Controllability of Spacecraft Systems in a Central Gravitational Field¹

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Abstract

The natural configuration space for spacecraft systems in a central gravitational field can be modeled by $SO(3) \times \mathbb{R}^3$, where the Special Orthogonal Group $SO(3)$ represents the attitude dynamics and $\mathbb{R}^3$ is for the orbital dynamics. While the attitude dynamics of the spacecraft system is affected by the orbital elements through the well-known gravity-gradient torque, it has not been noted before, to our knowledge, that the effects of attitude-orbit coupling can be used to change orbital motions by controlling the attitude. This controllability property is the main issue of this work. We shall prove that the control systems for spacecraft with reaction wheels or gas jets as attitude controllers are controllable.

1 Introduction

For a spacecraft system moving in a central gravitational field, the gas jets, or thrusters, acting directly on the orbital elements, are commonly used in orbital maneuvers. Typically, the satellite is modeled as a point mass and the orbits are simply Keplerian orbits. However, if a rigid body with finite extent is chosen to model the spacecraft, the orbital motion will be then coupled with the attitude dynamics. The Hamiltonian structure of such system and the effects of this coupling have been discussed in [28, 29, 31, 30]. With this coupling, it was then suspected that controlling the attitude motion may eventually affect the orbital motion. This then leads to the notion of controllability of the spacecraft system by using gas jets or reaction wheels which control *only* the attitude. This idea motivated our work in this paper.

The natural configuration manifold for spacecraft systems in a central gravitational field can be modeled as $SO(3) \times \mathbb{R}^3$, or abbreviated as $SE(3)$. The Special Orthogonal Group $SO(3)$ refers to the configuration space of attitude dynamics, while $\mathbb{R}^3$ is for the configuration space of orbital (or translational) dynamics. How to control the attitude motion and orbital motion (efficiently) during maneuver keeps being an important issue in spacecraft controller design. More specifically, various control strategies have been used in controlling the attitude. These include thrusters (gas jets), magnetic torquer, dual-spin technique, gravity-gradient torque, control-moment gyros, momentum wheels, and reaction wheels, see, e.g. [32, 33]. Concerning the efficiency, power, and accuracy of the control systems, an effective 3-axis stabilization scheme may use reaction wheels for attitude control with the momentum dumping through the thrusters. On the other hand, as described in the previous paragraph, orbital controls are mainly performed through gas jets. In previous design of such controllers, the gravity-gradient torque is sometimes included as disturbance torque or is used to assist the controller design. This gravity-gradient torque technique exhibits one of the coupling effects on the attitude from the orbital motion. However, as will be demonstrated later, this coupling also appears in orbital dynamical equations. Accordingly, how to assess this effect on orbital dynamics and how to properly use them to assist orbital control become an interesting question.

The controllability problem has attracted many investigators for its important role in control theory and applications. For a general nonlinear system, the sufficient and necessary conditions, i.e. the Lie algebra rank condition, for local accessibility (or weak controllability) for analytical systems was addressed in many places, such as [25, 14, 10]. In order to guarantee that a point p is in the interior of the reachable set from p for arbitrary small time, the notion of small time local controllability was introduced in [16]. Many sufficient conditions for the small time local controllability have been obtained in [6, 11, 9]. In [23], Sussman proved a general theorem accommodating these conditions by applying the idea of action of Lie groups on the state-space of the dynamical system. For general nonlinear systems the concept of (global) controllability is very sophisticated, cf. [5, 24]. Although there are no completely satisfactory criteria has been developed, there are fruitful results for some special classes of nonlinear systems. In [19], it has been proved that the controllability can be established for Poisson stable systems satisfying the Lie algebra rank condition. Some extension of this result, which only requires the drift vector field (i.e. the vector field with no controls) having the Poisson stability property, can be founded in [13], and [4]. These result-

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Figure 1: Spacecraft with Reaction Wheels in a Central Force Field

s have been applied in [8] to established the controllability property of the spacecraft system in which only attitude dynamics were considered. In this paper, a slight generalization of the controllability theory mentioned above will be constructed. The requirement of Poisson stability will be replaced by a weaker condition, i.e. *weakly positively Poisson stability*. This theorem will be then directly used to prove the controllability of the spacecraft systems, which include attitude-orbit coupling, controlled by using reaction wheels or gas jets in only attitude controllers. This is one of the main contributions of this work.

In fact, we shall show that for a dynamical system with weakly positively Poisson stable drift vector field, the Lie algebra rank condition will imply controllability. For the spacecraft system under consideration, its intrinsic Hamiltonian structure can be utilized to prove the weakly positively Poisson stability by applying Poincaré recurrence theorem. On the other hand, the work of checking the Lie algebra rank condition can be carried out with the aid of the general formula for the Lie bracket computation in $T^*T(SO(3)^m \times \mathbb{R}^n)$ developed in [17][18]. With these tools, the controllability can be then concluded.

The rest of this paper is organized as follows. A quick review of equations of motion for the spacecraft systems is given in Section 2. Formal definitions of controllability and weakly positive Poisson stability will be given in Section 3. The techniques developed in this section and the Lie bracket computation on jet bundles of $SO(3)$ are then used to prove the controllability of spacecraft systems in Section 4. In order to verify the Lie algebra rank condition, proper approximations for the gravity term are adopted. We shall prove that for some spacecraft *without any axis-symmetry* and satisfying some conditions, the controllability of the coupled system by using gas jets or reaction wheels in attitude controller can be concluded. Consequently, the coupling effects of attitude and orbit may be useful in controlling orbital motion for a spacecraft.

The result obtained in this work may be not intuitive at the first look. However, it has been noted, cf. [22], that the center of gravity of the assembly is different from the center of mass and the former may change its location with respect to different attitudes. This is one place where the coupling effects appear. From this observation, it is then natural that controlling the attitude may lead to some change of orbital motion. Accordingly, the controllability results are also physically appealing.

2 System Description

Consider a system consisting of a rigid body, or platform, with some symmetric rotors, moving in a central gravitational field, as depicted in Figure 1. Let E denote the center of a gravitational field (e.g. generated by the Earth) with gravitational constant μ . The system of the platform of mass m_0 and I

rotors of masses m_i , $i = 1, \dots, l$, moves around the center E . The total mass of the assembly B will be denoted by m . Let O denote the center of mass of the assembly. We attach an inertial reference frame $\{e_1, e_2, e_3\}$ to E and a body reference frame $\{b_1, b_2, b_3\}$ to O , respectively. The radius vector from E to O is represented by $r \in \mathbb{R}^3$ in inertial frame.

The attitude of the assembly can be modeled by an element $B \in SO(3)$, which represents the rotation of the body frame relative to the inertial frame. The special orthogonal group or rotation group $SO(3)$, is defined to be the set

$$SO(3) = \{B \in \mathbb{R}^{3 \times 3} | B^T B = I, \det(B) = 1\},$$

which is a subset of 3×3 nonsingular matrices. Here, and in what follows, I denotes the identity matrix of the appropriate dimension.

Let $so(3)$ denote the space of 3×3 skew-symmetric matrices. A natural isomorphism between $\mathbb{R}^3$ and $so(3)$ can be constructed as, with the operator,

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

The kinematic equation for a rotation matrix can be written as $\dot{B} = B\hat{\Omega}$, where $\hat{\Omega}$ is the instantaneous body angular velocity of the assembly.

For the i -th rotor, the location of its center of mass and orientation of its axis of symmetry will be denoted by $\tilde{d}_i$ and a unit vector u_i , respectively, both in body frame. The location of the center of mass of the platform will be denoted by $\tilde{d}_o$ in body frame. The motion of i -th rotor relative to the platform can be described by an angle θ_i , or equivalently, by an element in the circle group S^1 . As a consequence, the configuration space of the assembly can be modeled as $SO(3) \times T^n \times \mathbb{R}^3$.

