Disturbance Estimator and Its Application in Estimation of System Output Derivatives

Min-Shin Chen
Dept. of Mechanical Engineering
National Taiwan University
Taipei, Taiwan, ROC

Masayoshi Tomizuka
Dept. of Mechanical Engineering
University of California
Berkeley, CA 94709

Abstract

A disturbance estimator is proposed to estimate unmeasurable disturbances which may be arbitrarily time-varying and/or grow to infinity with a cone-bounded growth rate. We show that estimation of the disturbance is achieved to any desired accuracy within a finite period of time. The idea behind the disturbance estimator is then applied to the estimation of output derivatives for uncertain linear systems. Under an assumption that the system input is a state feedback control with bounded feedback gains, the proposed scheme can estimate output derivatives up to the $(r-1)$ 'th order, where r is the relative degree of the system. When r is larger than two, the proposed estimation scheme is first obtained in an unrealizable form, and then its approximation in a realizable form is suggested. The realizable scheme is verified to be effective by a computer simulation study.

1. Introduction

Conventionally, we solve the problem of disturbance rejection by first trying to understand the dynamic characteristics of the disturbance; in other words, to identify a disturbance model. This disturbance model is then utilized in the synthesis of a disturbance-rejecting controller. Such a design approach has been adopted by the Internal Model Controller [1] for disturbances generated by a linear autonomous system, and Repetitive Controller [2] for periodic disturbances. However, most disturbances encountered in the real world are arbitrarily time-varying and can not be modeled by simple mathematical model. To suppress arbitrarily time-varying disturbances, other types of controllers such as the Sliding Mode Controller (SMC) [3] and the Ultimate Boundedness Controller (UBC) [4] have been suggested. When the UBC and the SMC were first presented, a discontinuous "sign" function was used in the control law, which causes discontinuous control signals. Later development [5-7] suggested using "continuous" functions like the "saturation" function instead of the "sign" function so as to achieve disturbance rejection by smooth control signals. We conjecture that in achieving disturbance rejection, the "continuous" SMC or UBC must have somehow "estimated" the unknown disturbance and use the obtained estimate to cancel the disturbance. In other words, these controllers can in fact be regarded as disturbance estimators. In the first part of this paper, we substantiate the above statement by actually constructing a disturbance estimator based on the type of the "continuous" controllers proposed in [5-7] for a very simple system. The proposed estimator can estimate arbitrarily time-varying bounded disturbances as well as certain potentially unbounded disturbances.

The second part of this paper demonstrates an application of the proposed disturbance estimators: estimation of system output derivatives. This application is motivated by the following fact: in many control schemes, such as adaptive control [8,9] and robust output feedback control [10], the design and implementation of the controller become very complicated when the relative degree of the system is larger than one. One way to solve the problem is first to add an unrealizable postcompensator to the open-loop system so that the compensated system becomes relative degree one, and then to design the feedback controller for the compensated system. In real implementation, the unrealizable postcompensator is made "realizable" by inserting filters with very small time constants. The use of these small-time-constant filters is then legitimized by the argument of Singular Perturbation [11] when the time constants of the filters are small enough. In this paper, we present a different approach, based on the idea of disturbance estimation, to the realization of these "unrealizable" compensators without

using the small-time-constant filters.

We will use a special notation in the subsequent sections: given any function $g(\cdot)$ in Appendix A, define two constants

$$\bar{g}_i = \sup_{x \in N_i} |g(x)| \quad \text{and} \quad g_i' = \inf_{x \in N_i} g'(x)$$

where N_i is a bounded open set on the real line.

The remainder of this paper is arranged as follows: Section 2 presents an estimation scheme for bounded as well as potentially unbounded disturbances. Section 3 introduces a "normality" definition on the control input of a linear system. Section 4 presents a scheme to estimate system output derivatives for an uncertain linear system. Finally, Section 5 gives the conclusions.

