [v_{\rho-1}] \right] \end{aligned}$$

$$\begin{aligned}
&= e_o + \left(1 - \frac{1}{F(\tau s)}\right) M(s) L_1(s) \\
&\quad \left[v_1 + \frac{1}{\ell_1(s)} [v_2] + \dots + \frac{1}{\ell_1(s) \dots \ell_{p-2}(s)} [v_{p-1}] \right] \\
&\triangleq e_o + R
\end{aligned} \quad (4.10)$$

Now, since $\|(u_p)_t\|_\infty \leq \kappa \|(y_p)_t\|_\infty + \kappa$, it can be easily found that

$$\left\| \left(v_1 + \frac{1}{\ell_1(s)} [v_2] + \dots + \frac{1}{\ell_1(s) \dots \ell_{p-2}(s)} [v_{p-1}] \right)_t \right\|_\infty \leq k_7 \|(y_p)_t\|_\infty + k_7$$

Furthermore, since $\left\| \frac{1}{s} \left(1 - \frac{1}{F(\tau s)} \right) \right\|_\infty = 2\tau$ and $\|sM(s)L_1(s)\|_\infty = k_8$ for some $k_8 > 0$, we can conclude that

$$\begin{aligned}
\|(R)_t\|_\infty &\leq \left\| \frac{1}{s} \left(1 - \frac{1}{F(\tau s)} \right) \right\|_\infty \|sM(s)L_1(s)\|_\infty (k_7 \|(y_p)_t\|_\infty + k_7) \\
&\leq \tau(k_8 \|(y_p)_t\|_\infty + k_9) \\
&\leq \tau(k_{10} \|(e_o)_t\|_\infty + k_{10})
\end{aligned}$$

for some $k_9, k_{10} > 0$. Now, from (4.10) we have

$$\begin{aligned}
\|(e_o)_t\|_\infty &\leq \|(E_a)_t\|_\infty + \|(R)_t\|_\infty \\
&\leq \|(E_a)_t\|_\infty + \tau(k_{10} \|(e_o)_t\|_\infty + k_{10})
\end{aligned}$$

which implies that there exists a $\tau^* > 0$ such that $1 - \tau^* k_{10} > 0$ and for all $\tau \in (0, \tau^*)$,

$$\|(e_o)_t\|_\infty \leq \frac{\|(E_a)_t\|_\infty + \tau k_{10}}{1 - \tau k_{10}} \quad (4.11)$$

Since E_a is uniformly bounded, we readily conclude that e_o is uniformly bounded and all signals inside the closed loop system remain uniformly bounded.

Part III : Investigate the tracking performance of e_{ai} and e_o .

Since all signals inside the closed-loop system are uniformly bounded, we have

$$e_{ai} \in L_2 \cap L_\infty, \quad \dot{e}_{ai} \in L_\infty, \quad i = 1, \dots, \rho$$

Hence, by Barbalat's lemma, e_{ai} approaches zero asymptotically. By using the same argument given in Theorem 4.1, we can guarantee that e_{ai} actually converges to zero in finite time. Since e_{ai} reaches zero in finite time and $E_a = e_{a1} + \bar{e}_{a2} + \dots + \bar{e}_{a\rho}$, it can be concluded that E_a approaches zero exponentially. Now, from the fact of (4.11) and E_a approaching zero exponentially, it is clear that e_o will converge exponentially to a small residual set whose size depends on the design parameter τ . $\triangle\triangle$

5 NUMERICAL EXAMPLES

To demonstrate the effectiveness and the superior performance of the proposed controller, in this section a relative degree two examples is simulated via a computer. For a relative degree two plant, the example here is the time-varying plant which is described as follows :

$$(s^2 + a_1(t)s + a_2(t))[y_p](t) = u_p(t)$$

where $a_1(t)$ and $a_2(t)$ are the time-varying plant parameters. The design procedure of the adaptive variable structure controller is given as follows :

