Fault-Tolerant Ring Embedding in Faulty Arrangement Graphs

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Abstract

The arrangement graph $A_{n,k}$, which is a generalization of the star graph ($n - k = 1$), presents more flexibility than the star graph in adjusting the major design parameters: number of nodes, degree, and diameter. Previously the arrangement graph has proven hamiltonian. In this paper we further show that the arrangement graph remains hamiltonian even if it is faulty. Let $|F_e|$ and $|F_v|$ denote the numbers of edge faults and vertex faults, respectively. We show that $A_{n,k}$ is hamiltonian when (1) ($k = 2$ and $n - k \geq 4$, or $k \geq 3$ and $n - k \geq 4 + \lceil \frac{k}{2} \rceil$), and $|F_e| \leq k(n - k) - 2$, or (2) $k \geq 2$, $n - k \geq 2 + \lceil \frac{k}{2} \rceil$, and $|F_e| \leq k(n - k - 3) - 1$, or (3) $k \geq 2$, $n - k \geq 3$, and $|F_e| \leq k$.

1 Introduction

The star graph [1] has been recognized as an attractive alternative to the hypercube network. Unfortunately, there is a major drawback for the star graph: a rigorous restriction to its size (i.e., $n!$). Recently, the arrangement graph [4], which is a generalization of the star graph, has been proposed as a possible solution to the problem. The arrangement graph presents more flexibility than the star graph in terms of choosing the major design parameters: number of vertices, degree, and diameter, while preserving most of the nice properties of the star graph. Previous work on the arrangement graph can be found in [3] [4] [5] [6].

Since node faults and link faults may happen when a network is put in use, it is practically meaningful to consider faulty networks. Many related works have appeared in the literature, for example, [7] [8] [9] [10]. In [8], Latifi, Zheng, and Bagherzadeh showed that an n -dimensional hypercube with at most $n - 2$ link faults is hamiltonian. In [9], Rowley and Bose showed that with slight modifications a base- d undirected de Bruijn graph with at most $d - 1$ edge faults is hamiltonian. In [7], Fernandes, Friesen, and Kanevsky showed that a WK-recursive network of degree d with at most $d - 3$ link faults is hamiltonian. In [10], Tseng, Chang and Sheu showed that an n -dimensional star graph with at most $n - 3$ edge faults is hamiltonian.

Although Day and Tripathi [4] have shown that the arrangement graph is hamiltonian, they have assumed the arrangement graph is fault-free. In the remaining sections, we consider faulty arrangement graphs, and investigate their hamiltonicity.

2 Preliminaries

First we briefly review the arrangement graph, denoted by $A_{n,k}$, where $1 \leq k < n$.

Definition 1 The vertex set of $A_{n,k}$ is denoted by $\{p_1 p_2 \cdots p_k \mid p_i \in \{1, 2, \dots, n\} \text{ for all } 1 \leq i \leq k \text{ and } p_i \neq p_j \text{ for } i \neq j\}$. Vertex adjacency is defined as follows: $p_1 p_2 \cdots p_k$ is adjacent to $q_1 q_2 \cdots q_k$ if and only if $p_i \neq q_i$ for some $1 \leq i \leq k$ and $p_j = q_j$ for all $1 \leq j \leq k$ and $j \neq i$.

Often k is referred to as the dimensions of $A_{n,k}$, and the position of p_i is said to be the i th dimension. An edge of $A_{n,k}$ is said to be of the i th dimension if its two incident vertices differ between their corresponding arrangements in the i th dimension.

The structure of $A_{n,k}$ is recursive. We note that $A_{n,k}$ contains embedded $A_{n-k+r,r}$ for all $1 \leq r \leq k$. An embedded $A_{n-k+r,r}$ is conveniently denoted by $\langle s_1 s_2 \dots s_k \rangle_{n,r}$, where $s_i \in \{*, 1, 2, \dots, n\}$ for all $1 \leq i \leq k$ ($*$ represents a "don't care" symbol) and exactly r of $s_1, s_2, \dots, s_k$ are $*$. Two basic operations on $\langle s_1 s_2 \dots s_k \rangle_{n,r}$ are defined as follows.

Definition 2 Suppose $\langle s_1 s_2 \dots s_k \rangle_{n,r}$ is an embedded $A_{n-k+r,r}$ of $A_{n,k}$. For each $s_i = *$ ($1 \leq i \leq k$), we define the i -partition on $\langle s_1 s_2 \dots s_k \rangle_{n,r}$ to be an operation that partitions the embedded $A_{n-k+r,r}$ into $n - k + r$ embedded $A_{n-k+r-1,r-1}$'s, denoted by $\langle s_1 s_2 \dots s_{i-1} q s_{i+1} \dots s_k \rangle_{n,r-1}$, where q is a missing element of $\{1, 2, \dots, n\}$ in $s_1 s_2 \dots s_k$.