Let I_o and I_i denote, respectively, the moments of inertial tensor of the platform and of the i -th rotor about their centers of mass. The equations of motion for the spacecraft system with l reaction wheels in a central gravitational field can be derived by using the Euler-Lagrange method [17] or Newton-Euler method [30] and are expressed as follows,

$$\begin{aligned} \dot{B} &= B\hat{\Omega}, \\ I\hat{\Omega} + \dot{\eta} &= -\Omega \times (I\hat{\Omega} + \eta) + \int_B \frac{\mu(B^T r \times \tilde{q})}{|r + B\tilde{q}|^3} dm(\tilde{q}), \\ I_i \Omega^T u_i + I_i \dot{\theta}_i &= \tau_i, \quad \forall i = 1, \dots, l, \\ m\ddot{r} &= - \int_B \frac{\mu(r + B\tilde{q})}{|r + B\tilde{q}|^3} dm(\tilde{q}), \end{aligned} \quad (1)$$

where $I = \sum_{i=0}^l (I_i + m_i \tilde{d}_i^T \tilde{d}_i)$, I_i stands for the eigenvalue of I_i along the eigenvector u_i , τ_i is the torque generated by the actuator acting on the i -th rotor, and η denotes the total angular momentum of rotors relative to the body,

$$\eta = \sum_{i=1}^l \theta_i I_i u_i. \quad (2)$$

3 Poisson Stability and Controllability

Before answering the controllability problem for the spacecraft system, we need to give formal definitions and obtain some criteria to suit our needs. Consider a smooth vector field X on a smooth connected manifold M . Let ϕ^x denote its flow:

$$\phi^x : \mathbb{R} \times M \rightarrow M; (t, p) \mapsto \phi_t^x(p).$$

Assume that X is complete or $\phi_t^x(p)$ is defined for all $t \geq 0$. A point p in M is called *positively Poisson stable* for X if for all $T > 0$ and for any neighborhood U_p of p , there exists a time $t > T$ such that

$$\phi_t^x(p) \in U_p.$$

The vector field X is called *positively Poisson stable* if the set of positively Poisson stable points for X is dense in M .

A point $p \in M$ is called a *nonwandering point* of X if for all $T > 0$, and for any neighborhood U_p of p , there exists a time $t > T$ such that

$$\phi_t^x(U_p) \cap U_p \neq \emptyset,$$

where $\phi_t^x(U_p) = \{\phi_t^x(q) : q \in U_p\}$. A positively Poisson stable point is necessarily a nonwandering point. Let Γ_x denote the *nonwandering set*, which is defined to be the set of all nonwandering points of X . We have the following lemma.

Lemma 1 *The nonwandering set of a positively Poisson stable vector field is M .* $\square$

Proof: Let S denote the set of positively Poisson stable points for X . By definition, the closure of S , $\text{cl}(S)$, is equal to M . That is for any $p \in M$, given U_p , a neighborhood of p , we have $U_p \cap S \neq \emptyset$. This implies that there exists a positively Poisson stable point q . Let $U_q \in U_p$, be a neighborhood of q . We have $\forall T > 0, \exists t > T, \phi_t^x(q) \in U_q$. Hence $\phi_t^x(q) \in U_p$, which implies

$\phi_t^x(U_p) \cap U_p \neq \emptyset$. Thus p is a nonwandering point. Since p is arbitrary, we prove $\Gamma_x = M$. $\square$

From this lemma, the conditions for a vector field with the whole manifold M being its nonwandering set are weaker than the conditions for it being positively Poisson stable. Accordingly, a vector field will be called *weakly positively Poisson stable* if its nonwandering set is M .

Recall that in classical mechanics Liouville theorem tells us that the flow generated by a time-independent Hamiltonian system is volume-preserving. On the other hand, Poincaré recurrence theorem proved that for a volume-preserving, bijective, continuous map g on a bounded domain $D \subset \mathbb{R}^n$, every neighborhood U of each point in D contains a point q which returns to U after repeated application of the mapping, i.e., $g^n(q) \in U$ for some positive integer n . By combining the above observations, cf. [3, 27], we have the following theorem:

Theorem 1 *A time-independent Hamiltonian vector field on a manifold which is bounded is weakly positively Poisson stable.* $\square$

A dynamical polysystem, c.f. [19], is defined to be a collection of complete vector fields on M , which can be denoted by $\mathcal{F} = \{X_\alpha, \alpha \in I\}$ where I is an index set. An integral curve of $\mathcal{F}$ is a mapping $\Phi : [t_0, t_f] \rightarrow M$ such that there exist vector fields, $X_1, \dots, X_k$ in $\mathcal{F}$ and $t_0 < t_1 < \dots < t_k = t_f$, with the property that the restriction of Φ to $[t_{i-1}, t_i]$ is an integral curve of X_i for each $i = 1, 2, \dots, k$.

Let U be an open connected subset of M and $T \geq 0$. A point $q \in U$ is *locally $\mathcal{F}$ -reachable* from p at time T if associated to $\mathcal{F}$, there exists an integral curve $\Phi(t) \in U$ for all $t \in [0, T]$ such that $\Phi(0) = p$ and $\Phi(T) = q$. The set of all locally $\mathcal{F}$ -reachable point from p at time T will be denoted by $R_T^U(\mathcal{F}, p)$. Let $R_T^U(\mathcal{F}, p)$ denote $\cup_{0 \leq t \leq T} R_t^U(\mathcal{F}, p)$.

Definition 1 A dynamical polysystem $\mathcal{F}$ is said to be *locally accessible* if restricted to every open connected subset U of M , for all $T > 0$, $R_T^U(\mathcal{F}, p)$ has a nonempty interior subset for every $p \in U$. It is said to be *accessible* if for all $T > 0$, $R_T^M(\mathcal{F}, p)$ has a nonempty interior subset for every $p \in M$. $\square$

The local accessibility, which is sometimes called *locally weak controllability*, means that for every $p, q \in U$, any $T > 0$, there exists an integral curve in U steering p to q or (not necessary and) q to p within the interval $(0, T)$.

Definition 2 A dynamical polysystem $\mathcal{F}$ is said to be *locally controllable* if restricted to every open connected subset U of M and for every $p \in U$, $R_T^U(\mathcal{F}, p) = M$. It is said to be *controllable* if $R_T^M(\mathcal{F}, p) = M$ for every $p \in M$ and some $T > 0$. $\square$

Accordingly, local controllability (accessibility) implies controllability (accessibility). However, it is noteworthy that, in contrast with [10, 15, 7, 20], with the above definitions, (locally) controllability does not imply (locally) accessibility. This is because that for accessibility, the nonempty interior property must be satisfied for all $T > 0$ while for controllability, we only ask for some $T > 0$, $R_T^U(\mathcal{F}, p) = M$. For the brevity of notations, in the sequel, we shall adopt the convention that if T is suppressed, $R^U(\mathcal{F}, p)$ is to be understood as $R_T^U(\mathcal{F}, p)$ for some $T > 0$, while if U is suppressed, $U = M$, e.g., $R(\mathcal{F}, T, p)$ is to be understood as $R^M(\mathcal{F}, T, p)$.

The necessary and sufficient conditions for local accessibility are given by the following theorem:

Theorem 2 [10] *Let $\mathcal{L}(\mathcal{F})$ denote the smallest Lie algebra in $\mathcal{X}(M)$ that contains $\mathcal{F}$. The dynamical polysystem $\mathcal{F}$ is locally accessible if (and only if for the analytical case)*

$$\text{span}((\mathcal{L}(\mathcal{F}))(p)) = T_p M, \quad (3)$$

for all $p \in M$. $\square$

In the following, condition (3) will be called the *Lie algebra rank condition*. It has also been stated in [19] that for the analytical case, the polysystem $\mathcal{F}$ which is controllable must satisfy the Lie algebra rank condition. Therefore, (local) controllability will imply local accessibility for analytic vector fields.