2. Estimation of Unknown Disturbances

Consider the following system

$$\dot{x} = u + w \quad (1)$$

where u is a manipulable variable and w is a potentially unbounded disturbance to be estimated, which is given by

$$w = \gamma z + d \quad (2)$$

where γ and d are unknown time-varying variables satisfying

$$|d| \leq D^0, \quad |\dot{d}| \leq D^1 \quad (3a)$$

$$|\gamma| \leq \Delta^0, \quad |\dot{\gamma}| \leq \Delta^1 \quad (3b)$$

with $D^0, D^1, \Delta^0, \Delta^1$ being finite positive constants. In particular, we assume that D^0 and Δ^0 are known *a priori*. z is a measurable signal which may grow to infinity, but its growth rate is "cone-bounded"; i.e., there exist two finite positive constants α and β such that

$$|\dot{z}| \leq \alpha |z| + \beta \quad (3c)$$

When the signal w in (2) satisfies the conditions (3a) - (3c), it is called a "strictly regular" signal.

To estimate the disturbance w , we propose the following estimator:

$$u = -\sigma x - \rho g(x), \quad \rho = \rho_1 |z| + \rho_2 \quad (4a)$$

where σ is an arbitrary positive constant, $g(\cdot)$ can be any smooth function listed in Appendix A, and ρ_1 and ρ_2 are two positive constants satisfying

$$\rho_1 > \Delta^0 \quad \text{and} \quad \rho_2 > \max(D^0, \rho_1) \quad (4b)$$

Theorem 1: Consider the system (1) subject to (2) and (3). If the estimator (4) is applied to the system, then

I. there exists a finite time T_1 such that

$$x(t) \in N_1 \quad \text{for all } t \geq T_1$$

$$\text{where } N_1 = \left\{ x : |x| < n, n = g^{-1} \left[\sup_{z \in \mathbb{R}} \frac{\Delta^0 |z| + D^0}{\rho_1 |z| + \rho_2} \right] \right\} \quad (5)$$

Notice that since $|g(x)| \leq 1$ for all x , inequalities in (4b) are essential for the existence of n defined in N_1 .

II. there exists a finite time T_2 such that

$$\dot{x}(t) \in N_2 \quad \text{for all } t \geq T_2$$

$$\text{where } N_2 = \left\{ \dot{x} : |\dot{x}| < n', n' = \sup_{z \in \mathbb{R}} \frac{(c\alpha + \Delta^1)|z| + (c\beta + D^1)}{g_1'(\rho_1 |z| + \rho_2)} \right\} \quad (6)$$

and $c = \rho_1 \bar{g}_1 + \Delta^0$.

Proof: see Appendix B.

Remark: We can make n defined in N_1 arbitrarily small if ρ_1 and ρ_2 in (4b) are chosen sufficiently large. Also notice that the ratio $\bar{g}_1/g_1'$ becomes arbitrarily small at the same time since

$$\frac{\bar{g}_1}{g_1'} = \begin{cases} \frac{n(n+\delta)}{\delta} & \text{when } g = g_1 \text{ in Table 1} \\ n & \text{when } g = g_2 \\ \frac{n}{1-q} & \text{when } g = g_3 \end{cases}$$

Therefore, n' defined in N_2 is also arbitrarily small if ρ_1 and ρ_2 are large enough. With the above understanding, we can restate Theorem 1 as follows. The first part of the theorem says that x is regulated into an open neighborhood of the origin, and this neighborhood can be made arbitrarily small if ρ_1 and ρ_2 are chosen sufficiently large. The second part of the theorem states that the estimator (4) drives $-u$ to "track" the disturbance w in the following sense: given any $n' > 0$, there exist sufficiently large ρ_1 and ρ_2 in (4b) such that $|w+u|$ becomes smaller than n' within a finite period of time. This latter part of the theorem is especially important since it allows us to estimate state-dependent uncertainties in a control problem before the stability of the feedback system is established.

Theorem 1 can be extended to the case when the disturbance is a linear combination of potentially unbounded signals; that is, w in (2) is expressed as

$$w = \sum_{i=1}^q \gamma_i z_i + d = \Gamma^T Z + d \quad (7)$$

where $0 < q < \infty$, $\Gamma^T = [\gamma_1, \dots, \gamma_q]$, $Z^T = [z_1, \dots, z_q]$. We now give a formal definition of "strict regularity" for the signal w in (7).

Definition 1: The signal w in (7) is "strictly regular" if the following conditions are satisfied:

$$|d| \leq D^0, \quad |\dot{d}| \leq D^1 \quad (8a)$$

$$\|\Gamma\| \leq \Omega^0, \quad \|\dot{\Gamma}\| \leq \Omega^1 \quad (8b)$$

$$\|Z\| \leq \alpha \|Z\| + \beta \quad (8c)$$

where $D^0, D^1, \Omega^0, \Omega^1, \alpha$ and β are all finite positive constants.