Definition 3 Suppose $\langle s_1 s_2 \dots s_k \rangle_{n,r}$ represents an embedded $A_{n-k+r,r}$ of $A_{n,k}$. Let $I = (i_1, i_2, \dots, i_m)$, where $1 \leq i_j \leq k$ and $s_{i_j} = *$ for all $1 \leq j \leq m$. We

define the I -partition on $\langle s_1 s_2 \dots s_k \rangle_{n,r}$ to be a sequence of operations that perform i_1 -partition, i_2 -partition, ..., i_m -partition, sequentially, on $\langle s_1 s_2 \dots s_k \rangle_{n,r}$. After the I -partition, the embedded $A_{n-k+r,r}$ is partitioned into $(n-k+r)(n-k+r-1) \dots (n-k+r-m+1)$ embedded $A_{n-k+r-m,r-m}$'s.

Suppose $\langle s_1 s_2 \dots s_k \rangle_{n,r}$ and $\langle t_1 t_2 \dots t_k \rangle_{n,r}$ represents two embedded $A_{n-k+r,r}$'s of $A_{n,k}$. They are said to be *adjacent* if there exists some $1 \leq j \leq k$ so that $s_j \neq *$, $t_j \neq *$, and $s_j \neq t_j$, and $s_i = t_i$ for all $1 \leq i \leq k$ and $i \neq j$. Further, we use $\text{dif}(\langle s_1 s_2 \dots s_k \rangle_{n,r}, \langle t_1 t_2 \dots t_k \rangle_{n,r}) (= j)$ to indicate the different position between $s_1 s_2 \dots s_k$ and $t_1 t_2 \dots t_k$.

Definition 4 Let

$A_0, A_1, \dots, A_{n(n-1)(n-2) \dots (n-k+r+1)-1}$ represent embedded $A_{n-k+r,r}$'s of $A_{n,k}$ that result from executing a $(i_1, i_2, \dots, i_{k-r})$ -partition on $A_{n,k}$, where $i_1 i_2 \dots i_{k-r}$ is a permutation of $k-r$ elements from $\{1, 2, \dots, k\}$. If A_i is adjacent to $A_{(i-1) \bmod n(n-1)(n-2) \dots (n-k+r+1)-1}$ and $A_{(i+1) \bmod n(n-1)(n-2) \dots (n-k+r+1)-1}$ for all $0 \leq i \leq n(n-1)(n-2) \dots (n-k+r+1)-1$, then $A_0, A_1, \dots, A_{n(n-1)(n-2) \dots (n-k+r+1)-1}$ are said to form an r -ring ($0 \leq r \leq k$), denoted by $R_r = [A_0, A_1, \dots, A_{n(n-1)(n-2) \dots (n-k+r+1)-1}]$.

If we regard each A_i as a supervertex called r -vertex, $r \geq 1$, and define a superedge called r -edge between A_i and $A_{(i+1) \bmod n(n-1)(n-2) \dots (n-k+r+1)-1}$, then R_r forms a cycle of $n(n-1)(n-2) \dots (n-k+r+1)$ r -vertices. Actually, each r -edge consists of $(n-k+r-1)(n-k+r-2) \dots (n-k)$ edges of $A_{n,k}$. Each r -vertex can be partitioned into $n-k+r$ $(r-1)$ -vertices, and any two of them are connected with an $(r-1)$ -edge. So, each r -vertex can be viewed as a complete graph of $n-k+r$ $(r-1)$ -vertices. For easy reference, we use K_{n-k+r}^{r-1} to denote such a complete graph in subsequent discussion.

Definition 5 We define an i -partition on $R_r = [A_0, A_1, \dots, A_{n(n-1)(n-2) \dots (n-k+r+1)-1}]$ to be a sequence of operations that perform an i -partition on $A_0, A_1, \dots, A_{n(n-1)(n-2) \dots (n-k+r+1)-1}$, respectively, where $1 \leq i \leq k$ and $r \geq 1$.

Lemma 2.1 For $A_{n,k}$ with $n-k \geq 3$ (resp. $n-k = 2$), an R_r of (maximal) length $n(n-1)(n-2) \dots (n-k+r+1)$ contains an R_{r-1} of (maximal) length $n(n-1)(n-2) \dots (n-k+r+1)(n-k+r)$, where $r \geq 1$ (resp. $r \geq 2$).

