AN APPROXIMATE MAXIMUM ENTROPY SOLUTION FOR
BEARING ESTIMATION OF COHERENT SOURCES

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ABSTRACT

This paper presents a new method to find an approximate solution of the maximum entropy (ME) method for bearing estimation. It is especially suitable for the coherent source problem. This solution clearly shows that the ME spectrum is an all-pole spectrum and the coefficients of its denominator polynomial can be found from the estimation error of signals. Since no spatial smoothing is required, it always provides a spectrum with whole array aperture, therefore it is possible to find a solution for two closely spaced sources with a 3-element array, based on single snap-shot data. Depending on how signal is estimated, it is possible to find a superresolution spectrum with DFT only.

INTRODUCTION

Maximum entropy method (MEM) was introduced by Burg for superresolution spectrum analysis. Developments of its theory were well documented in [1]. Although MEM is related to the linear prediction theory closely, it is a non-linear optimization problem in general. Its solution can be found by the technique of Lagrange multiplier [2,3,4]. Algorithms to implement the MEM can be found in [3]. In this paper, an approximate solution of the MEM was found. This solution clearly shows that the ME spectrum is an all-pole spectrum. Although how to implement this solution optimally is not known yet, simulation results do show that it may

find a solution for two closely spaced sources with a 3-element array. It may provide spectrums with resolution higher than the well known MUSIC algorithm [5], since this solution can be found by using single snap-shot of data without spatial smoothing; that is, a larger order can be used in this solution.

Bearing estimation of coherent sources has a problem of inverting a singular covariance matrix [6]. To solve this problem, spatial smoothing must be applied. Reduction of aperture size is unavoidable in using spatial smoothing. Theoretically speaking, this is exact the problem of spectrum estimation of sinusoidal processes based on a single set of measurement data. By linear prediction theory, to find a reliable AR spectrum of the sinusoidal process [7,8], the order of linear prediction filter must be less than the data length. In this paper the problem of bearing estimation of coherent sources is treated exactly same as the spectral estimation of sinusoidal processes based on single snap-shot of data, since multiple snap-shot of data do not change the nature of this problem.

APPROXIMATE MAXIMUM ENTROPY SOLUTION

Since MEM is formulated on time domain conventionally, the usual time domain notations are used here for convenience. Following the standard notations in spectrum estimation and stationarity assumption, the power spectrum of a signal with autocorrelation sequence (ACS) $r(n)$ is

$$P(f) = \sum_{N=0}^{N-1} r(n) e^{-j2\pi fnT} \quad ; n = 0..N-1 \quad (1)$$

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From the ME theory, $P(f)$ is to be found by maximizing

$$H = \int \ln P(f) df \text{ and constrained by}$$

$$r(n) = \int P(f) e^{j2\pi nT} df \quad ; n = 0..N-1 \quad (2)$$

By Lagrange multiplier technique, this object function can be formulated to be an unconstrained one[9] as

$$J = \int \ln P(f) df - \left| r(n) - \lambda_n \sum_{n=0}^{N-1} \int P(f) e^{j2\pi nT} df \right|^2 \quad (3)$$

where λ_n are the Lagrange multipliers. Consider the discrete version of this objective function

$$J = \delta \sum_{m=1}^M \ln P_m - \lambda \sum_{n=0}^{N-1} \left| r(n) - \delta \sum_{m=1}^M P_m e^{j\omega_m nT} \right|^2 \quad (4)$$

where $P_m = p(f_m)$, $\omega_m = 2\pi f_m$, $\delta = 2\pi m\delta$. Note that the Lagrange multipliers are set to be a single variable λ , this essentially makes the estimation error of $r(n)$ be equally weighted. To find the solution of P_m , a set of non-linear equations must be solved, which are

$$\begin{aligned} \frac{\partial J}{\partial P_i} &= \frac{\delta f}{P_i} - 2\lambda \operatorname{Re} \left[\sum_{n=0}^{N-1} \left[r(n) - \delta \sum_{m=1}^M P_m e^{j\omega_m nT} \right] e^{-j\omega_i nT} \right] \\ &= 0 \quad ; i = 1..M \end{aligned} \quad (5)$$

The IDFT of P_m is an estimate of $r(n)$ which can be defined as

$$\hat{r}(n) = \delta \sum_{m=1}^M P_m e^{j\omega_m nT} \quad (6)$$

Equation (5) can be rewritten as

$$\begin{aligned} \frac{\delta f}{P_i} &= 2\lambda \operatorname{Re} \left[\sum_{n=0}^{N-1} \left[r(n) - \hat{r}(n) \right] e^{-j\omega_i nT} \right] \\ &= 0 \quad ; i = 1..M \end{aligned} \quad (7)$$

Equation (5) is a complicated equation of P_i , since P_i exist inside the bracket of (5) when $m = i$. However, P_i exist on the left-hand side of (7) only, a solution of (7) can be found to be

$$\begin{aligned} P_i &= \frac{\delta f}{2\lambda \operatorname{Re} \left[\sum_{n=0}^{N-1} \left[r(n) - \hat{r}(n) \right] e^{-j\omega_i nT} \right]} \\ ; i &= 1..M \end{aligned} \quad (8)$$

This is an approximate solution of the ME spectrum, since P_i are not exactly solved from (4), and the spectrum found is named approximate maximum entropy (AME) spectrum here. Equation (8) tells us that P_i is proportional to the inverse of the discrete Fourier transform (DFT) of the estimation error sequence

$$\alpha(n) = r(n) - \hat{r}(n) \quad ; n = 0..N-1; \quad (9)$$

This result does not provide any direct solution of P_i , however, it provides a general way to find P_i ; that is, an estimate of the ACS, $\hat{r}(n)$, in stead of the interested spectrum must be found first; then the estimation error of the ACS can be used to find the spectrum indirectly. Since the assumed known data is the ACS $r(n)$, $\hat{r}(n)$ must be estimate from $r(n)$. This is a problem similiar to the signal reconstruction problem.