Next we consider a general nonlinear control system:

$$\dot{x} = f(x) + \sum_{i=1}^j g_i(x) u_i; \quad x \in M, \quad (4)$$

where $f, g_1, \dots, g_j$ are analytic vector fields on M , and the control $u : \mathbb{R}^+ \rightarrow \mathcal{U}$, a bounded subset of $\mathbb{R}^j$, is a measurable function. Let $\mathcal{X}^a(M)$ denote the set of all analytical vector fields on M . Let $\mathcal{F} \equiv \{f + \sum_{i=1}^j g_i u_i \mid u \in \mathcal{U}\}$ define a dynamical polysystem. An integral curve of $\mathcal{F}$ essentially corresponds to a trajectory of (4) with piecewise constant inputs. The smallest Lie algebra in $\mathcal{X}^a(M)$ that contains $\mathcal{F}$ is equivalent to the Lie algebra generated by $\{f, g_1, \dots, g_j\}$.

It was proved in [21] that for $f = 0$, and $\mathcal{F}$ defined above being symmetric, i.e., for any $X \in \mathcal{F}$, also $-X \in \mathcal{F}$, then the local accessibility implies controllability. However, the control systems we are dealing with are not symmetric. The method of analyzing the controllability of the spacecraft system in the subsequent sections is based on the following controllability theorem:

Theorem 3 Suppose that f is a weakly positively Poisson vector field. The system (4) is controllable if and only if it is locally accessible. $\square$

Before proving the theorem, we introduce the following lemma.

Lemma 2 [12] Let $\mathcal{F}$ be a dynamical polysystem. Then

$$\text{cl}(\mathcal{R}(\mathcal{F}, p)) = \text{cl}(\mathcal{R}(\text{conv}(\mathcal{F}), p)), \quad \forall p \in M, \quad (5)$$

where $\text{conv}(\mathcal{F})$ denote the convex hull of $\mathcal{F}$. $\square$

In fact, the condition (5) says that $\mathcal{F}$ and its convex hull $\text{conv}(\mathcal{F})$ are equivalent by the definition in [12]. Now we are ready to prove the main theorem.

Proof of Theorem 3² We only need to prove local accessibility implies controllability. Let $\mathcal{E} = \mathcal{F} \cup \{-f\}$ and $\mathcal{L}(\mathcal{E})$ denotes the Lie algebra generated by $\mathcal{E}$. Then $\text{conv}(\mathcal{E})$ is symmetric and by the accessibility condition $\text{span } \mathcal{L}(\mathcal{E})(p) = T_p(M)$, $\forall p \in M$. Also $\text{span } \mathcal{L}(\text{conv}(\mathcal{E}))(p) = T_p(M)$. Since $\text{conv}(\mathcal{E})$ is symmetric, by previous remark on symmetric systems, we have $\mathcal{R}(\text{conv}(\mathcal{E}), p) = M$, $\forall p \in M$. From the above lemma, $\text{cl}(\mathcal{R}(\mathcal{E}, p)) = M$, $\forall p \in M$, also. Since $\text{span } \mathcal{L}(\mathcal{E})(p) = \text{span } \mathcal{L}(-\mathcal{E})(p) = T_p(M)$, $\mathcal{R}(-\mathcal{E}, y)$ has a nonempty interior for all $y \in M$. Hence, for any $q \in M$, there exists $r \in \text{int}(\mathcal{R}(-\mathcal{E}, q)) \cap \mathcal{R}(\mathcal{E}, p)$. This implies $q \in \mathcal{R}(\mathcal{E}, p)$. As a result, $\mathcal{R}(\mathcal{E}, p) = M$, $\forall p \in M$.

Now, we prove that for arbitrary $p, q \in M$, $q \in \mathcal{R}(\mathcal{F}, p)$.

Let $z \in \text{int } \mathcal{R}(-\mathcal{F}, q)$ and $w \in \text{int } \mathcal{R}(\mathcal{F}, p)$. Since $\mathcal{R}(\mathcal{E}, w) = M$, there exists an integral curve of $\mathcal{E}$ joining w to z , i.e.,

$$\exists t_1, \dots, t_k > 0 \text{ and } X_1, \dots, X_k \in \mathcal{E} \text{ s.t.}$$

$$z = \phi_{t_1}^{X_1} \circ \dots \circ \phi_{t_k}^{X_k}(w).$$

If $X_i, i = 1, \dots, k$, belong to $\mathcal{F}$, we have $q \in \mathcal{R}(\mathcal{F}, p)$. If there are some X_i such that $X_i = -f \in \mathcal{E}$, we now try to "correct" the integral curve by using the property of weakly positive Poisson stability of f . Without loss of generality, assume that $X_1 = -f$. We can find a neighborhood of z , say U_z , in the interior of $\mathcal{R}(-\mathcal{F}, q)$ such that

$$(\phi_{t_1}^{X_1} \circ \dots \circ \phi_{t_k}^{X_k})^{-1}(U_z) \subset \text{int}(\mathcal{R}(\mathcal{F}, p)), \quad (6)$$

with $w \in (\phi_{t_1}^{X_1} \circ \dots \circ \phi_{t_k}^{X_k})^{-1}(U_z)$.

Now the weakly positively Poisson stability of f implies that for U_z and t_1 , there exists $T_1 > t_1$ such that

$$\phi_{T_1}^f(U_z) \cap U_z \neq \emptyset.$$

Accordingly, there exists ξ and $\bar{\xi}$ such that $\xi, \bar{\xi} \in U_z \subset \text{int}(\mathcal{R}(-\mathcal{F}, q))$ and $\xi = \phi_{T_1}^f(\bar{\xi})$. From (6), we can find $s \in \text{int}(\mathcal{R}(\mathcal{F}, p))$, such that

$$\bar{\xi} = \phi_{t_1}^{-f} \circ \dots \circ \phi_{t_k}^{X_k}(s).$$

As a result,

$$\begin{aligned} \xi &= \phi_{T_1}^f \circ \phi_{t_1}^{-f} \circ \dots \circ \phi_{t_k}^{X_k}(s) \\ &= \phi_{T_1-t_1}^f \circ \dots \circ \phi_{t_k}^{X_k}(s). \end{aligned}$$

Since $\xi \in \text{int}(\mathcal{R}(-\mathcal{F}, q))$ and $s \in \text{int}(\mathcal{R}(\mathcal{F}, p))$, we proved

$$q = \phi^{\mathcal{F}} \circ \phi_{T_1-t_1}^f \circ \dots \circ \phi_{t_k}^{X_k} \circ \phi^{\mathcal{F}}(s),$$

where $\phi^{\mathcal{F}}$ denote a flow of $\mathcal{F}$. Other possible $-f$ in $X_2, \dots, X_k$ can be treated in an analogous way. As a consequence, q can be reached from p by some integral curve of $\mathcal{F}$. This completes the proof of the theorem. $\square$

In [4, 19], it has been proved that the controllability can be established for Poisson stable systems satisfying the Lie algebra rank condition. The theorem above extends their results to the weakly positively Poisson stable systems. Now, we combine Theorem 1 and 3 to yield the following theorem.

Theorem 4 Consider a control system on M ,

$$\dot{x} = X_H + \sum_{i=1}^j g_i u_i, \quad (7)$$

where X_H is an analytical Hamiltonian vector field with time-independent Hamiltonian H , and $g_i, i = 1, \dots, j$ are analytical vector fields on M . Assume M is bounded, then the system (7) is controllable (with piecewise constant inputs) iff the Lie algebra rank condition is satisfied, i.e.,

$$\text{span } \mathcal{L}(X_H, g_1, \dots, g_j)(p) = T_p M, \quad \forall p \in M. \quad \square$$

This theorem will be directly used in the following sections to prove the controllability of the spacecraft system by using gas jets.