Proof:

Suppose $R_r = [A_0, A_1, \dots, A_{n(n-1)(n-2) \dots (n-k+r+1)-1}]$. First a partition on R_r is executed. Then, for $0 \leq i \leq n(n-1)(n-2) \dots (n-k+r+1)-1$, two $(r-1)$ -vertices, say X_i and Y_i , are determined in each A_i so that $Y_{(i-1) \bmod n(n-1)(n-2) \dots (n-k+r+1)-1}$ and $X_{(i+1) \bmod n(n-1)(n-2) \dots (n-k+r+1)-1}$ are connected to X_i and Y_i through $(r-1)$ -edges, respectively. The existence of X_i and Y_i is assured by $n-k+r \geq 4$ (because

$n-k \geq 3$ and $r \geq 1$, or $n-k = 2$ and $r \geq 2$). Since each A_i can be viewed as a complete graph K_{n-k+r}^{r-1} , there exists a path, denoted by P_i , from X_i to Y_i that passes all $(r-1)$ -vertices in A_i exactly once. Clearly, all the $(r-1)$ -vertices encountered along the path: $P_0, (Y_0, X_1), P_1, (Y_1, X_2), \dots, P_{n(n-1)(n-2) \dots (n-k+r+1)-1}, (Y_{n(n-1)(n-2) \dots (n-k+r+1)-1}, X_0)$, form an R_{r-1} of the desired length. **Q.E.D.**

Remark. In the rest of this paper, X_i and Y_i as described above are referred to as the *entry* $(r-1)$ -vertex and the *exit* $(r-1)$ -vertex of A_i , respectively.

Given a graph G , we use $V(G)$ and $E(G)$ to represent the sets of vertices and edges of G , respectively. We say G is *Connected* if there exists a path in G between any two of its vertices. The *degree* of $v \in V(G)$, denoted by $\deg(v)$, is the number of edges incident with it. A cycle (path) in G is called a *hamiltonian cycle (path)* if it contains every vertex of G exactly once. G is said to be *hamiltonian* if it contains a hamiltonian cycle. A hamiltonian graph G is said to be k *vertex (edge) fault-tolerant hamiltonian* if it remains hamiltonian after removing at most k vertices (edges) [11]. G is said to be *hamiltonian-connected* if there exists a hamiltonian path between every two vertices of G .

Definition 6 Suppose G is a hamiltonian graph. The *vertex (resp. edge) fault-tolerant hamiltonicity* of G , denoted by $H_v(G)$ (resp. $H_e(G)$), is defined as $\max\{k \mid G \text{ is } k \text{ vertex (resp. edge) fault-tolerant hamiltonian}\}$.

Lemma 2.2 [2] If $|V(G)| \geq 3$ and $\deg(u) + \deg(v) \geq |V(G)|$ for every two non-adjacent vertices u, v of G , then G is hamiltonian.

Lemma 2.3 [2] If G is a connected graph and $\deg(u) + \deg(v) \geq |V(G)| + 1$ for every two non-adjacent vertices u, v of G , then G is hamiltonian-connected.

Lemma 2.4 Removing any $d-3$ edges $(x_1, y_1), (x_2, y_2), \dots, (x_{d-3}, y_{d-3})$ from K_d (the complete graph of d vertices) results in a hamiltonian graph (denoted by K'_d). Further, if for each (x_i, y_i) , $1 \leq i \leq d-3$, there exists another (x_j, y_j) , $j \neq i$, so that they are not incident to the same vertex, then K'_d is hamiltonian-connected.

Proof: We first prove that K'_d is hamiltonian by showing $\deg(x_i) + \deg(y_i) \geq d$ for all $1 \leq i \leq d-3$. Let p and q denote the number of removed edges, exclusive of (x_i, y_i) , that are incident to x_i and y_i , respectively. Clearly, we have $p+q \leq d-4$, and so $\deg(x_i) + \deg(y_i) = (d-1-(p+1)) + (d-1-(q+1)) = 2d-4-(p+q) \geq 2d-4-(d-4) = d$. By Lemma 2.2, K'_d is hamiltonian. It can be proved similarly that K'_d is hamiltonian-connected if the condition is satisfied. We have $p+q \leq d-5$ under the condition, and thus $\deg(x_i) + \deg(y_i) \geq d+1$. By Lemma 2.3, K'_d is hamiltonian-connected. **Q.E.D.**

Slight modifications of the proof above can lead to the following lemma.

Lemma 2.5 Suppose $(x_1, y_1), (x_2, y_2), \dots, (x_{d-2}, y_{d-2})$ are any $d-2$ edges of K_d . If for each (x_i, y_i) , $1 \leq i \leq d-2$, there exists another (x_j, y_j) , $j \neq i$, so that they are not incident to the same vertex, then removing $d-2$ edges from K_d results in a hamiltonian graph.

The following two lemmas can be proved similarly.

Lemma 2.6 Removing any $d-4$ edges from K_d results in a hamiltonian-connected graph.

Lemma 2.7 Removing any $d-6$ edges and any two vertices of K_d results in a hamiltonian-connected graph.

3 Fault-Free Hamiltonian Cycles in $A_{n,k}$ with Faulty Edges

Let F_e denote the set of faulty edges. According to the dimensions where faulty edges belong to, F_e can be partitioned into k subsets, denoted by D_i , $i = 1, 2, \dots, k$, where each D_i is the set of faulty edges that belong to the i th dimension.

Lemma 3.1 Suppose $|F_e| = k(n-k) - 2$, let $S = a_1 a_2 \dots a_k$ be a permutation of $1, 2, \dots, k$ so that $|D_{a_1}| \geq |D_{a_2}| \geq \dots \geq |D_{a_k}|$. Then, we have $|D_{a_k}| \leq n-3$ if $k=1$, and $|D_{a_k}| \leq (n-k) - 1$ if $k \geq 2$.