It can be checked that Eqs.(8a) ~ (8c) are equivalent to the following conditions:

$$|d| \leq D^0, \quad |\dot{d}| \leq D^1 \quad (9a)$$

$$|\gamma_i| \leq \omega_i^0, \quad |\dot{\gamma}_i| \leq \omega_i^1, \quad i=1, \dots, q \quad (9b)$$

$$|z_i| \leq \sum_{j=1}^q \alpha_{ij} |z_j| + \beta_i, \quad i=1, \dots, q \quad (9c)$$

where $\omega_i^0, \omega_i^1, \alpha_{ij}$ and β_i are all finite positive constants.

Notice that using (8a) and (8b), we obtain the following upper bound on w :

$$|w| \leq \Omega^0 \|Z\| + D^0 \equiv \bar{w} \quad (10)$$

While using (9a) and (9b), we obtain another upper bound on w :

$$|w| \leq \sum_{i=1}^q \omega_i^0 |z_i| + D^0 \equiv \underline{w} \quad (11)$$

Corresponding to the upper bound $\bar{w}$ in (10), we propose the following estimator:

$$\dot{u} = -\sigma x - \rho g(x), \quad \rho = \rho_1 \|Z\| + \rho_2 \quad (12a)$$

$$\rho_1 > \Omega^0 \quad \text{and} \quad \rho_2 > \max(D^0, \rho_1) \quad (12b)$$

where σ and $g(\cdot)$ are as given in (4). In case the disturbance w is characterized by (9a) ~ (9c), we propose another estimator based on the upper bound $\underline{w}$ in (11):

$$\dot{u} = -\sigma x - \rho g(x), \quad \rho = \sum_{i=1}^q \rho_{1i} |z_i| + \rho_2 \quad (13a)$$

where

$$\rho_{1i} > \omega_i^0 \quad \text{and} \quad \rho_2 > \max(D^0, \rho_{1i}) \quad (13b)$$

Similar estimation results as those stated in Theorem 1 can be obtained for the estimators (12) and (13), respectively. Details can be found in Reference [12].

Example 1: In this example, we examine the effects of different estimator parameters on the estimation results. In particular, we study the estimation of a sinusoidal disturbance in the system (1); that is,

$$d = \sin(\omega t), \quad \gamma = 0$$

in Eq.(2), where ω is the disturbance frequency. In this case,

$$D^0 = 1 \quad \text{and} \quad D^1 = \omega$$

in Eq.(3). We use the estimator (4) with

$$\rho_1=0, \quad \rho_2=2, \quad \sigma=0.01, \quad g(x) = \frac{x}{1+x+0.01}$$

and simulate the system (1) and (2) for selected disturbance frequencies $\omega = 1, 50, 100$. Figure 1 shows the corresponding estimates of d . It is seen that as ω increases, both the phase lag and the magnitude mismatch between $-u$ and d become larger.

We now utilize Part II of Theorem 1 to study how the estimation results can be improved. First notice that in the present example,

$$\bar{g}_1 = \frac{n}{n+\delta} = \frac{D^0}{\rho} \quad \text{and} \quad g_1' = \frac{\delta}{(n+\delta)^2}$$

where n is defined in (5). It follows from (6) that

$$n' = \frac{\omega \rho \delta}{(\rho-1)^2}$$

The above equation shows how each parameter affects the accuracy of disturbance estimation: the larger the parameter ρ or the smaller δ , the smaller the estimation error. Figure 2 shows the effects of using different values of ρ on the estimation results: when ρ is increased to 10, a satisfactory estimate of the disturbance $\sin(100t)$ is obtained. We can also make the estimation error small by using a smaller δ in the smooth function $g(\cdot)$ as shown in Figure 3 where a satisfactory estimate of the disturbance $\sin(100t)$ is obtained when $\delta=0.001$.

Example 2: Consider the system (1) subject to an exponentially growing disturbance w , where

$$\gamma(t) = \sin(5t), \quad z(t) = e^t \quad \text{and} \quad d(t) = 0$$

in Eq.(2), and

$$u = -0.001x - 2e^t g(x), \quad g(x) = \frac{x}{1+x+0.01}$$

The simulation result shown in Figure 4 confirms Theorem 1 that $-u$ does track the disturbance w even though w grows to infinity asymptotically.

