Two problems on butterfly graphs

Shien-Ching Hwang and Gen-Huey Chen
Department of Computer Science and Information Engineering,
National Taiwan University, Taipei, TAIWAN

Abstract

The cycle partition problem and the pancycle problem on butterfly graphs are studied in this paper. Suppose $G=(V, E)$ is a graph and $\{V_1, V_2, \dots, V_s\}$ is a partition of V . We say that $\{V_1, V_2, \dots, V_s\}$ forms a cycle partition of G if each subgraph of G induced by V_i contains a cycle of length $|V_i|$, where $1 \leq i \leq s$. A cycle partition $\{V_1, V_2, \dots, V_s\}$ is λ -uniform if $|V_1|=|V_2|= \dots =|V_s|=\lambda$. G has λ -complete uniform cycle partitions if G has $m\lambda$ -uniform cycle partitions for all $1 \leq m \leq (r+n)/2$ and m dividing $|V|/\lambda$. Let $BF(k, r)$ denote the r -dimensional k -ary butterfly graph. For the cycle partition problem, we construct a lot of uniform cycle partitions for $BF(k, r)$. Besides, we construct r -complete uniform cycle partitions for $BF(2, r)$, and kr -complete uniform cycle partitions for $BF(k, r)$. For the pancycle problem, given any pair of n and r we can determine if there exists a cycle of length n in $BF(2, r)$, and construct it if it exists. The results of this paper reveal that the butterfly graphs are superior in embedding rings. They can embed rings of almost all possible lengths. Besides, there are many situations in which they can embed the most rings of the same length.

1. Introduction

Butterfly graphs [13] were defined as the underlying graphs of FFT networks [12] which can perform the fast Fourier transform (FFT) very efficiently. In [2], butterfly graphs were shown to be an instance of Cayley graphs. Cayley graphs [1], which represent a category of symmetric and regular graphs derivable from permutation groups, have been shown very suitable to serve as network topologies. For convenience we use $BF(k, r)$ to denote the r -dimensional k -ary butterfly graph. A formal definition of $BF(k, r)$ is provided in the next section.

Cycles in interconnection networks are useful to many applications such as indexing, embedding linear arrays and rings [15], computing fast Fourier transforms [8], *etc.* In [3], two edge-disjoint hamiltonian cycles were constructed in $BF(2, r)$. In [18], $BF(k, r)$ was shown to be hamiltonian-connected if r is odd, and hamiltonian-laceable if r is even. In this paper, two problems relating to

cycles are studied; one is the cycle partition problem and the other is the pancycle problem. Suppose $G=(V, E)$ is a graph and $\{V_1, V_2, \dots, V_s\}$ is a partition of V , where V and E are the vertex set and edge set of G , respectively. We say that $\{V_1, V_2, \dots, V_s\}$ forms a cycle partition of G if each subgraph of G induced by V_i contains a cycle of length $|V_i|$, where $1 \leq i \leq s$. The cycle partition problem on G requires a cycle partition of G . On the other hand, given a positive integer n the pancycle problem on G asks if G contains a cycle of length n , and constructs it if it exists.

Previously the two problems were studied on several interconnection networks. For example, the cycle partition problem on star graphs and arrangement graphs was solved in [16] and [6], respectively. The pancycle problem on arrangement graphs was solved in [6]. The results of [10] can provide a partial solution to the pancycle problem on $BF(2, r)$, because cube-connected cycles are subgraphs of $BF(2, r)$ (see [7]). The pancycle problem on $BF(2, r)$ was also discussed in [17]. More information about the pancycle problem on a variety of interconnection networks can be found in [17]. Thus far, the two problems are open for $BF(2, r)$.

A cycle partition $\{V_1, V_2, \dots, V_s\}$ is said to be λ -uniform (or uniform with cycle length λ) if $|V_1|=|V_2|= \dots =|V_s|=\lambda$. G has λ -complete uniform cycle partitions if G has $m\lambda$ -uniform cycle partitions for all $1 \leq m \leq |V|/\lambda$ and $m(|V|/\lambda)$, where $m(|V|/\lambda)$ is the abbreviation of m dividing $|V|/\lambda$. That is, G has uniform cycle partitions whose cycle lengths include the most multiples of λ . For the cycle partition problem, we construct a lot of uniform cycle partitions for $BF(k, r)$. Besides, we construct r -complete uniform cycle partitions for $BF(2, r)$, and kr -complete uniform cycle partitions for $BF(k, r)$. For the pancycle problem, given any pair of n and r we can determine if there exists a cycle of length n in $BF(2, r)$, and construct it if it exists.