Proof: We have $|D_{a_k}| \leq \frac{|F_e|}{k} \leq \frac{(k(n-k)-2)}{k}$. Thus the maximal value of $|D_{a_k}|$ is $\lfloor \frac{(k(n-k)-2)}{k} \rfloor$, which is equal to $n-3$ as $k=1$, and $(n-k) - 1$ as $k \geq 2$. **Q.E.D.**

Lemma 3.2 Suppose $U = \langle u_1 u_2 \dots u_k \rangle_{n,r}$, $V = \langle v_1 v_2 \dots v_k \rangle_{n,r}$, and $W = \langle w_1 w_2 \dots w_k \rangle_{n,r}$, are arbitrary three consecutive r -vertices in R_r , where $r \geq 1$. Also let $p = \text{dif}(U, V)$ and $q = \text{dif}(V, W)$. If $u_p \neq w_q$, then after executing a partition on R_r each $(r-1)$ -vertex of V is adjacent to one $(r-1)$ -vertex of U or one $(r-1)$ -vertex of W .

Proof: Without loss of generality, we assume that a j -partition is executed on R_r , where $1 \leq j \leq k$. Hence, we have $u_j = v_j = w_j = *$. Since $p = \text{dif}(U, V)$ and $q = \text{dif}(V, W)$, we have $u_p \neq v_p$, $v_q \neq w_q$, $u_i = v_i$ for all $1 \leq i \leq k$ and $i \neq p$, and $v_i = w_i$ for all $1 \leq i \leq k$ and $i \neq q$. Suppose $u_p \neq w_q$, and after the j -partition, there exists an $(r-1)$ -vertex, say $V' = \langle v_1 v_2 \dots v_{j-1} v' v_{j+1} \dots v_k \rangle_{n,r-1}$, of V that is not adjacent to any $(r-1)$ -vertex of U and W . Thus, we have $v' = u_p$ for otherwise V' is adjacent to some $(r-1)$ -vertex of U . Similarly, we have $v' = w_q$. This implies $u_q = v' = w_q$, which contradicts our assumption. **Q.E.D.**

For $1 \leq i \leq k-1$, an i -edge of $A_{n,k}$ is said to be *fault-free* if it does not contain faulty r -edges. In the rest of this section, we use $E_{i,j}$ to denote the set of faulty i -edges that belong to the j th dimension.

Lemma 3.3 Suppose $|F_e| = k(n-k) - 2$, and let $S = a_1 a_2 \dots a_k$ be a permutation of $1, 2, \dots, k$ so that $|D_{a_1}| \geq |D_{a_2}| \geq \dots \geq |D_{a_k}|$. If $|D_{a_1}| = |D_{a_2}| = \dots = |D_{a_{k-2}}| = n-k$, there is a fault-free R_2 in $A_{n,k}$.

Proof: First we construct fault-free $R_{k-1}, R_{k-2}, \dots, R_3$ for the faulty $A_{n,k}$. We only need to apply a $(a_1, a_2, \dots, a_{k-3})$ -partition to the faulty $A_{n,k}$. Initially, an a_1 -partition is executed on the faulty $A_{n,k}$, and so a K_n^{k-1} is obtained. We note that all edges of the K_n^{k-1} are $(k-1)$ -edges and $|E_{k-1,a_1}| \leq |D_{a_1}| = n-k$ of them are faulty. By Lemma 2.4, the K_n^{k-1} with $|E_{k-1,a_1}|$ faulty $(k-1)$ -edges removed is hamiltonian. Hence, a fault-free R_{k-1} can be generated.

In general, when an a_i -partition is executed on a fault-free R_{k-i+1} for $2 \leq i \leq k-3$, each $(k-i+1)$ -vertex of the fault-free R_{k-i+1} forms a K_{n-i+1}^{k-i} . Since $|E_{k-i,a_i}| \leq |D_{a_i}| = n-k$, each K_{n-i+1}^{k-i} contains $n-k$ faulty $(k-i)$ -edges at most. Now that each K_{n-i+1}^{k-i} contains $n-i+1 \geq n-k+4$ $(k-i)$ -vertices, Lemma 2.6 assures that it is hamiltonian-connected, even if all faulty $(k-i)$ -edges are removed. As a result, there is a fault-free hamiltonian path for each K_{n-i+1}^{k-i} that goes from the entry $(k-i)$ -vertex to the exit $(k-i)$ -vertex. All these hamiltonian paths interleaved with $(k-i)$ -edges form a fault-free R_{k-i} .