²The following proof is mainly based on an unpublished note of W. P. Dayawansa.

4 Controllability of Spacecraft System

With the tools developed in the previous sections, we are now to prove the controllability result for the spacecraft system.

4.1 Spacecraft System with Reaction Wheels

First, we consider the system of spacecraft with reaction wheels. The equations of motion were given by (1). To apply Theorem 1, it is natural to transform the dynamics on TQ to the dynamics on T^*Q . We accomplish it through the following process. With conjugate momenta defined by

$$\Pi = I\Omega + \eta, \quad \mathbf{h} = \eta + \sum_{i=1}^l I_i \mathbf{n}_i \mathbf{n}_i^T \Omega, \quad \mathbf{p} = m\dot{\mathbf{r}},$$

where η is defined in (2), we can transform the dynamical equations (1) to

$$\begin{aligned} \dot{B} &= B(J^{-1}(\Pi - \mathbf{h}))^\wedge, \\ \dot{\Pi} &= \Pi \times J^{-1}(\Pi - \mathbf{h}) + \int_B \frac{\mu(B^T \mathbf{r} \times \bar{\mathbf{q}})}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} d\mathbf{m}(\bar{\mathbf{q}}), \\ \dot{\mathbf{r}} &= \frac{\mathbf{p}}{m}, \\ \dot{\mathbf{p}} &= - \int_B \frac{\mu(\mathbf{r} + B\bar{\mathbf{q}})}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} d\mathbf{m}(\bar{\mathbf{q}}), \\ \dot{\mathbf{h}} &= \tau, \end{aligned} \quad (8)$$

where $\tau = \sum_{i=1}^l \tau_i \mathbf{n}_i$. Here, Π denotes the total angular momentum of the assembly and $J \triangleq I - \sum_{i=1}^l I_i \mathbf{n}_i \mathbf{n}_i^T$.

For the case of no control inputs, i.e., $\tau_i = 0$ (sometimes this is called the case of free rotors), then $\mathbf{h}$ is constant. Hence, the dynamical system described above is indeed a canonical Hamiltonian system with associated Hamiltonian H :

$$H(B, B\bar{\Pi}, \mathbf{r}, \mathbf{p}) = \frac{1}{2} \langle \Pi - \mathbf{h}, J^{-1}(\Pi - \mathbf{h}) \rangle + \frac{1}{2} \frac{\mathbf{p}^2}{m} - \int_B \frac{\mu}{|\mathbf{r} + B\bar{\mathbf{q}}|} d\mathbf{m}(\bar{\mathbf{q}}). \quad (9)$$

Accordingly, the following proposition is obtained:

Proposition 1 The dynamics of the spacecraft system with free rotors can be expressed as a canonical Hamilton's equation in $(B, B\bar{\Pi}, \mathbf{r}, \mathbf{p})$ -coordinates with Hamiltonian (9), where $\mathbf{h}$ is constant. Moreover, when $\mathbf{r}$ is finite and bounded away from zero, then the system is weakly positively Poisson stable. $\square$

Proof The restriction on $\mathbf{r}$ implies that the energy manifold is bounded, and then the system is weakly positively Poisson stable by theorem 1. $\square$

For the controllability of system (8) being of concern, we look at the corresponding conservation law of angular momentum. In fact, it can be easily verified that

$$B\Pi + \mathbf{r} \times \mathbf{p} = \mathbf{k} \quad (10)$$

is the total angular momentum with respect to the inertial frame and is conserved for (8). Accordingly, the relationship among $\Pi, \mathbf{r}, \mathbf{p}$ is invariant for any control input τ . We thus conclude that the control system (8) is not controllable.

However, the phase variables that we are concerned are not $(B, B\bar{\Pi}, \mathbf{r}, \mathbf{p})^T$, where Π is the total angular momentum of the spacecraft and reaction wheels, but instead $(B, B\bar{\Pi}_s, \mathbf{r}, \mathbf{p})^T$, where $\Pi_s = J\Omega$ denotes the angular momentum of the spacecraft with locked reaction wheels. The dynamical equations (1) can be transformed into a system in terms of the variables $(B, B\bar{\Pi}_s, \mathbf{r}, \mathbf{p})$ as follows.

$$\begin{aligned} \dot{B} &= B\widehat{J^{-1}\Pi_s}, \\ \dot{\Pi}_s &= (\Pi_s + \mathbf{h}) \times J^{-1}\Pi_s + \int_B \frac{\mu(B^T \mathbf{r} \times \bar{\mathbf{q}})}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} d\mathbf{m}(\bar{\mathbf{q}}) - \tau, \\ \dot{\mathbf{r}} &= \frac{\mathbf{p}}{m}, \\ \dot{\mathbf{p}} &= - \int_B \frac{\mu(\mathbf{r} + B\bar{\mathbf{q}})}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} d\mathbf{m}(\bar{\mathbf{q}}), \\ \dot{\mathbf{h}} &= \tau. \end{aligned} \quad (11)$$

In contrast with the system (8), the constant of motion for system (11) becomes

$$B(\Pi_s + \mathbf{h}) + \mathbf{r} \times \mathbf{p} = \mathbf{k}, \quad (12)$$

where the relationship among $B, \Pi_s, \mathbf{r}, \mathbf{p}$ is no longer constrained since $\mathbf{h}$ can be controlled through τ . It is noteworthy that there is no obvious Hamiltonian structure for the spacecraft system (11) with zero input. Thus Theorem 1 can not be directly applied to system (11). While the system (8) is not controllable, its property of weakly positive Poisson stability can be used to help us in establishing such property for (11). This is achieved through the notion of topological equivalence.

Definition 3 [2, 1] Topological flows ϕ and ψ on phase space M, N , respectively, are said to be *topologically equivalent* if there is a homeomorphism $\Phi: N \rightarrow M$ which carries the trajectories of one flow to the trajectories of the other while preserving the direction of motion. $\square$

Proposition 2 Consider vector fields X, Y and associated flow ϕ^x and ϕ^y , respectively. If ϕ^x and ϕ^y are topologically equivalent, then the weakly positive Poisson stability of X implies the weakly positive Poisson stability of Y , and vice versa. $\square$

Proof By the topological equivalence, there is a homeomorphism Φ with $M = \Phi(N)$ such that for any $y \in N$ and every neighborhood $V_y \subset N$, the set $U_x = \Phi(V_y)$ is a neighborhood of $x = \Phi(y)$. Since X is weakly positively Poisson stable, for some $T < \infty$, $\phi_T^x(U_x) \cap U_x \neq \emptyset$. From the property that Φ carries the trajectories of Y to the trajectories of X while preserves the direction of motion, we know $\phi_t^y(V_y) \cap V_y \neq \emptyset$ for some $t \leq T$. $\square$

Corollary 1 The drift vector field of system (11) is weakly positively Poisson stable. $\square$

Proof: The corollary follows if we can establish the topological equivalence between the flows of the drift dynamics for (8) and (11). Let

$$\Phi: \begin{pmatrix} B \\ \Pi_s \\ \mathbf{r} \\ \mathbf{p} \end{pmatrix} \mapsto \begin{pmatrix} B \\ \Pi \\ \mathbf{r} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} B \\ \Pi_s + \mathbf{h} \\ \mathbf{r} \\ \mathbf{p} \end{pmatrix},$$

where $\mathbf{h}$ is a constant. The mapping Φ is obviously homeomorphism. Now we prove that the transformation Φ carries the trajectories. Let ϕ_s^x and ϕ_s^y denote the flows of the drift dynamics (8) and (11), respectively. For arbitrary $y = (B, \Pi_s, \mathbf{r}, \mathbf{p})^T$, we need to verify $\frac{d}{dt}(\Phi \circ \phi_s^y)|_y = X(\Phi(y))$. In particular,

$$\begin{aligned} \frac{d}{dt} \Phi \circ \phi_s^y \left(\begin{pmatrix} B \\ \Pi_s \\ \mathbf{r} \\ \mathbf{p} \end{pmatrix} \right) &= \mathbf{1}_{12 \times 12} \begin{pmatrix} B\mathbf{J}^{-1}\Pi_s \\ (\Pi_s + \mathbf{h}) \times \mathbf{J}^{-1}\Pi_s + \int_B \frac{\mu(\mathbf{B}^T \mathbf{r} \times \bar{\mathbf{q}})}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} dm(\bar{\mathbf{q}}) \\ \mathbf{p} \\ - \int_B \frac{\mu(\mathbf{r} + B\bar{\mathbf{q}})}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} dm(\bar{\mathbf{q}}) \end{pmatrix} \\ &= X \left(\begin{pmatrix} B \\ \Pi_s + \mathbf{h} \\ \mathbf{r} \\ \mathbf{p} \end{pmatrix} \right). \end{aligned}$$