The rest of this paper is organized as follows. In the next section, $BF(k, r)$ is formally defined, and some other necessary definitions and notations are introduced. The cycle partition problem is solved in Sections 3 and 4. In Section 3, a lot of uniform cycle partitions for $BF(k, r)$ are constructed. Besides, r -complete uniform cycle partitions for $BF(2, r)$ are also constructed. In Section 4, kr -complete uniform cycle partitions for $BF(k, r)$ are constructed. The

pancycle problem is solved in Section 5. Given any input pair of n and r , it can be determined if there exists a cycle of length n in $BF(2, r)$. If the cycle exists, it can be constructed as well. Finally, this paper concludes with some remarks in Section 6.

2. Definitions and notations

$BF(k, r)$ contains r levels, numbered $0, 1, \dots, r-1$, respectively. Each level has k^r vertices labeled with k^r k -ary sequences of r symbols. Hence there are totally $r \times k^r$ vertices in $BF(k, r)$. A vertex of level l labeled with $\beta_0\beta_1\cdots\beta_{r-1}$ is denoted by $\langle l, \beta_0\beta_1\cdots\beta_{r-1} \rangle$, where $0 \leq l \leq r-1$ and $0 \leq \beta_j \leq k-1$ for all $0 \leq j \leq r-1$. Each vertex $\langle l, \beta_0\beta_1\cdots\beta_{r-1} \rangle$ is connected to $2k$ vertices: $\langle l-1 \pmod{r}, \beta_0\beta_1\cdots\beta_{l-2}i\beta_{l+1}\cdots\beta_{r-1} \rangle$ and $\langle l+1 \pmod{r}, \beta_0\beta_1\cdots\beta_{l-1}i\beta_{l+1}\cdots\beta_{r-1} \rangle$ for all $0 \leq i \leq k-1$. In the rest of this paper, we assume that all arithmetic computations with respect to the symbols and the levels are performed modulo k and modulo r , respectively. Formally $BF(k, r)$ is defined as follows.

Definition 1. The vertex set of $BF(k, r)$ is $\{\langle l, \beta_0\beta_1\cdots\beta_{r-1} \rangle \mid l \in \{0, 1, \dots, r-1\} \text{ and } \beta_j \in \{0, 1, \dots, k-1\} \text{ for all } 0 \leq j \leq r-1\}$. The edge set of $BF(k, r)$ is $\{\langle l, \beta_0\beta_1\cdots\beta_{r-1} \rangle, \langle l+1, \beta'_0\beta'_1\cdots\beta'_{r-1} \rangle \mid l \in \{0, 1, \dots, r-1\} \text{ and } \beta_j = \beta'_j \text{ for all } 0 \leq j \leq r-1 \text{ and } j \neq l\}$. Further $(\langle l, \beta_0\beta_1\cdots\beta_{r-1} \rangle, \langle l+1, \beta'_0\beta'_1\cdots\beta'_{r-1} \rangle)$ is said to be an i -edge if $\beta'_l = \beta_l + i$, where $0 \leq i \leq k-1$.

Remarks. $BF(k, r)$ following Definition 1 is referred to as wrapped butterfly graphs in [13].

Clearly $BF(k, r)$ is regular of degree $2k$ and its edges exist only between adjacent levels. Level 0 and level $r-1$ are considered adjacent. Since $BF(k, 1)$ consists of k loops and $BF(1, r)$ is a cycle of length r , we consider $BF(k, r)$ with $k \geq 2$ and $r \geq 2$ throughout this paper. The structure of $BF(2, 3)$ is illustrated in Figure 1, where level 0 is replicated.

Definition 2. Two sequences $\beta_0\beta_1\cdots\beta_{r-1}$ and $\beta'_0\beta'_1\cdots\beta'_{r-1}$ are said to *differ in dimension* j , denoted by $\text{dif}(\beta_0\beta_1\cdots\beta_{r-1}, \beta'_0\beta'_1\cdots\beta'_{r-1}) = j$, if $\beta_j \neq \beta'_j$ and $\beta_i = \beta'_i$ for all $0 \leq i \leq r-1$ and $i \neq j$.

Definition 3. A cycle C in $BF(k, r)$ is said to be an i -cycle if the edges of C are all i -edges.

Figure 1. The structure of $BF(2, 3)$ with level 0 replicated.