Next we show how to obtain a fault-free R_2 from a fault-free $R_3 = [A_0, A_1, \dots, A_{n(n-1)(n-2)\dots(n-k+4)-1}]$. An a_{k-2} -partition is first executed on R_3 so that each A_i forms a K_{n-k+3}^2 , where $0 \leq i \leq n(n-1)(n-2)\dots(n-k+4) - 1$. There are totally $|E_{2,a_{k-2}}|$ ($|D_{a_{k-2}}| = n-k$) faulty 2-edges contained in all A_i 's. If $|E_{2,a_{k-2}}| < n-k$ or $|E_{2,a_{k-2}}|$ faulty 2-edges are distributed over two or more A_i 's, then it is assured by Lemma 2.6 that each K_{n-k+3}^2 with faulty 2-edges removed is hamiltonian-connected. Consequently, as described in the proof of Lemma 2.1, a fault-free R_2 can be generated by properly selecting the entry 2-vertex and the exit 2-vertex for each A_i .

On the other hand, if $|E_{2,a_{k-2}}| = n-k$ and all faulty 2-edges are located in some A_t , then it is assured by Lemma 2.4 that the K_{n-k+3}^2 resulting from A_t with $n-k$ faulty 2-edges removed is hamiltonian. That is, the K_{n-k+3}^2 contains a fault-free hamiltonian cycle, denoted by $C = (X_0, X_1, \dots, X_{n-k+2}, X_0)$, where each X_j ($0 \leq j \leq n-k+2$) is a 2-vertex of A_t . Since there are $n-k+2$ 2-edges between each pair of A_i and $A_{(i+1) \bmod n(n-1)(n-2)\dots(n-k+4)}$, there are $n-k+2$ 2-vertices in C that are adjacent to $n-k+2$ 2-vertices of $A_{(t-1) \bmod n(n-1)(n-2)\dots(n-k+4)}$, and there are $n-k+2$ 2-vertices in C that are adjacent to $n-k+2$ 2-vertices of $A_{(t+1) \bmod n(n-1)(n-2)\dots(n-k+4)}$. Thus, there must exist a 2-edge, say $(X_l, X_{(l+1) \bmod n-k+3})$, of C so that X_l and $X_{(l+1) \bmod n-k+3}$ are adjacent to 2-vertices of $A_{(t-1) \bmod n(n-1)(n-2)\dots(n-k+4)}$ and $A_{(t+1) \bmod n(n-1)(n-2)\dots(n-k+4)}$, respectively. Since the other K_{n-k+3}^2 's that result from all A_j 's with $j \neq t$ are hamiltonian-connected and their 2-edges are not faulty, a fault-free R_2 can be generated by taking X_l and $X_{(l+1) \bmod n-k+3}$ as the entry and exit 2-vertices of A_t , respectively, and properly selecting the entry and exit 2-vertices for all A_j 's with $j \neq t$. **Q.E.D.**

For ease of description, an $R_r = [A_0, A_1, \dots, A_{n(n-1)(n-2)\dots(n-k+r+1)-1}]$ is said to be *good* if after a partition each $(r-1)$ -vertex of A_i is adjacent to one $(r-1)$ -vertex of $A_{(i-1) \bmod n(n-1)(n-2)\dots(n-k+r+1)}$ or one $(r-1)$ -vertex of $A_{(i+1) \bmod n(n-1)(n-2)\dots(n-k+r+1)}$.

Lemma 3.4 Suppose $|F_e| = k(n-k) - 2$, and let $S = a_1 a_2 \dots a_k$ be a permutation of $1, 2, \dots, k$ so that $|D_{a_1}| \geq |D_{a_2}| \geq \dots \geq |D_{a_k}|$. If $|D_{a_1}| = |D_{a_2}| = \dots = |D_{a_{k-2}}| = n-k > 5$ and $|D_{a_{k-1}}| = n-k-1$, there is a fault-free good R_1 in $A_{n,k}$.

Proof: With the aid of Lemmas 2.4, 3.2 and Lemma 3.3, the result can be shown. **Q.E.D.**

Lemma 3.5 Suppose $|F_e| = k(n-k) - 2$, and let $S = a_1 a_2 \dots a_k$ be a permutation of $1, 2, \dots, k$ so that $|D_{a_1}| \geq |D_{a_2}| \geq \dots \geq |D_{a_k}|$. When $k=1$, there is a fault-free hamiltonian cycle for $A_{n,k}$ if $|D_{a_1}| = n-3$. When $k=2$, there is a fault-free hamiltonian cycle for $A_{n,k}$ if $n-k \geq 3$ and $|D_{a_1}| = |D_{a_2}| = n-3$. When $k \geq 3$, there is a fault-free hamiltonian cycle for $A_{n,k}$ if $n-k > 5$, $|D_{a_{k-2}}| = n-k$, and $|D_{a_{k-1}}| = |D_{a_k}| = n-k-1$.

Proof: When $k=1$, $A_{n,1}$ is a complete graph of n vertices. By Lemma 2.4, $A_{n,1}$ with $|D_{a_1}| = n-3$ faulty edges removed hamiltonian.