Accordingly, ϕ^x and ϕ^y are topologically equivalent, and then the corollary follows from Proposition 2. $\square$

As stated before, we are concerned with the controllability property on the space $(B, B\Pi_s, \mathbf{r}, \mathbf{p})^T$. The dynamics of $\mathbf{h}$ in (11) can be in fact decoupled from the dynamics of $(B, B\Pi_s, \mathbf{r}, \mathbf{p})$ by using the law of conservation of angular momentum (12). Accordingly, for constant $\mathbf{k}$, we reformulate the system as follows.

$$\begin{aligned} \dot{B} &= B\mathbf{J}^{-1}\Pi_s, \\ B\dot{\Pi}_s &= B\mathbf{J}^{-1}\Pi_s \Pi_s + \\ &B \left(B^T(\mathbf{k} - \mathbf{r} \times \mathbf{p}) \times \mathbf{J}^{-1}\Pi_s + \int_B \frac{\mu(B^T \mathbf{r} \times \bar{\mathbf{q}})}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} dm(\bar{\mathbf{q}}) \right) - B\dot{\tau}, \\ \dot{\mathbf{r}} &= \frac{\mathbf{p}}{m}, \\ \dot{\mathbf{p}} &= - \int_B \frac{\mu(\mathbf{r} + B\bar{\mathbf{q}})}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} dm(\bar{\mathbf{q}}), \end{aligned} \quad (13)$$

where the canonical representation on $TT^*(SO(3) \times \mathbb{R}^3)$ (cf. [18]) has been adopted. The drift dynamics of (13) is also weakly positively Poisson stable. This follows from the corresponding property of the drift dynamics of (11). In fact, for arbitrary constant $\mathbf{k}$ in (13), we can find $\mathbf{h} = B^T(\mathbf{k} - \mathbf{r} \times \mathbf{p}) - \Pi_s$, which is also a constant such that the drift dynamics of (13) is the same as that of (11).

Now, we prove the controllability of dynamics (13) by applying Theorem 3. The Lie algebra rank condition for (13) can be checked as follows. Since the Lie algebra rank condition is invariant under control feedback, we can choose

$$\tau = -B^T(\mathbf{k} - \mathbf{r} \times \mathbf{p}) \times \mathbf{J}^{-1}\Pi_s - \int_B \frac{\mu(B^T \mathbf{r} \times \bar{\mathbf{q}})}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} dm(\bar{\mathbf{q}}) - \mathbf{u}, \quad (14)$$

Here, we assume that τ can control three independent directions of rotation. With the feedback (14), system (13) can be put into the following form:

$$\begin{aligned} \dot{B} &= B\mathbf{J}^{-1}\Pi_s, \\ B\dot{\Pi}_s &= B\mathbf{J}^{-1}\Pi_s \Pi_s + B\dot{\mathbf{u}}, \\ \dot{\mathbf{r}} &= \frac{\mathbf{p}}{m}, \\ \dot{\mathbf{p}} &= - \int_B \frac{\mu(\mathbf{r} + B\bar{\mathbf{q}})}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} dm(\bar{\mathbf{q}}). \end{aligned} \quad (15)$$

Furthermore, by writing $\mathbf{u} = \mathbf{b}_1 u_1 + \mathbf{b}_2 u_2 + \mathbf{b}_3 u_3$, we could express the above equation in the form of (4) with

$$f = \begin{pmatrix} B\mathbf{J}^{-1}\Pi_s \\ B\mathbf{J}^{-1}\Pi_s \Pi_s \\ \frac{\mathbf{p}}{m} \\ - \int_B \frac{\mu(\mathbf{r} + B\bar{\mathbf{q}})}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} dm(\bar{\mathbf{q}}) \end{pmatrix}, \quad g_i = \begin{pmatrix} 0 \\ B\mathbf{b}_i \\ 0 \\ 0 \end{pmatrix}, \quad \forall i = 1, 2, 3,$$

where $\mathbf{b}_1 = (1, 0, 0)^T$, $\mathbf{b}_2 = (0, 1, 0)^T$, $\mathbf{b}_3 = (0, 0, 1)^T$. By applying the techniques developed in [17, 18], the Lie bracket of f and g_i results

$$f_i \triangleq [f, g_i] = \begin{pmatrix} -B\mathbf{J}^{-1}\mathbf{b}_i \\ -B\mathbf{J}^{-1}\mathbf{b}_i \Pi_s \\ 0 \\ 0 \end{pmatrix}, \quad \text{for } i = 1, 2, 3.$$

Similarly, we have the following vector fields generated by the Lie bracket operation, $\forall i = 1, 2, 3$,

$$F_i \triangleq [f, f_i] = \begin{pmatrix} -B(\mathbf{J}^{-1}\Pi_s \times \mathbf{J}^{-1}\mathbf{b}_i) \\ -B(\mathbf{J}^{-1}\Pi_s \times \mathbf{J}^{-1}\mathbf{b}_i) \Pi_s \\ 0 \\ \mu \int_B \left(\frac{1}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} - \frac{3(\mathbf{r} + B\bar{\mathbf{q}})\mathbf{r}^T}{|\mathbf{r} + B\bar{\mathbf{q}}|^5} \right) B\bar{\mathbf{q}} dm(\bar{\mathbf{q}}) \mathbf{J}^{-1}\mathbf{b}_i \end{pmatrix}, \quad (16)$$

$$G_i \triangleq [f, F_i] = \begin{pmatrix} * \\ * \\ * \\ -\frac{F_{i,4}}{m} \end{pmatrix},$$

where “*” denotes the “don’t care” terms and $F_{i,4}$ is the fourth component of the vector field F_i given by

$$\mu \int_B \left(\frac{1}{|\mathbf{r} + B\bar{\mathbf{q}}|^3} - \frac{3(\mathbf{r} + B\bar{\mathbf{q}})\mathbf{r}^T}{|\mathbf{r} + B\bar{\mathbf{q}}|^5} \right) B\bar{\mathbf{q}} dm(\bar{\mathbf{q}}) \mathbf{J}^{-1}\mathbf{b}_i.$$

From the above formulae, it is noted that the Lie algebra rank condition is satisfied if the three vectors $F_{1,4}$, $F_{2,4}$, $F_{3,4}$ can span the 3-dimensional space. To this end, we rewrite $\sum_{i=1}^3 c_i F_{i,4}$ with $c_i \in \mathbb{R}$ as follows,

$$\mu B \int_B \underbrace{\left(\frac{\bar{\mathbf{q}}}{|\lambda + \bar{\mathbf{q}}|^3} - \frac{3\lambda\lambda^T \bar{\mathbf{q}}}{|\lambda + \bar{\mathbf{q}}|^5} + \frac{3\bar{\mathbf{q}}\bar{\mathbf{q}}^T \lambda}{|\lambda + \bar{\mathbf{q}}|^5} \right)}_P dm(\bar{\mathbf{q}}) \sum_{i=1}^3 c_i \mathbf{J}^{-1}\mathbf{b}_i \triangleq \mu B P \sum_{i=1}^3 c_i \mathbf{J}^{-1}\mathbf{b}_i \quad (17)$$

where $\lambda = B^T \mathbf{r}$. Now, it becomes obvious that the non-singularity of matrix P at every $(B, \mathbf{r})$ implies the Lie algebra rank condition which, in turn, implies the controllability. However, for an asymmetric molecule, which consists of six point masses with two on each principal axis (c.f. [31]), matrix P can be computed to be singular for some configurations. Therefore, a more detailed analysis is necessary.