Definition 4. Two disjoint cycles in $BF(k, r)$ are said to be *adjacent* if there exists an edge between them.

Definition 5. Suppose C and C' are two adjacent cycles in

$BF(k, r)$. Two edges (u, v) and (v, u') of $BF(k, r)$ form a *crossing pair* between C and C' if u and u' are with the same level, v and v' are with the same level, (u, v) is an edge of C , and (u', v') is an edge of C' . Moreover, the crossing pair is *with level* l if u and u' are with level l and v and v' are with level $l+1$.

3. Uniform cycle partitions

In this section a lot of uniform cycle partitions for $BF(k, r)$ are constructed. Suppose (u, v) is an i -edge of $BF(k, r)$, where u is with level l and v is with level $l+1$. We let $v = f_i(u)$ and $u = f_i^{-1}(v)$, and define $f_i^0(u) = u$, $f_i^j(u) = f_i(f_i^{j-1}(u))$, and

$f_i^{-j}(u) = f_i^{-1}(f_i^{-j+1}(u))$ for all $j \geq 1$. It is easy to see that $(u, f_i^0(u), f_i^1(u), f_i^2(u), \dots, f_i^{r-1}(u), f_i^r(u) = u)$ forms a 0-cycle of length r . Such a 0-cycle, which is induced by u , is denoted by $C_0(u)$. The following Lemma is easy to verify.

Lemma 1. Suppose $u = \langle l, \beta_0\beta_1\cdots\beta_{r-1} \rangle$ and $v = \langle l', \beta'_0\beta'_1\cdots\beta'_{r-1} \rangle$ are two distinct vertices of $BF(k, r)$. Then $C_0(u) = C_0(v)$ if $\beta_0\beta_1\cdots\beta_{r-1} = \beta'_0\beta'_1\cdots\beta'_{r-1}$, and $C_0(u) \cap C_0(v)$ is empty else.

According to Lemma 1, there are k^r vertex-disjoint (disjoint for short) 0-cycles of each length r in $BF(k, r)$, as stated in the following lemma.

Lemma 2. There are k^r disjoint 0-cycles of each length r in $BF(k, r)$.

For each 0-cycle we assign the vertex of level 0 to be the *cycle leader*. In subsequent discussion, when $C_0(u)$ is mentioned, u is the cycle leader. Besides, we use $C_0(\beta_0\beta_1\cdots\beta_{r-1})$ instead of $C_0(\langle 0, \beta_0\beta_1\cdots\beta_{r-1} \rangle)$ to simplify the notation.

Lemma 3. In $BF(k, r)$, $C_0(\beta_0\beta_1\cdots\beta_{r-1})$ is adjacent to $r(k-1)$ disjoint 0-cycles, i.e., $C_0(\beta_0\beta_1\cdots\beta_{j-1}(\beta_j+i)\beta_{j+1}\cdots\beta_{r-1})$ for all $1 \leq i \leq k-1$ and $0 \leq j \leq r-1$.

Proof. Suppose C is an arbitrary 0-cycle in $BF(k, r)$, where ϵ is the cycle leader. There are $r(2k-2)$ vertices of $BF(k, r)$, i.e., $f_i(f_i^j(\epsilon))$ and $f_i^{-1}(f_i^j(\epsilon))$ for all $1 \leq i \leq k-1$ and $0 \leq j \leq r-1$, which are adjacent to one vertex of C but not contained in C . These $r(2k-2)$ vertices can induce $r(2k-2)$ 0-cycles that are adjacent to C . In the following, we show that among them, only $r(k-1)$ are disjoint.

Suppose $\epsilon = \langle 0, \beta_0\beta_1\cdots\beta_{r-1} \rangle$. It is easy to see that $C_0(f_i(\epsilon))$, $C_0(f_i^j(f_i^j(\epsilon)))$, $C_0(f_i^{-1}(\epsilon))$, and $C_0(f_i^{-1}(f_i^j(\epsilon)))$ have cycle leaders $\langle 0, (\beta_0+i)\beta_1\cdots\beta_{r-1} \rangle$, $\langle 0, \beta_0\beta_1\cdots\beta_{j-1}(\beta_j+i)\beta_{j+1}\cdots\beta_{r-1} \rangle$, $\langle 0, \beta_0\beta_1\cdots(\beta_{r-1}-i) \rangle$, and $\langle 0, \beta_0\beta_1\cdots\beta_{j-2}(\beta_{j-1}-i)\beta_j\cdots\beta_{r-1} \rangle$, respectively, where $1 \leq i \leq k-1$ and $1 \leq j \leq r-1$. It is not difficult to see that all different cycle leaders form the set $\{\langle 0, \beta_0\beta_1\cdots\beta_{j-1}(\beta_j+i)\beta_{j+1}\cdots\beta_{r-1} \rangle \mid i \in \{1, 2, \dots, k-1\} \text{ and } j \in \{0, 1, \dots, r-1\}\}$ whose size is $r(k-1)$. This lemma thus follows as a consequence of Lemma 1.