When $k=2$, we first apply an a_1 -partition to $A_{n,2}$ to form a K_n^1 . Since $|E_{1,a_1}| \leq |D_{a_1}| = n-3$, the K_n^1 contains at most $n-3$ faulty 1-edges. By the aid of Lemma 2.4, a fault-free $R = [A_0, A_1, \dots, A_{n-1}]$, can be determined in $A_{n,2}$. We use $X = \langle x_1 x_2 \dots x_k \rangle_{n,1}$, $Y = \langle y_1 y_2 \dots y_k \rangle_{n,1}$, and $Z = \langle z_1 z_2 \dots z_k \rangle_{n,1}$ to denote arbitrary three consecutive 1-vertices of the R_1 . Since $x_{dif}(X,Y) \neq z_{dif}(Y,Z)$, it is assured by Lemma 3.2 that the R_1 is good. We then apply an a_2 -partition to the R_1 so that each A_i ($0 \leq i \leq n-1$) forms a K_{n-1}^0 . Since there are totally $|E_{0,a_2}| = |D_{a_2}| = n-3$ faulty edges contained in all A_i 's, each K_{n-1}^0 contains $n-3$ faulty edges at most. Below we construct a fault-free hamiltonian cycle for $A_{n,2}$ according to three situations.

The first situation is that all $n-3$ faulty edges are located in the same A_t for some $0 \leq t \leq n-1$, and there exists a faulty edge, say (u, v) , so that the other $n-4$ faulty edges are incident to u or v . We note that $n-2$ vertices of A_t are connected to $A_{(t-1) \bmod n}$, and $n-2$ vertices of A_t are connected to $A_{(t+1) \bmod n}$. Moreover, since the R_1 is good, one vertex of A_t is connected to $A_{(t-1) \bmod n}$ only, another is connected to $A_{(t+1) \bmod n}$ only, and the others are each connected to both $A_{(t-1) \bmod n}$ and $A_{(t+1) \bmod n}$. Thus, it can be assured that one of u and v is connected to $A_{(t-1) \bmod n}$ and the other is connected to $A_{(t+1) \bmod n}$, and v is adjacent to a vertex, say v' , of $A_{(t+1) \bmod n}$.

We then show that there exists a fault-free hamiltonian path between u and v in A_t . Since such a path exists for the trivial case of $n=4$, we assume $n \geq 5$.

Let K' denote the resulting K_{n-1}^0 of A_t with $n-3$ faulty edges removed, and $H = K' - \{u, v\}$ be a complete graph of $n-3$ vertices. Since $n \geq 5$, H contains two or more vertices. Let x and y be two vertices of H that are adjacent to u and v , respectively. Clearly a hamiltonian path between x and y in H combined with edges (u, x) and (y, v) constitutes a fault-free hamiltonian path between u and v in A_t .

Since all the other A_i 's with $i \neq t$ do not contain faulty edges, they are hamiltonian-connected. A fault-free R_0 , i.e., a fault-free hamiltonian cycle, in $A_{n,2}$ can be generated by taking u and v as the entry and exit vertices of A_t , respectively, and properly selecting the entry and exit vertices for all A_i 's with $i \neq t$.

The second situation is that all $n-3$ faulty edges are located in the A_t , but the faulty edge (u, v) mentioned above does not exist. By Lemma 2.5, the resulting K_{n-1}^0 of A_t with $n-3$ faulty edges removed contains a fault-free hamiltonian cycle. Let f and g be any two adjacent vertices in the hamiltonian cycle. Since all other A_i 's with $i \neq t$ are hamiltonian-connected, a fault-free hamiltonian cycle for $A_{n,2}$ can be generated by taking f and g as the entry and exit vertices of A_t , respectively, and properly selecting the entry and exit vertices for all A_i 's with $i \neq t$.

The third situation is that $n-3$ faulty edges are distributed over two or more A_i 's. By Lemma 2.4, each resulting K_{n-1}^0 with faulty edges removed is hamiltonian-connected. As described in the proof of Lemma 2.1, a fault-free hamiltonian cycle for $A_{n,2}$ can be generated by properly selecting the entry and exit vertices for each A_i .

When $k \geq 3$, we have $|D_{a_1}| = |D_{a_2}| = \dots = |D_{a_{k-3}}| = n-k$ because $|F_e| = k(n-k) - 2$ and $|D_{a_1}| \geq |D_{a_2}| \geq \dots \geq |D_{a_k}|$. Since constructing a fault-free hamiltonian cycle for $A_{n,k}$ is very similar to the case of $k=2$, we only show the difference. As before an R_1 in $A_{n,k}$ is first obtained by executing a $(a_1, a_2, \dots, a_{k-1})$ -partition. Like the case of $k=2$, a good R_1 is necessary for constructing a fault-free hamiltonian cycle for $A_{n,k}$. But, unlike the case of $k=2$, a good R_1 is now assured by Lemma 3.4 because $n-k > 5$. **Q.E.D.**

Similar to the proof of the above, we have the following.