3. "Normal" Feedback Control

In a control problem, parameter mismatches between the true system and the nominal system can result in system uncertainties which are potentially unbounded. The purpose of this section is to investigate under what conditions this type of system uncertainties are "strictly regular" as defined in Definition 1 so that they can be estimated by the estimation scheme introduced in the previous Section. We will show that for a linear system as long as the control input is "normal" in the sense of Definition 2 given below, the uncertainties resulting from parameter mismatches are "strictly regular".

Consider the following SISO system:

$$\dot{x} = Ax + bu + d, \quad y = cx \quad (14)$$

where $x \in R^n$, d is a bounded disturbance, y is the system output and u is the control input. We introduce a "normality" condition on the control input u as follows.

Definition 2: The control input u in (14) is a *normal state feedback control* if

$$u(t) = k_1^T(t)x(t) + k_2(t)$$

where $k_1(t) \in R^n$ and $k_2(t) \in R^1$ satisfy

$$\|k_1(t)\| \leq \bar{k}_1 < \infty, \quad \|k_2(t)\| \leq \bar{k}_2 < \infty$$

In the following Lemma, we study a class of signals associated with the system (14), and show that these signals are "strictly regular" if the control input u in (14) is "normal" as defined above. We state the lemma without showing its proof since the lemma can be easily verified.

Lemma 1: Given the system (14), consider a signal w given by

$$w = Gx + d \quad (15)$$

where x is the state of the system (14), and $G \in R^n$ and $d \in R^1$ may be time-varying signals satisfying

$$|d| \leq D^0, \quad |\dot{d}| \leq D^1$$

$$\|G\| \leq \Delta^0, \quad \|\dot{G}\| \leq \Delta^1$$

If the control input u in (14) is a "normal" state feedback control, then w in (15) is "strictly regular".

4. Estimation of System Output Derivatives

We now present an application of the disturbance estimator proposed in previous Sections: estimation of output derivatives for uncertain linear systems. We assume that the state of the uncertain system is directly accessible.

Consider an SISO system described by

$$\dot{x} = \bar{A}x + \bar{b}u + d, \quad y = \bar{c}x \quad (16)$$

where $x \in R^n$ is the system state, d is the external disturbance, u is the control input, y is the system output, and

$$\bar{A} = A + \Delta A, \quad \bar{b} = b + \Delta b, \quad \bar{c} = c + \Delta c$$

with (A, b, c) representing nominal system matrices and $(\Delta A, \Delta b, \Delta c)$ unknown matrix mismatches. We make the following assumptions on the system (22):

(A1) Both the nominal system and the true system are of relative degree $r(>1)$; therefore,

$$cA^{r-1}b \neq 0, \quad \text{and} \quad cA^k b = 0 \quad \text{for } k = 0, 1, \dots, r-2$$

$$\bar{c}\bar{A}^{r-1}\bar{b} \neq 0, \quad \text{and} \quad \bar{c}\bar{A}^k \bar{b} = 0, \quad \text{for } k = 0, 1, \dots, r-2$$

(A2) The disturbance d has bounded time derivatives up to the $(r-1)$ 'th order; i.e.,

$$|d| \leq D^0, \quad |\dot{d}| \leq D^1, \dots, |d^{(r-1)}| \leq D^{r-1}$$

(A3) The uncertainty upper bounds are characterized by the following inequalities:

$$\|G_i^T\| \leq \eta_i$$

$$|d_i| \leq D_i^0 \quad \text{and} \quad |\dot{d}_i| \leq D_i^1, \quad i = 1, \dots, r-1$$

where $G_i = \bar{c}\bar{A}^i - cA^i$ and $d_i = \sum_{j=1}^i \bar{c}\bar{A}^{i-j} d^{(j-1)}$

In particular, η_i 's and D_i^0 's are known *a priori*.