We continue to show that merging adjacent 0-cycles can result in larger cycles. For this purpose, a condensation of $BF(k, r)$, denoted by $CBF_0(k, r)$, needs to be introduced first. Intuitively, each 0-cycle $C_0(\beta_0\beta_1\cdots\beta_{r-1})$ in $BF(k, r)$ is represented by a vertex $\beta_0\beta_1\cdots\beta_{r-1}$ of $CBF_0(k, r)$, and every two adjacent 0-cycles $C_0(\beta_0\beta_1\cdots\beta_{r-1})$ and $C_0(\beta'_0\beta'_1\cdots\beta'_{r-1})$ in $BF(k, r)$ define an edge $(\beta_0\beta_1\cdots\beta_{r-1}, \beta'_0\beta'_1\cdots\beta'_{r-1})$ of $CBF_0(k, r)$. By the aid of Lemma 3, $CBF_0(k, r)$ can be defined as follows.

Definition 6. The vertex set of $CBF_0(k, r)$ is $\{\beta_0\beta_1\cdots\beta_{r-1} \mid \beta_j \in \{0, 1, \dots, k-1\} \text{ for all } 0 \leq j \leq r-1\}$. The edge set of $CBF_0(k, r)$ is $\{(\beta_0\beta_1\cdots\beta_{r-1}, \beta'_0\beta'_1\cdots\beta'_{r-1}) \mid \beta_j = \beta'_j \text{ for } 0 \leq j \leq r-2 \text{ and } \beta_{r-1} \neq \beta'_{r-1}\}$.

The following lemma can be shown easily.

Lemma 4. Suppose $C_0(x)$ and $C_0(y)$ are arbitrary two adjacent 0-cycles in $BF(k, r)$. There is a unique crossing pair with level $\text{dif}(x, y)$ between them.

A path $(x_1, x_2, \dots, x_m)$ of length $m-1$ in $CBF_0(k, r)$ is said to be *feasible* if $\text{dif}(x_{i-1}, x_i) \neq \text{dif}(x_i, x_{i+1})$ for all $2 \leq i \leq m-1$.

Lemma 5. A feasible path $(x_1, x_2, \dots, x_m)$ of length $m-1$ in $CBF_0(k, r)$ can generate a cycle of length mr in $BF(k, r)$ which contains the vertices of $C_0(x_1), C_0(x_2), \dots, C_0(x_m)$, where $2 \leq m \leq k'$.

Proof. We first show that a cycle of length $2r$ can be obtained by merging $C_0(x_1)$ and $C_0(x_2)$. Without loss of generality, suppose $\text{dif}(x_1, x_2) = l_1$. By Lemma 4 there is a unique crossing pair between $C_0(x_1)$ and $C_0(x_2)$. We assume that the crossing pair is (u_1, v_2) and (v_1, u_2) , where u_1 and u_2 are with the same level l_1 and v_1 and v_2 are with the same level l_1+1 . There is a cycle of length $2r$ in $BF(k, r)$, which contains the crossing pair and all the edges of $C_0(x_1)$ and $C_0(x_2)$ but (u_1, v_1) and (u_2, v_2) .

We then continue to merge $C_0(x_3)$. Suppose $\text{dif}(x_2, x_3) = l_2$. By Lemma 4 a unique crossing pair, say (w_2, t_3) and (t_2, w_3) , exists between $C_0(x_2)$ and $C_0(x_3)$, where w_2 and w_3 are with the same level l_2 and t_2 and t_3 are with the same level l_2+1 . Since $l_1 \neq l_2$, there is a cycle of length $3r$ in $BF(k, r)$, which contains the crossing pair, all the edges of the cycle above but (w_2, t_2) , and all the edges of $C_0(x_3)$ but (w_3, t_3) .

The merging can proceed with $C_0(x_4), C_0(x_5), \dots, C_0(x_m)$ sequentially in the same way. Finally, a cycle of length mr in $BF(k, r)$ can be obtained after $C_0(x_m)$ is merged.