One point worth to mention is that $\hat{r}(n)$ is an estimate of the true ACS in stead of the given measured data $r(n)$. There are many techniques to do signal reconstruction. One simple method to find $\hat{r}(n)$ is given in next section. From the theory of MUSIC algorithm, this result has a common requirement of finding an estimate of the "noise" vector $\alpha(n)$, although the AME solution does not provide a clear way to find it.

ESTIMATE OF $\hat{r}(n)$

An easy way to estimate $\hat{r}(n)$ from $r(n)$ is by defining

$$q_m = \sum_{N=0}^{N-1} r(n) e^{-j\omega_m nT} ; m = 1..M, \text{ and}$$

$$\bar{r}(n) = \frac{1}{N} \sum_{k \in I} q_k e^{j\omega_k nT} \quad (10)$$

where I is a set of k 's, for which $|q_k| > \beta$ and β is a threshold to discriminate the noise background in q_m from signal responses. The intermediate spectrum of q_m is a low resolution one, since N is a small number in general. It is well known that from the DFT and IDFT pair of (10), $\bar{r}(n)$ might be exactly equal to $r(n)$, if the discrimination threshold is not applied. In theory, the value of β can be set by a priori knowledge about the noise level or by the sidelobe level of signals in q_m .

To get a minimum error estimate of $\hat{r}(n)$ from $\bar{r}(n)$, the distance between $\bar{r}(n)$ and $r(n)$ is minimized by a scaling factor γ as

$$\min_{\gamma} \sum_{n=0}^{N-1} |r(n) - \gamma \bar{r}(n)|^2 \quad (11)$$

Then $\hat{r}(n) = \gamma \bar{r}(n)$ is used as the desired estimate of true ACS. This minimization can be done easily and it essentially reduces the sensitivity of what value of β is used in (10).

Note that the algorithm given above to estimate $\hat{r}(n)$ is not a general one, since it is not an estimator found in any optimum sense; however, from previous simulation results, this algorithm is good enough for cases similar to the examples given in next section. Because the AME solution does not put any clear condition on $\hat{r}(n)$, other reasonable methods to find $\hat{r}(n)$ might yield an equally good AME solution also. In general, the optimum method to find $\hat{r}(n)$ is not known.

SIMULATIONS

To show performances of the AME solution, two simulation examples of coherent source bearing estimation are given here. They represent the cases that are solvable and unsolvable with conventional bearing estimation techniques. Since time domain formulation

as used in previous sections, some simple conversions of time to spatial domain variables are necessary, the major one is the spatial frequency variable $u = \sin \theta$, where θ is the angle measured from array broadside.

The first example has two sources with bearings $u_1 = 0.0$, $u_2 = 0.25$ and their associated phases $\phi_1 = 0.0$, $\phi_2 = 30$ degrees. Each source has a SNR of 20 dB. The receiving array has 6 elements which are spaced at half wavelength. This is a solvable case by applying spatial smoothing and the well known MUSIC algorithm. Results of applying these techniques are given in Fig. 1 with an order of 4. Results of ten test runs, based on single snap-shot data, are overlaid together. It can be found that there are many spurious peaks due to excessive order used. Therefore, the same data were processed again with an order of 3 and the results are shown in Fig 2. It can be found that the two sources are resolved in most cases. The same set of data were processed by the AME technique, its results are given in Fig. 3. It can be found that spurious peaks exist due to a large order of 6 being used also. To cure this problem, the coefficients $\alpha(n)$ estimated from ten snap-shot data were averaged to be a new estimate of $\alpha(n)$, the result is given in Fig. 4. It can be found that this result is much better than the previous results, since very sharp peaks were pointed to the target directions correctly.

The second example has a similar scenario as the first example. The target bearings are $u_1 = 0.2$, $u_2 = -0.25$ and their associated phases are $\phi_1 = 0.0$, $\phi_2 = 30$ degrees. The receiving array has 3 elements only. This is a unsolvable case by applying conventional techniques. Results of the AME solution are given in Fig. 5, in which the curves with lower resolution are results of simple DFT angular spectrum. It can be found that both targets are resolved well based on single snap-shot data only.

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Figure 1. The MUSIC angular spectrum of two target with a 6-element array. The targets have equal SNR of 20 dB and their (bearing, phase) are $(0.0, 0^\circ)$ and $(0.25, 30^\circ)$. Spatial smoothing is applied to get an order-4 covariance matrix. Results of ten test runs are overlaid.

Figure 2. Same conditions as in Figure 1, except the order used is 3.

Figure 3. The AME spectrum of the same signals used in Figure 1.

Figure 4. The averaged AME solution from results of 10 snap-shot data given in Figure 3.

Figure 5. The AME spectrum of two targets with a 3-element array. The targets have equal SNR of 20 dB and their (bearing, phase) are $(0.2, 0^\circ)$ and $(-0.25, 30^\circ)$. The lower resolution curves are the DFT spectrum.