Lemma 3.6 Suppose $|F_e| = k(n-k) - 2$, and let $S = a_1 a_2 \dots a_k$ be a permutation of $1, 2, \dots, k$ so that $|D_{a_1}| \geq |D_{a_2}| \geq \dots \geq |D_{a_k}|$. When $k=2$, there is a fault-free hamiltonian cycle for $A_{n,k}$ if $|D_{a_1}| = n-k \geq 3$ and $|D_{a_2}| = n-k-2$. When $k \geq 3$, there is a fault-free hamiltonian cycle for $A_{n,k}$ if $n-k \geq 3$ and either $|D_{a_{k-1}}| = n-k$ and $|D_{a_k}| = n-k-2$ or $|D_{a_{k-2}}| = n-k$, $|D_{a_{k-1}}| = n-k-1$, and $|D_{a_k}| = n-k-2$.

An R_r is said to be *healthy* if after a partition its every r -edge contains at least three $(r-1)$ -edges that are not faulty.

Lemma 3.7 Suppose $|F_e| = k(n-k) - 2$, and let $S = a_1 a_2 \dots a_k$ be a permutation of $1, 2, \dots, k$ so that $|D_{a_1}| \geq |D_{a_2}| \geq \dots \geq |D_{a_k}|$. When $k \geq 3$ and $n \geq$

$\frac{3k+8}{2}$, there is a fault-free hamiltonian cycle for $A_{n,k}$ if $|D_{a_k}| \leq n - k - 3$.

Proof: We first prove by induction that healthy rings $R_{k-1}, R_{k-2}, \dots, R_1$ can be generated if a $(a_1, a_2, \dots, a_{k-1})$ -partition is applied to $A_{n,k}$. To begin, an a_1 -partition is applied to $A_{n,k}$, and so a K_{n-1}^{k-1} results. Since $|F_e| = k(n-k) - 2$, we have $|D_{a_1}| \leq k(n-k) - 2$. Let S_{k-1} denote the set of those $(k-1)$ -edges in $A_{n,k}$ that each contain $n-4$ or more faulty edges. We have $|S_{k-1}| \leq n-3$ for otherwise $|F_e| \geq (n-4)|S_{k-1}| > (n-4)(n-3) > k(n-k)$, which is a contradiction. It is assured by Lemma 2.4 that K_{n-1}^{k-1} with the set S_{k-1} of $(k-1)$ -edges removed is hamiltonian. That is, an R_{k-1} can be obtained whose $(k-1)$ -edges each contain at most $n-5$ faulty edges. Since each $(k-1)$ -edge of the R_{k-1} comprises $n-2$ $(k-2)$ -edges, the R_{k-1} is healthy.

Now we assume a healthy $R_{k-j+1} = [A_0, A_1, \dots, A_{n(n-1)(n-2)\dots(n-j+2)-1}]$ is obtained after applying a $(a_1, a_2, \dots, a_{j-1})$ -partition to $A_{n,k}$, where $2 \leq j \leq k-1$. Then an a_j -partition is applied to the R_{k-j+1} , and so each A_i ($0 \leq i \leq n(n-1)(n-2)\dots(n-j+2)-1$) forms a K_{n-j+1}^{n-j} . We note that $|D_{a_j}| \leq \lfloor \frac{k(n-k)-2}{j} \rfloor$. Let S_{k-j} denote the set of those $(k-j)$ -edges in all A_i 's that each contain $n-j-3$ or more faulty edges. We have $|S_{k-j}| \leq n-j-3$ for otherwise $|D_{a_j}| \geq (n-j-3)|S_{k-j}| > (n-j-3)(n-j-3) > \lfloor \frac{k(n-k)-2}{j} \rfloor$, which is a contradiction. Then Lemma 2.6 assures that each resulting K_{n-j+1}^{n-j} with the set S_{k-j} of $(k-j)$ -edges removed is hamiltonian-connected. Since the R_{k-j+1} is healthy, there are at least three fault-free $(k-j)$ -edges between each pair of A_i and $A_{(i+1) \bmod n(n-1)(n-2)\dots(n-j+2)}$. As a result, we can determine the entry and exit $(k-j)$ -vertices, say X_i and Y_i , respectively, for each A_i so that $X_i \neq Y_i$ and the $(k-j)$ -edges $(Y_{(i-1) \bmod n(n-1)(n-2)\dots(n-j+2)}, X_i)$ and $(Y_i, X_{(i+1) \bmod n(n-1)(n-2)\dots(n-j+2)})$ are not faulty. An R_{k-j} can be generated thus for $A_{n,k}$ whose each $(k-j)$ -edge contains at most $n-j-4$ faulty edges (and therefore at most $n-j-4$ faulty $(k-j-1)$ -edges). Since each $(k-j)$ -edge of the R_{k-j} comprises $n-j-1$ $(k-j-1)$ -edges, the R_{k-j} is healthy.