Let $\mathbf{J}^{-1}\mathbf{b}_i \triangleq \bar{\mathbf{b}}_i$ and I denote the coefficient of inertia of the assembly, i.e., $I \triangleq \int_B \bar{\mathbf{q}} \bar{\mathbf{q}}^T dm(\bar{\mathbf{q}})$. We have $I = \frac{1}{2} \text{tr}(I) \mathbf{1} - \mathbf{I}$ where $\mathbf{I}$ is the total moment of inertia of the assembly and is assumed to be diagonal. Hence I is a diagonal matrix and has three different eigenvalues, say I_1, I_2, I_3 . For most applications, the body size of the spacecraft is very small comparing to the size of the orbital radius. We could thus approximate our vector fields based on the small parameter $\epsilon \triangleq \frac{\max|\bar{\mathbf{q}}|}{|\lambda|}$ which represent the ratio of body size to the radius. Taking Taylor expansion with respect to ϵ , the matrix P can be expressed as

$$\begin{aligned} P &= \underbrace{\frac{-3I\lambda}{|\lambda|^5}}_{\mathcal{O}(\epsilon^2)} + \underbrace{\int_B \frac{-3\bar{\mathbf{q}}(\bar{\mathbf{q}}^T \bar{\mathbf{q}})}{2|\lambda|^5} dm(\bar{\mathbf{q}})}_{\mathcal{O}(\epsilon^2)} + \underbrace{\int_B \frac{15\bar{\mathbf{q}}(\lambda^T \bar{\mathbf{q}})^2}{2|\lambda|^7} dm(\bar{\mathbf{q}})}_{\mathcal{O}(\epsilon^2)} + \\ &\underbrace{\frac{15\lambda\lambda^T I\lambda}{|\lambda|^7}}_{\mathcal{O}(\epsilon^2)} + \underbrace{\int_B \frac{15\lambda\lambda^T \bar{\mathbf{q}}(\bar{\mathbf{q}}^T \bar{\mathbf{q}})}{2|\lambda|^7} dm(\bar{\mathbf{q}})}_{\mathcal{O}(\epsilon^2)} + \underbrace{\int_B \frac{-105\lambda\lambda^T \bar{\mathbf{q}}(\lambda^T \bar{\mathbf{q}})^2}{2|\lambda|^9} dm(\bar{\mathbf{q}})}_{\mathcal{O}(\epsilon^2)} + \\ &\underbrace{\frac{3I\lambda}{|\lambda|^5}}_{\mathcal{O}(\epsilon^2)} + \underbrace{\int_B \frac{-15\bar{\mathbf{q}}\bar{\mathbf{q}}^T \lambda(\lambda^T \bar{\mathbf{q}})}{|\lambda|^7} dm(\bar{\mathbf{q}})}_{\mathcal{O}(\epsilon^2)} + \mathcal{O}(\epsilon^4). \end{aligned} \quad (18)$$

For the second-order approximations, it can be found that the determinant of P is

$$\det P = \frac{-9}{|\lambda|^5} (I_1 - I_2)(I_2 - I_3)(I_3 - I_1) \lambda_1 \lambda_2 \lambda_3,$$

with $\lambda = (\lambda_1, \lambda_2, \lambda_3)^T$. Accordingly, for nonzero $\lambda_1, \lambda_2, \lambda_3$, and asymmetric body, the Lie algebra rank condition is satisfied. However, in order to apply Theorem 3, Lie algebra rank condition needs to be satisfied for all $\lambda_1, \lambda_2, \lambda_3$. Complete analysis will be given after the following main theorem is stated.

Theorem 5 Suppose the spacecraft system (13) with three-axes reaction wheels, i.e., the effects of reaction wheels are on three independent axes. Suppose also that $\mathbf{r}$ is finite and sufficiently large comparing to the size of the spacecraft. Then the spacecraft system is controllable if the spacecraft has three distinct principal moments of inertia and its mass distribution satisfies the constraints

$$\int_B \bar{q}^T (-1 + 5\mathbf{b}_i \mathbf{b}_i^T) \bar{q} \bar{q}^T dm(\bar{q}) \neq \mathbf{b}_i, \forall i = 1, 2, 3. \quad (19)$$

Moreover, if the total effects of reaction wheels are only on one-axis or two-independent axis, then the spacecraft system is neither controllable nor accessible.

Proof The weakly positive Poisson stability of the drift dynamics of (13) has been proved. For the Lie algebra rank condition, it is easily checked that $g_i, f_i, \forall i = 1, 2, 3$, span $T_{(B, \tilde{\Pi}, \mathbf{r}, \mathbf{p})} T^*SO(3)$. It suffices to seek some vector fields which span $T_{(\mathbf{r}, \mathbf{p})} T^*\mathbb{R}^3$.

The matrix P in (18) can be expressed as component form of

$$\frac{3}{|\lambda|^5} \underbrace{\begin{pmatrix} \frac{5\lambda_1\lambda_2\lambda_3}{|\lambda|^2}(\mathbf{I}_3 - \mathbf{I}_2) & (\frac{5\lambda_1^2\lambda_3}{|\lambda|^2} - \lambda_3)(\mathbf{I}_1 - \mathbf{I}_3) & (\frac{5\lambda_2^2\lambda_3}{|\lambda|^2} - \lambda_2)(\mathbf{I}_2 - \mathbf{I}_1) \\ (\frac{5\lambda_1^2\lambda_3}{|\lambda|^2} - \lambda_3)(\mathbf{I}_3 - \mathbf{I}_2) & \frac{5\lambda_1\lambda_2\lambda_3}{|\lambda|^2}(\mathbf{I}_1 - \mathbf{I}_3) & (\frac{5\lambda_2^2\lambda_1}{|\lambda|^2} - \lambda_1)(\mathbf{I}_2 - \mathbf{I}_1) \\ (\frac{5\lambda_2^2\lambda_1}{|\lambda|^2} - \lambda_2)(\mathbf{I}_3 - \mathbf{I}_2) & (\frac{5\lambda_1^2\lambda_2}{|\lambda|^2} - \lambda_1)(\mathbf{I}_1 - \mathbf{I}_3) & \frac{5\lambda_1\lambda_2\lambda_3}{|\lambda|^2}(\mathbf{I}_2 - \mathbf{I}_1) \end{pmatrix}}_{C(\epsilon^2)} \\ + \underbrace{\begin{pmatrix} c'_{11} & c'_{12} & c'_{13} \\ c'_{21} & c'_{22} & c'_{23} \\ c'_{31} & c'_{32} & c'_{33} \end{pmatrix}}_{C(\epsilon^3)}.$$

The proof of the theorem is conducted through the following four steps. The idea is to find three independent vectors on the tangent space corresponding to $\mathbf{r}$ and $\mathbf{p}$, respectively.

Step 1: We show that for every configuration $(B, B\tilde{\Pi}, \mathbf{r}, \mathbf{p})^T$ there exists at least one vector $\nu \in \mathbb{R}^3$ satisfying $\nu \nparallel \lambda$ and $\nu \nperp \lambda$.