(1) Reference model and reference input :

$$M(s) = \frac{1}{(s+1)(s+2)}, \quad r_m(t) = 10\sin(t)$$

(2) Augmented signal and auxiliary errors :

$$\begin{aligned}
y_a(t) &= M(s)L_1(s) \left[-v_1 + \frac{1}{L_1(s)} [u_p] \right](t) \\
e_{a1}(t) &= e_o(t) - y_a(t) \\
e_{a2}(t) &= -[v_1]_{av} + \frac{1}{\ell_1(s)} [u_p](t) \\
L_1(s) &= (s+1.5) \\
e_o(t) &= y_p(t) - y_m(t) \\
[v_1]_{av} &= \frac{1}{\left(\frac{1}{20\pi} s + 1\right)^2} [v_1](t)
\end{aligned}$$

(3) The adaptive variable structure controller :

$$\begin{aligned}
v_1(t) &= -\text{sgn}(e_{a1}(t))(\beta_{11}(t)\xi_1(t) + \beta_{12}(t)m(t) + \beta_{13}(t)) \\
u_p(t) &= -\text{sgn}(e_{a2}(t))(\beta_{21}(t)\xi_2(t) + \beta_{22}(t)m(t) + \beta_{23}(t)) \\
\dot{w}_1(t) &= -2w_1(t) + u_p(t) \\
\dot{w}_2(t) &= -2w_2(t) + y_p(t) \\
w(t) &= [r_m(t), w_1(t), w_2(t), y_p(t)]^T \\
\xi_1(t) &= \frac{1}{L_1(s)} [|w|](t) \\
\xi_2(t) &= \frac{1}{F(\tau s)} [|w|](t) \\
\dot{m}(t) &= -0.1m(t) + 0.1(|u_p(t)| + |y_p(t)| + 1), \quad m(0) = 0.5
\end{aligned}$$

(4) Adaptation law :

$$\begin{aligned}
\dot{\beta}_{i1}(t) &= |e_o(t)|\xi_i(t) \\
\dot{\beta}_{i2}(t) &= |e_o(t)|m(t) \\
\dot{\beta}_{i3}(t) &= |e_o(t)|, \quad i = 1, 2, 3
\end{aligned}$$

with

$$\begin{aligned}
\beta_{11}(0) &= 15, \beta_{12}(0) = 3, \beta_{13}(0) = 1 \\
\beta_{21}(0) &= 15, \beta_{22}(0) = 3, \beta_{23}(0) = 1
\end{aligned}$$

Two simulation cases are studied in this example :

Case 1 : The time-varying parameters $a_1(t)$ and $a_2(t)$ are chosen to be -6 and $2\sin(\mu t)$ respectively, which is the same as that in [11]. However, the time-varying rate μ is set to be 3 which is much faster than the simulation example in [11]. The well behaved tracking performance and transient response are observed in Figure 5-1.

Case 2 : In the second case of simulation, $a_1(t)$ and $a_2(t)$ are chosen to be $-4\cos(\mu t)$ and $2\sin(\mu t)$ with $\mu = 30$ respectively such that both plant parameters are fast time-varying. This plant is more complex than that presented in the first case. All the designs remain the same as those in the first case. Figure 5-2 shows that the control objective is still achieved as expected.

6 CONCLUDING REMARKS

In this paper, a new model reference adaptive control approach has been proposed to solve the MRAC problem for linear fast time-varying unknown plants. Controllers are designed incorporating the concept of variable structure and normalization signal. Unlike the existing approaches, which deal with the same problem here but only with slowly time-varying parameters, no restriction on the time-varying rate of the plant parameters are needed in our new adaptive variable structure schemes. It is shown that without any PE condition, the output tracking error will converge to zero at finite time for relative degree one plants and to a small residual set exponentially for plants with relative degree greater than one. Furthermore, it is noted that the control scheme also guarantees the much improved transient behavior and tracking performance as compared with those from the traditional MRAC approaches even when the plant is time-invariant.

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Fig 5.1 (a) (time:sec)

Fig 5.1 (b) (time:sec)

Fig 5.1 (c) (time:sec)

Fig 5.1 (d) (time:sec)

Fig 5.2 (a) (time:sec)

Fig 5.2 (b) (time:sec)

Fig 5.2 (c) (time:sec)

Fig 5.2 (d) (time:sec)