Suppose $\alpha_1, \alpha_2, \dots, \alpha_{r-s}$ are constants from $\{0, 1, \dots, k-1\}$. It is not difficult to see that the subgraph of $CBF_0(k, r)$ induced by $\{\beta_0\beta_1\cdots\beta_{s-1}\alpha_1\alpha_2\cdots\alpha_{r-s} \mid \beta_j \in \{0, 1, \dots, k-1\} \text{ for all } 0 \leq j \leq s-1\}$ is isomorphic to $CBF_0(k, s)$, where $2 \leq s \leq r$. Hence, a feasible path of length $m-1$ in $CBF_0(k, s)$ can generate k^{r-s} disjoint feasible paths of each length $m-1$ in $CBF_0(k, r)$. Combining Lemma 5, we have the following lemma.

Lemma 6. A feasible path of length $m-1$ in $CBF_0(k, s)$ can generate k^{r-s} disjoint cycles of each length mr in $BF(k, r)$,

where $2 \leq m \leq k^s$ and $2 \leq s \leq r$.

Lemma 7. k^s/m disjoint feasible paths of each length $m-1$ in $CBF_0(k, s)$ can generate k^{s+1}/tm disjoint feasible paths of each length $tm-1$ in $CBF_0(k, s+1)$, where $2 \leq m \leq k^s$, $1 \leq s \leq r-1$, $1 \leq t \leq k$, and $t \mid k$.

Proof. First we note that the structure of $CBF_0(k, s+1)$ is recursive. Specifically, $CBF_0(k, s+1)$ can be partitioned into k subgraphs, denoted by $CBF_0^i(k, s+1)$ for $i=0, 1, \dots, k-1$, where $CBF_0^i(k, s+1)$ is induced by $\{\beta_0\beta_1\cdots\beta_{s-1}i \mid \beta_j \in \{0, 1, \dots, k-1\} \text{ for all } 0 \leq j \leq s-1\}$. Clearly, each $CBF_0^i(k, s+1)$ is isomorphic to $CBF_0(k, s)$. A path $(x_1, x_2, \dots, x_m)$ of length $m-1$ in $CBF_0(k, s)$ can generate a path $(x_1i, x_2i, \dots, x_mi)$ of length $m-1$ in each $CBF_0^i(k, s+1)$.

Suppose $P_1, P_2, \dots, P_{k^s/m}$ are k^s/m disjoint feasible paths of each length $m-1$ in $CBF_0(k, s)$. Correspondingly, there are k^s/m disjoint feasible paths of each length $m-1$ in each $CBF_0^i(k, s+1)$, denoted by $P_1^i, P_2^i, \dots, P_{k^s/m}^i$, which are generated from $P_1, P_2, \dots, P_{k^s/m}$, respectively.

We note that the first (last) vertex of P_z^c is adjacent to the first (last) vertex of P_z^{c+1} , where $0 \leq c \leq k-2$ and $1 \leq z \leq k^s/m$.

Hence, $P_z^0, P_z^1, \dots, P_z^{k-1}$ can be joined into k/t disjoint feasible paths of each length $tm-1$, where $1 \leq t \leq k$ and $t \mid k$. Consequently, there are totally $(k^s/m) \times (k/t) = k^{s+1}/tm$ disjoint feasible paths of each length $tm-1$ in $CBF_0(k, s+1)$.

Lemma 8. $BF(k, r)$ with even k contains $k'/mt_1t_2\cdots t_{r-2}$ disjoint cycles of each length $rmt_1t_2\cdots t_{r-2}$, where $m > 0$ is even, $m \mid 2k$, $1 \leq t_i \leq k$ and $t_i \mid k$ for all $1 \leq i \leq r-2$.

Proof. We note that the structure of $CBF_0(k, 2)$ contains a $k \times k$ grid as a spanning subgraph. Suppose $m > 0$ is an even integer and $m \mid 2k$. There are $2k/m$ feasible paths of each length $m-1$ in any two adjacent rows of $CBF_0(k, 2)$, as shown in Figure 2. There are totally $(2k/m) \times (k/2) = k^2/m$ disjoint feasible paths of each length $m-1$ in $CBF_0(k, 2)$. According to Lemma 7, there are k^3/mt_1 disjoint feasible paths of each length mt_1-1 in $CBF_0(k, 3)$, where $1 \leq t_1 \leq k$ and $t_1 \mid k$. Repeatedly applying Lemma 7 to $CBF_0(k, 3), CBF_0(k, 4), \dots, CBF_0(k, r)$, we finally reach a conclusion that $CBF_0(k, r)$ contains $k'/mt_1t_2\cdots t_{r-2}$ disjoint feasible paths of each length $mt_1t_2\cdots t_{r-2}-1$, where $1 \leq t_i \leq k$ and $t_i \mid k$ for all $1 \leq i \leq r-2$. This lemma then follows as a consequence of Lemma 6.