- Case 1: $|\lambda_1|, |\lambda_2|, |\lambda_3| > \bar{\epsilon}$, where $\bar{\epsilon}$ is a small number. This condition in fact says that $\lambda_1, \lambda_2, \lambda_3$ are all reasonably large. Under this circumstance, the matrix P is nonsingular. Hence, by suitably assigning c_1, c_2, c_3 , we can find $\nu = P \sum_{i=1}^3 c_i \bar{\mathbf{b}}_i$ such that $\nu \nparallel \lambda$ and $\nu \nperp \lambda$.
- Case 2: Without loss of generality, assume $\lambda_1 \in (-\bar{\epsilon}, \bar{\epsilon})$, $|\lambda_2|, |\lambda_3| > \bar{\epsilon}$, or λ_1 small and λ_2, λ_3 reasonably large. Let $\sum_{i=1}^3 c_i \bar{\mathbf{b}}_i = [c'_1, c'_2, 0]^T$. Obviously, c'_1, c'_2 can be chosen arbitrarily. We compute

$$\nu = P \begin{pmatrix} c'_1 \\ c'_2 \\ 0 \end{pmatrix} \approx \frac{3}{|\lambda|^5} \begin{pmatrix} -c'_2(\mathbf{I}_1 - \mathbf{I}_3)\lambda_3 \\ c'_1(\mathbf{I}_3 - \mathbf{I}_2)(\frac{5\lambda_2^2\lambda_3}{|\lambda|^2} - \lambda_3) \\ c'_1(\mathbf{I}_3 - \mathbf{I}_2)(\frac{5\lambda_1^2\lambda_2}{|\lambda|^2} - \lambda_2) \end{pmatrix}.$$

Hence we have

$$\nu \cdot \lambda \approx \frac{12c'_1(\mathbf{I}_3 - \mathbf{I}_2)}{|\lambda|^5} \lambda_2 \lambda_3, \\ \nu \times \lambda \approx \begin{pmatrix} c'_1(\mathbf{I}_3 - \mathbf{I}_2)(\lambda_2^2 - \lambda_3^2) \\ -c'_2(\mathbf{I}_1 - \mathbf{I}_3)\lambda_3^2 \\ -c'_2(\mathbf{I}_1 - \mathbf{I}_3)\lambda_2 \lambda_3 \end{pmatrix}.$$

Accordingly, it is evident that we can choose c'_1, c'_2 such that $\nu \nparallel \lambda$ and $\nu \nperp \lambda$.

- Case 3: $\lambda_1, \lambda_2 \in (-\bar{\epsilon}, \bar{\epsilon})$, $\lambda_3 \approx |\lambda|$. Assume either λ_1 or λ_2 is nonzero, say $\lambda_2 \neq 0$. By a similar argument as the above case, we can choose $\sum_{i=1}^3 c_i \bar{\mathbf{b}}_i = [c'_1, 0, 0]^T$ such that $\nu = P \sum_{i=1}^3 c_i \bar{\mathbf{b}}_i$ satisfies $\nu \nparallel \lambda$ and $\nu \nperp \lambda$. When $\lambda_1 = \lambda_2 = 0$, the above arguments do not work. We need to look at the third-order approximation, i.e., the terms of $O(\epsilon^3)$ in P need to be used. We have

$$\lambda^T P \\ = \frac{-9}{|\lambda|^5} \lambda^T \mathcal{I} \lambda + \frac{1}{|\lambda|^5} \int_B \left(-6\bar{q}^T \bar{q} \bar{q}^T \lambda + \frac{30(\lambda^T \bar{q})^2 \bar{q}^T \lambda}{|\lambda|^2} \right) dm(\bar{q}) + O(\epsilon^4), \\ \approx \left\{ \frac{6}{|\lambda|^5} \int_B \bar{q}^T (-1 + \frac{5\lambda \lambda^T}{|\lambda|^2}) \bar{q} \bar{q}^T dm(\bar{q}) \right\} \lambda,$$

where the fact $\lambda \parallel \mathcal{I} \lambda$ has been used. Now, let $\nu_1 = P \sum_{i=1}^3 c_i \bar{\mathbf{b}}_i$, we have

$$\lambda^T \nu_1 \approx \left\{ \frac{6}{|\lambda|^5} \int_B \bar{q}^T (-1 + 5\mathbf{b}_3 \mathbf{b}_3^T) \bar{q} \bar{q}^T dm(\bar{q}) \right\} \bar{\mathbf{b}}_3 \sum_{i=1}^3 c_i \bar{\mathbf{b}}_i.$$

Thus, if conditions (19) are satisfied, we can choose $c_i, i = 1, 2, 3$, such that $\nu_1 \nparallel \lambda$. Now, if $\nu_1 \nparallel \lambda$, we are done and the desired $\nu = \nu_1$. Otherwise, by similar argument as in Case 2 above, we can find ν_2 such that $\nu_2 \nparallel \lambda$. If $\nu_2 \nperp \lambda$, we let $\nu = \nu_2$. If $\nu_2 \perp \lambda$, then we pick $\nu = \nu_1 + \nu_2$. Obviously, $\nu \nparallel \lambda$ and $\nu \nperp \lambda$. As a result, ν is the desired vector.

In summary, we make the following conclusion: let Δ_1 denote the distribution spanned by vector fields $F_i, i = 1, 2, 3$ in (16) then there exists a vector field $V_1, V_1 \neq 0$, and $V_{1,4}(B, B\tilde{\Pi}, \mathbf{r}, \mathbf{p}) = B\nu$ with $\nu \nparallel \lambda$ and $\nu \nperp \lambda$, such that

$$\Delta_1 \supset \text{span}\{(*, *, 0, V_1)^T\}$$

for every point of $(B, \tilde{\Pi}, \mathbf{r}, \mathbf{p})$.

Step 2: With ν found in Step 1, here we prove that $\lambda \times \nu$ is in $\text{span}(\mathcal{L}(f, g_i))$. Taking the Lie bracket of F_i and f_i , we obtain

$$\Delta_2 \triangleq \text{span}\left\{ \sum_{i=1}^3 c_i F_i, \sum_{j=1}^3 l_j f_j \right\} = \\ \text{span}\left\{ \sum_{i=1}^3 \sum_{j=1}^3 \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \right. \\ \left. - \mu B(-l_j \bar{\mathbf{b}}_j) P(B, \mathbf{r}) c_i \bar{\mathbf{b}}_i + \mu B \frac{d}{dt} \bigg|_{t=\mathbf{r}} P(B e^{-t l_j \bar{\mathbf{b}}_j}, \mathbf{r}) c_i \bar{\mathbf{b}}_i \right\}.$$

For the scalars $c_i, i = 1, 2, 3$ corresponding to the vector ν obtained in the previous step, we have

$$\left[\sum_{i=1}^3 c_i F_i, \sum_{j=1}^3 l_j f_j \right]_4 = \mu B \sum_{j=1}^3 l_j \bar{\mathbf{b}}_j \nu - \mu B \left(D_\lambda P \cdot \sum_{j=1}^3 l_j \bar{\mathbf{b}}_j \lambda \right) \nu,$$

where $D_\lambda P$ denotes the derivative of P with respect to λ . In fact, as shown in (17), P is function of $\lambda = B^T \mathbf{r}$ only. Now, we pick l_j such that $\sum_{j=1}^3 l_j \bar{\mathbf{b}}_j = \kappa_1 \lambda$. Then

$$\left[\sum_{i=1}^3 c_i F_i, \sum_{j=1}^3 l_j f_j \right] = \mu B \kappa_1 (\lambda \times \nu).$$

As a result, if we define a vector field V_2 such that $V_{2,4}(B, B\tilde{\Pi}, \mathbf{r}, \mathbf{p}) = B(\lambda \times \nu)$, then we have

$$\Delta_1 + \Delta_2 \supset \text{span}\{(*, *, 0, V_1)^T, (*, *, 0, V_2)^T\}.$$

Step 3: Now we proceed to construct the third independent vector field. By similar process as in the above step, we compute the Lie bracket

$$\Delta_3 \triangleq \text{span}\left\{ \left[\sum_{i=1}^3 c_i F_i, \sum_{j=1}^3 l_j f_j \right], \sum_{k=1}^3 d_k f_k \right\},$$

where $c_i, l_j, i, j = 1, 2, 3$ were given in Step 1 and 2, respectively. By choosing $d_k, k = 1, 2, 3$ such that $\sum_{k=1}^3 d_k \bar{\mathbf{b}}_k = \kappa_2 \lambda$, the bracket can be found to be

$$\left[\left[\sum_{i=1}^3 c_i F_i, \sum_{j=1}^3 l_j f_j \right], \sum_{k=1}^3 d_k f_k \right]_4 = \mu^2 \kappa_1 \kappa_2 B(\lambda \times (\lambda \times \nu)).$$

Now, let a vector field V_3 be such that $V_{3,4} = B(\lambda \times (\lambda \times \nu))$. Since $\{\nu, \lambda \times \nu, \lambda \times (\lambda \times \nu)\}$ forms a basis in $\mathbb{R}^3$, we conclude that the distribution $\Delta_1 + \Delta_2 + \Delta_3$ span the tangent space corresponding to the component $\mathbf{p}$.