According to the discussion above, a healthy R_1 can be obtained after applying a $(a_1, a_2, \dots, a_{k-1})$ -partition to $A_{n,k}$. Then an a_k -partition is applied to the R_1 , and its each 1-vertex forms a K_{n-k+1}^0 . Since there are totally $|D_{a_k}| \leq n-k-3$ faulty edges, Lemma 2.6 assures that each resulting K_{n-k+1}^0 with faulty edges removed is hamiltonian-connected. Since R_1 is healthy, a fault-free R_0 can be generated for $A_{n,k}$ with the same arguments as above. **Q.E.D.**

With similar arguments to prove the above lemmas, we have the following results.

Lemma 3.8 Suppose $|F_e| = k(n-k) - 2$, and let $S = a_1 a_2 \dots a_k$ be a permutation of $1, 2, \dots, k$ so that $|D_{a_1}| \geq |D_{a_2}| \geq \dots \geq |D_{a_k}|$. When $k \geq 3$, there is a

fault-free hamiltonian cycle for $A_{n,k}$ if $n \geq \frac{3k+8}{2}$ and $|D_{a_{k-2}}| = |D_{a_{k-1}}| = |D_{a_k}| = n - k - 1$.

Lemma 3.9 Suppose $|F_e| = k(n-k) - 2$, and let $S = a_1 a_2 \dots a_k$ be a permutation of $1, 2, \dots, k$ so that $|D_{a_1}| \geq |D_{a_2}| \geq \dots \geq |D_{a_k}|$. When $k \geq 3$, there is a fault-free hamiltonian cycle for $A_{n,k}$ if $n \geq \frac{3k+8}{2}$ and either $|D_{a_{k-1}}| = |D_{a_k}| = n - k - 2$, $|D_{a_{k-2}}| = |D_{a_{k-1}}| = n - k - 1$ and $|D_{a_k}| = n - k - 2$.

Theorem 3.10 When $k = 2$ and $n - k \geq 4$, or $k \geq 3$ and $n \geq \frac{3k+8}{2}$ (or $n - k \geq 4 + \lceil \frac{k}{2} \rceil$), we have $H_e(A_{n,k}) = k(n-k) - 2$, which is the best.

Proof: Since the degree of $A_{n,k}$ is $k(n-k)$, $H_e(A_{n,k}) \leq k(n-k) - 2$. With the aid of Lemmas 3.5, 3.6, 3.7, 3.8, 3.9, it can be shown that $H_e(A_{n,k}) \geq k(n-k) - 2$. **Q.E.D.**

The following results can be obtained similarly.

Theorem 3.11 When $k \geq 2$ and $n \geq \frac{3k+4}{2}$ (or $n - k \geq 2 + \lceil \frac{k}{2} \rceil$), we have $H_e(A_{n,k}) = k(n-k-3) - 1$.

Theorem 3.12 When $k \geq 2$ and $n - k \geq 3$, we have $H_e(A_{n,k}) \geq k$.

References

- [1] S. B. Akers, D. Harel and B. Krishnamurthy, "The star graph: an attractive alternative to the n -cube," *Proceedings of the International Conference on Parallel Processing*, 1987, pp. 393-400.
- [2] F. Buckley and F. Harary, "Distance in Graphs," Addison-Wesley, 1989.
- [3] K. Day and A. Tripathi, "Characterization of node disjoint paths in arrangement graphs," *Technical Report TR 91-43*, Computer Science Department, University of Minnesota, 1991.
- [4] K. Day and A. Tripathi, "Arrangement graphs: a class of generalized star graphs," *Information Processing Letters*, vol. 42, no. 5, pp. 235-241, 1992.
- [5] K. Day and A. Tripathi, "Embedding of cycles in arrangement graphs," *IEEE Transactions on Computers*, vol. 41, no. 8, pp. 1002-1006, 1992.
- [6] K. Day and A. Tripathi, "Embedding grids, hypercubes, and trees in arrangement graphs," *Proceedings of the International Conference on Parallel Processing*, 1993, pp. III-65-III-72.
- [7] R. Fernandes, D. K. Friesen, and A. Kanevsky, "Embedding rings in recursive networks," *Proceedings of the International Conference on Parallel Processing*, 1994, pp. I-51-I-58.
- [8] S. Latifi, S. Q. Zheng, and N. Bagherzadeh, "Optimal ring embedding in hypercubes with faulty links," *Proceedings of the IEEE Symposium on Fault-Tolerant Computing*, 1992, pp. 178-184.

- [9] R. A. Rowley and B. Bose, "Fault-tolerant ring embedding in deBruijn networks," *IEEE Transactions on Computers*, vol. 12, no. 2, pp. 1480-1486, 1993.
- [10] Y. C. Tseng, S. H. Chang, and J. P. Sheu, "Fault-tolerant ring embedding in star graphs," *Proceedings of the International Parallel Processing Symposium*, 1996, pp. 660-665.
- [11] M. Paoli, W. W. Wong, and C. K. Wong, "Minimum k -hamiltonian graphs II," *Journal of Graph Theory*, vol. 10, no. 1, pp. 79-95, 1986.