The objective is to obtain estimates of the system output derivatives up to the $(r-1)$ 'th order, i.e., $\dot{y}, \ddot{y}, \dots, y^{(r-1)}$. We notice that if there are no parameter mismatches and no external disturbances; in other words,

$$\Delta A = 0, \quad \Delta b = 0, \quad \Delta c = 0 \quad \text{and} \quad d = 0,$$

the first $(r-1)$ 'th derivatives of y can be obtained by successive differentiations of y in (16):

$$y^{(i)} = cA^i x \quad i = 1, \dots, r-1 \quad (17)$$

Equation (17) is an open-loop estimator for $y^{(i)}$. However, such an open-loop estimator is never practical due to the ever presence of system uncertainties. In the more realistic case, $y^{(i)}$ is given by

$$y^{(i)} = \bar{c}\bar{A}^i x + \bar{c}\bar{A}^{i-1}d + \dots + \bar{c}d^{(i-1)} \\ = cA^i x + G_i x + d_i \quad i = 1, \dots, r-1 \quad (18)$$

where G_i 's and d_i 's are as defined in Assumption (A3). The last two terms in (18) are unknown. However, if we can estimate $G_i x + d_i$, we obtain an estimate of $y^{(i)}$ in (18). Therefore, we propose the following output derivative estimators based on the disturbance estimator (12) in Section 2.

Unrealizable Output Derivative Estimator:

$$\dot{z}_i = cA^i x + v_i, \quad i = 1, \dots, r-1 \quad (19a)$$

where v_i is the estimator input given by

$$v_i = \sigma e_i + \rho_i g(e_i), \quad \rho_i = \rho_{1i} \|x\| + \rho_{2i} \quad (19b)$$

σ and $g(\cdot)$ are as given in (4), ρ_{1i} and ρ_{2i} are two positive constants satisfying

$$\rho_{1i} > \eta_i, \quad \rho_{2i} > \max(D_i^0, \rho_{1i}) \quad (19c)$$

and e_i is the estimation error defined by

$$e_i = y^{(i-1)} - z_i \quad (19d)$$

Theorem 2: Consider the system (16) subject to Assumptions (A1) - (A3), and the output derivative estimators (19). If the control input u in (16) is a "normal" state feedback control, then given any $\epsilon_i > 0$, there exist large enough ρ_{1i} and ρ_{2i} in (19c) and a finite time $T_i > 0$ such that

$$|\dot{z}_i(t) - y^{(i)}(t)| < \epsilon_i \quad \text{for all } t \geq T_i, \quad i = 1, \dots, r-1$$

Proof: see Appendix D.

Theorem 2 states that when the control input u is "normal", $\dot{z}_i$ in (19a) becomes a very accurate estimate of $y^{(i)}$ if we use large enough ρ_{1i} and ρ_{2i} in (19c). However, since the proposed estimators (19) are unrealizable (because of e_i defined in (19d)), we suggest the following realizable estimators to approximate the unrealizable ones.

Realizable Output Derivative Estimator:

Define another set of estimation errors

$$\hat{e}_1 = e_1 = y - z_1 \\ \hat{e}_i = \dot{z}_{i-1} - z_i, \quad i = 2, \dots, r-1 \quad (19d')$$

and use $\hat{e}_i$ instead of e_i to compute v_i in (19b).

Remark: The approximation of e_i by $\hat{e}_i$ is motivated by the following observation: since $\hat{e}_1 = e_1$, Theorem 2 says that $\dot{z}_1$ can be made arbitrarily close to $\dot{y}$ within a finite period of time. When estimating $\ddot{y}$ by the second estimator ($i=2$) in (19), we need $\dot{y}$ as a reference to calculate e_2 in (19d). Unfortunately, $\dot{y}$ is not accessible, we therefore use $\dot{z}_1$ as the reference instead. This motivates the proposition of $\hat{e}_2$ in (19d'). Since $\dot{z}_1$ is infinitesimally close to $\dot{y}$, we expect that Theorem 2 still holds for $i=2$. By continuing this argument up to $i=r-1$, we expect that all the results of Theorem 2 are retained when e_i is replaced by $\hat{e}_i$. Notice that no formal proof is given for the results of this realizable output derivative estimator. However, simulation study verifies the effectiveness of this approximation scheme.

Example 3: Consider a system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -6+\Delta_1 & 0 & 1+\Delta_2 \\ 13 & -7 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1+\Delta_3 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} w, \quad y = x_1$$

where the control input u , the disturbance w and parameter mismatches Δ_i are

$$u = 5\cos(3t), \quad w = \cos(t)$$

$$\Delta_1 = -5, \quad \Delta_2 = 4 \quad \text{and} \quad \Delta_3 = 0.6$$

We use the estimators (19) to estimate the output derivatives $\dot{y}$ and $\ddot{y}$ with the following parameters

$$g(e) = \frac{e}{1 + 0.01e}, \quad \sigma = 0.001$$

$$\rho_1 = 2 \quad \text{and} \quad \rho_2 = 7|x_1| + 7|x_2| + 2$$

and replace e_i in (19d) by $\hat{e}_i$ in (19d'). The time history of the system output y is shown in Figure 5. The estimates of $\dot{y}$ and $\ddot{y}$ are shown in Figure 6 and 7. An accurate estimate of $\dot{y}$ is obtained after $t > 1.1$, and after another 0.1 second or so we obtain an accurate estimate of $\ddot{y}$.