Step 4: Now, we need to prove that the distribution $\text{span}(\mathcal{L}(f, g_i))$ span the tangent space corresponding to the component $\mathbf{r}$. By a similar argument as in the previous steps, we can show that the distribution

$$\text{span}\left\{ \sum_{i=1}^3 c_i G_i, \left[\sum_{i=1}^3 c_i G_i, \sum_{j=1}^3 l_j f_j \right], \left[\left[\sum_{i=1}^3 c_i G_i, \sum_{j=1}^3 l_j f_j \right], \sum_{k=1}^3 d_k f_k \right] \right\}$$

where c_i, l_j, d_k are chosen in Step 1-3, spans the tangent space corresponding to $\mathbf{r}$. Consequently, we have the following conclusion:

$$\text{span}\{g_1, g_2, g_3, f_1, f_2, f_3, \\ \sum_{i=1}^3 c_i F_i, \left[\sum_{i=1}^3 c_i F_i, \sum_{j=1}^3 l_j f_j \right], \left[\left[\sum_{i=1}^3 c_i F_i, \sum_{j=1}^3 l_j f_j \right], \sum_{k=1}^3 d_k f_k \right], \\ \sum_{i=1}^3 c_i G_i, \left[\sum_{i=1}^3 c_i G_i, \sum_{j=1}^3 l_j f_j \right], \left[\left[\sum_{i=1}^3 c_i G_i, \sum_{j=1}^3 l_j f_j \right], \sum_{k=1}^3 d_k f_k \right] \} \\ = T_{(B, B\tilde{\Pi}, \mathbf{r}, \mathbf{p})} T^*SE(3),$$

This complete the proof for the controllability of the system with three-axis reaction wheels.

On the other hand, for the case of one-axis or two-axis reaction wheels, there exists some vector c such that

$$c^T B^T(k - r \times p) - c^T \Pi_s = c^T h = 0.$$

Therefore, there exists some constant of motion $c^T B^T(k - r \times p) - c^T \Pi_s$, which is invariant under any control input. Accordingly, the system (11) with one-axis or two-axis reaction wheels cannot be accessible, and hence not controllable. $\square$

The conditions stated in the above theorem mainly require that the shape of the spacecraft should be not symmetric. In fact, for a totally symmetric spacecraft, the orbital dynamics is decoupled from attitude dynamics and is thus uncontrollable from the torque affecting the attitude motion. By choosing suitable masses and sizes of the molecule mentioned before, it has been checked that the conditions (19) can be satisfied. This shows that there are indeed some physical cases that the above theorem can be applied. Now we turn our attention to the controllability problem for the spacecraft system with gas jets.

4.2 Spacecraft System with Gas Jets

The controllability analysis for spacecraft system with gas jets is similar to that for the system with reaction wheels. Define conjugate momenta

$$\Pi = I\Omega \quad \text{and} \quad p = m\dot{r}.$$

The dynamical equations expressed as a canonical form on $T^*(SO(3) \times \mathbb{R}^3)$ are:

$$\begin{aligned} \dot{B} &= B\dot{I}^{-1}\Pi, \\ B\dot{\Pi} &= B\dot{I}^{-1}\Pi\dot{\Pi} + B\left(\Pi \times I^{-1}\Pi + \int_B \frac{\mu(B^T r \times \bar{q})}{|r + B\bar{q}|^3} dm(\bar{q})\right) + B\dot{\tau}, \\ \dot{r} &= \frac{p}{m}, \\ \dot{p} &= - \int_B \frac{\mu(r + B\bar{q})}{|r + B\bar{q}|^3} dm(\bar{q}). \end{aligned} \quad (20)$$

Theorem 4 can be applied to prove the controllability of system (20). To this end, we first consider the Hamiltonian structure of the drift vector field of (20). By setting $\tau = 0$, system (20) is a Hamiltonian system with the following Hamiltonian,

$$H(B, B\dot{\Pi}, r, p) = \frac{1}{2} \langle \Pi, \Pi \rangle + \frac{1}{2} \frac{p^2}{m} - \int_B \frac{\mu}{|r + B\bar{q}|} dm(\bar{q}). \quad (21)$$

Now, we check the Lie algebra rank condition. By choosing the control feedback

$$\tau = -\Pi \times I^{-1}\Pi - \int_B \frac{\mu(B^T r \times \bar{q})}{|r + B\bar{q}|^3} dm(\bar{q}) + u, \quad (22)$$

system (20) can be transformed into the system (15) with Π_s replaced by Π , and J replaced by I . Accordingly, the verification of Lie algebra rank condition is exactly the same as in the previous subsection. Therefore, we have the following theorem.

Theorem 6 Suppose the spacecraft system (20) with three-axis gas jets, i.e., the effects of gas jets are on three independent axes. Suppose also that r is finite and sufficiently large comparing to the size of the spacecraft. Then the spacecraft system is controllable if the spacecraft has three distinct principal moments of inertia and its mass distribution satisfies the constraints

$$\int_B \bar{q}^T (-1 + 5b_i b_i^T) \bar{q} \bar{q}^T dm(\bar{q}) \neq b_i, \quad \forall i = 1, 2, 3. \quad (23)$$

The above two Theorems 5 and 4 essentially tell us that the orbital motion can be controlled by suitably adjusting the attitude motion. In fact, we may define the center of gravity of the assembly to be, cf. [22], the vector r_G such that

$$\frac{r_G}{|r_G|^3} = \int_B \frac{r + B\bar{q}}{|r + B\bar{q}|^3} dm(\bar{q}).$$

Thus the center of gravity is the place where the total gravitational force acts on. It can be proved that while the center of mass is fixed in the assembly, the center of gravity varies with respect to different orientations of the assembly. From this observation, it is then natural that changing the attitude (relative to the center of mass) may lead to the change of the orbit.

5 Conclusions

In this paper, the equations of motion for spacecraft system with reaction wheels or with gas jets in a central gravitational field are concerned. For these dynamics, we are interested in investigating the possibility of controlling the orbit by attitude controllers. The formal definitions of the controllability and weakly positive Poisson stability has been addressed. We showed that if the

drift vector field of the system is weakly positively Poisson stable and the Lie algebra rank condition is satisfied, controllability can be concluded. For the above mechanical systems, their intrinsic Hamiltonian structures were then used to guarantee the weakly positive Poisson stability. On the other hand, the Lie algebra rank condition was verified with the aid of the techniques in computing of Lie brackets on second tangent bundle of $SO(3) \times \mathbb{R}^3$. The main results of this paper state that it is possible to change the orbit of the spacecraft by controlling the attitude. The observation of such attitude-orbit coupling effects not only helps us to understand the dynamics of the spacecraft system but also may lead to new technologies in spacecraft control.

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