6. Conclusions

In this paper, a disturbance estimator is proposed to estimate unmeasurable disturbances which may be arbitrarily time-varying and may grow to infinity with a cone-bounded growth rate. It is shown that the disturbance estimation can be achieved to any desired accuracy within a finite period of time. The proposed disturbance estimator is then applied to the estimation of output derivatives of an uncertain linear system. Given a nominal state space description of the system, if the control input satisfies a "normality" condition, a scheme is proposed to estimate the derivatives of the system output up to the $(r-1)$ 'th order, where r is the relative degree of the system. When r is larger than two, this estimation scheme is first obtained in an unrealizable form, and then its approximation in a realizable form is suggested. The realizable scheme is verified to be effective by a computer simulation study.

We mention that a similar output derivative estimation scheme has been constructed for the case when the system state is not directly accessible and when only a nominal transfer function of the system is available [12].

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Appendix A

Table 1 Smooth Functions		
	$\ s\ > \epsilon$	$\ s\ \leq \epsilon$
$g_1(s)$	$\frac{s}{ s + \delta}$	$\frac{s}{ s + \delta}$
$g_2(s)$	$\text{sign}(s)$	$\frac{s}{\epsilon}$
$g_3(s)$	$\text{sign}(s)$	$\epsilon^{q-1} s^{1-q} \text{sign}(s)$

In Table 1,

$$\text{sign}(s) = \begin{cases} 1 & s \geq 0 \\ -1 & s < 0 \end{cases}$$

ϵ and δ are small positive numbers and $q \in [0, 1]$.

Appendix B

Proof of Theorem 1: let $V_1 = \frac{1}{2} x^2$, and take the derivative of V_1 :

$$\begin{aligned} \dot{V}_1 &= x \dot{x} \\ &= x (-\alpha x - \rho g(x) + \gamma z + d) \\ &\leq -\alpha x^2 - |x| \rho (g(|x|) - \frac{\Delta^0 |z| + D^0}{\rho_1 |z| + \rho_2}) \\ &\leq -\sigma n^2 \quad \text{for all } x \in N_1^c \end{aligned} \quad (\text{B.1})$$

where N_1 is defined in (5). Equation (B.1) shows that as long as $|x| \geq n$, the magnitude of x decreases with a nonzero rate. Therefore, we conclude that after a finite period of time, say $T_1 (\leq V_1(0)/(\sigma n^2))$, x is driven into N_1 and stays inside N_1 thereafter.

To study how $\dot{x}$ behaves, let $V_2 = \frac{1}{2} \dot{x}^2$ and differentiate V_2 :

$$\begin{aligned} \dot{V}_2 &= \dot{x} \ddot{x} \\ &= \dot{x} (-\alpha \dot{x} - \rho g'(\dot{x}) \dot{x} - \rho_1 g(x) \frac{d}{dt} |z| + \dot{\gamma} z + \gamma \dot{z} + \dot{d}) \end{aligned}$$

Notice that for any finite dimensional vector Z , we have the following inequality

$$\left| \frac{d}{dt} \|Z\| \right| \leq \|\dot{Z}\|, \quad \text{for all } Z \neq 0 \quad (\text{B.2})$$

Although inequality (B.2) does not hold for $Z=0$, we will show in Appendix C that we can use (B.2) for all Z in the remainder of the proof. Using (B.2), we obtain

$$\begin{aligned} \dot{V}_2 &\leq -\alpha \dot{x}^2 - |\dot{x}|^2 \rho g'(\dot{x}) + |\dot{x}| \rho_1 |\dot{z}| g(x) + |\dot{x}| \Delta^1 |z| \\ &\quad + \Delta^0 |\dot{x}| \cdot |z| + |\dot{x}| D^1 \end{aligned}$$

From Part I of the theorem, we know that x enters N_1 after $t > T_1$. This implies that after $t > T_1$, we have

$$\begin{aligned} \dot{V}_2 &\leq -\alpha \dot{x}^2 - |\dot{x}|^2 \rho g_1' + |\dot{x}| \rho_1 |\dot{z}| \bar{g}_1 + \Delta^1 |\dot{x}| \cdot |z| \\ &\quad + \Delta^0 |\dot{x}| \cdot |z| + |\dot{x}| D^1 \end{aligned}$$

Finally, it follows from the regularity assumption of z that

$$\begin{aligned} \dot{V}_2 &\leq -\alpha \dot{x}^2 - |\dot{x}| \rho(z) g_3' (|\dot{x}| - \frac{(c\alpha + \Delta^1) |z| + (c\beta + D^1)}{g_1' (\rho_1 |z| + \rho_2)}) \\ &\leq -\sigma n'^2 \quad \text{for all } \dot{x} \in N_2^c \end{aligned} \quad (\text{B.3})$$

where $c = \rho_1 \bar{g}_1 + \Delta^0$, and N_2 is defined in (6). From (B.3), we conclude that $\dot{x}$ is driven into N_2 after a finite period of time, say $T_2 (\leq T_1 + V_2(T_1)/(\sigma n'^2))$, and stays inside N_2 thereafter. This concludes the proof.

Appendix C

We show that even though (B.2) is valid only when $Z \neq 0$, we can

in fact use (B.2) for all values of Z if we slightly modify the estimator (4a).

Recall that in Theorem 1 the variable u is given by

$$u = -\sigma x - \rho g(x) \quad \rho = \rho_1 |z| + \rho_2 \quad (C.1)$$

Since the function $|z|$ used in ρ is not differentiable at $z=0$, we replace $|z|$ in (C.1) by another function which is differentiable for all z :

$$\phi(z) = \begin{cases} |z| & |z| \geq \varepsilon \\ \frac{1}{2\varepsilon} z^2 + \frac{\varepsilon}{2} & |z| < \varepsilon \end{cases} \quad (C.2)$$

where ε is an arbitrary positive constant. Note that $\phi(z)$ is a C^1 function satisfying

$$(i) \quad \phi(z) \geq |z|, \quad (ii) \quad |\phi'(z)| \leq 1 \quad \text{for all } z \quad (C.3)$$

We replace (C.1) by

$$u = -\sigma x - \bar{\rho} g(x) \quad \bar{\rho} = \rho_1 \phi(z) + \rho_2 \quad (C.4)$$

where σ , ρ_1 and ρ_2 are as defined in (4) and $\phi(\cdot)$ is given by (C.2). It can be shown [12] that when u is given by (C.4), we can still obtain the same results as stated in Theorem 1, with the only difference that N_1 in (5) is replaced by N'_1 , where

$$N'_1 = \left\{ x : |x| < n, \quad n = g^{-1} \left[\sup_{z \in K} \frac{\Delta^0 |z| + D^0}{\rho_1 \phi(z) + \rho_2} \right] \right\}$$

Since ε in (C.2) can be an arbitrary positive number, by choosing an arbitrarily small ε , we actually make $\phi(\cdot)$ in (C.2) infinitesimally close to the absolute function $| \cdot |$ used in (9a), and N'_1 infinitesimally close to N_1 . Hence, for practical purpose, we use $| \cdot |$ instead of the $\phi(\cdot)$ function throughout this paper, and allow Eq.(B.2) to be used for all z .

Appendix D

Proof of Theorem 2: from (18) and (19a), the dynamics of e_i are governed by

$$\dot{e}_i = -v_i + G_i x + d_i, \quad i = 1, \dots, r-1 \quad (D.1)$$

Since the control input u is "normal", Lemma 1 guarantees that the uncertainty $G_i x + d_i$ in (D.1) is "strictly regular". Invoking Theorem 1, we conclude that $v_i - G_i x - d_i (= \dot{z}_i - y^{(i)})$ can be made arbitrarily small within a finite period of time if sufficiently large ρ_{1i} and ρ_{2i} in (19c) are used. This concludes the proof.

Figure 1. Disturbance estimation with different disturbance frequencies

Figure 2. Estimation of disturbance with different ρ 's

Figure 3. Estimation of disturbance with different δ 's

Figure 4. Estimation of an exponentially growing disturbance

Figure 5. Estimation of system output derivatives

Figure 6. Estimation of system output derivatives

Figure 7. Estimation of system output derivatives