Figure 2. $2k/m$ disjoint feasible paths of each length $m-1$ in two adjacent rows of $CBF_0(k, 2)$.

Lemma 9. Suppose there exist m^{s-1} disjoint feasible paths of each length $m-1$ in $CBF_0(m, s)$, where $1 \leq m \leq k$, $m \mid k$, and

$1 \leq s \leq r$. Then, there exist $k'/mt_1t_2 \dots t_{r-s}$ disjoint cycles of each length $rm t_1 t_2 \dots t_{r-s}$ in $BF(k, r)$, where $1 \leq t_i \leq k$ and $t_i | k$ for all $1 \leq i \leq r-s$.

Proof. Since the structure of $CBF_0(k, s)$ has s dimensions of each length k , $CBF_0(k, s)$ can be partitioned into $(k/m) \times (k/m) \times \dots \times (k/m) = k^s/m^s$ subgraphs which each are isomorphic to $CBF_0(m, s)$. As a result, there are totally $m^{s-1} \times k^s/m^s = k^s/m$ disjoint feasible paths of each length $m-1$ in $CBF_0(k, s)$. Similar to the proof of Lemma 8, it can be shown that there are $k'/mt_1t_2 \dots t_{r-s}$ disjoint cycles of each length $rm t_1 t_2 \dots t_{r-s}$ in $BF(k, r)$, where $1 \leq t_i \leq k$ and $t_i | k$ for all $1 \leq i \leq r-s$.

Since $BF(k, r)$ contains $r \times k^r$ vertices, the following theorem is immediate according to Lemma 2, Lemma 8, and Lemma 9.

Theorem 1. $BF(k, r)$ has uniform cycle partitions whose cycle lengths include the following values: (1) r ; (2) $rm t_1 t_2 \dots t_{r-2}$, provided k is even, where $m > 0$ is even, $m | 2k$, $1 \leq t_i \leq k$ and $t_i | k$ for all $1 \leq i \leq r-2$; (3) $rm t_1 t_2 \dots t_{r-s}$, provided there exist m^{s-1} disjoint feasible paths of each length $m-1$ in $CBF_0(m, s)$, where $1 \leq m \leq k$, $m | k$, $1 \leq s \leq r$, $1 \leq t_i \leq k$ and $t_i | k$ for all $1 \leq i \leq r-s$.

Since the uniform cycle partitions proposed in Theorem 1 are based on 0-cycles whose length is r , their cycle lengths are multiples of r . Next, we continue to show that $BF(2, r)$ has r -complete uniform cycle partitions. Before doing so, we briefly review the Gray code [14] first.

A cyclically ordered sequence of 2^d distinct binary codewords forms a d -dimensional Gray code, denoted by $G(d)$, if every two adjacent codewords differ in exactly one dimension. There is a recursive method to construct $G(d)$ as follows. Initially, let $G(1) = (0, 1)$. For $i > 1$, $G(i) = (0G(i-1), 1G^R(i-1))$, where $0G(i-1)$ denotes the sequence obtained by prefixing 0 to $G(i-1)$ and $1G^R(i-1)$ denotes the sequence obtained by prefixing 1 to the reverse of $G(i-1)$.

Theorem 2. $BF(2, r)$ has r -complete uniform cycle partitions.

Proof. We show that $BF(2, r)$ has mr -uniform cycle partitions for all $1 \leq m \leq 2^r$ and $m | 2^r$. Clearly, $G(r)$ corresponds to a feasible cycle of length 2^r in $CBF_0(2, r)$. The feasible cycle can be split into $2^r/m$ disjoint paths of each length $m-1$ which can further induce $2^r/m$ disjoint cycles of each length mr in $BF(2, r)$ according to Lemma 6, where $1 \leq m \leq 2^r$ and $m | 2^r$. This completes the proof.

4. kr -complete uniform cycle partitions

In the previous section we constructed uniform cycle partitions for $BF(k, r)$ by merging 0-cycles. Now we show in this section that uniform cycle partitions can be constructed as well by merging i -cycles, where $i \neq 0$. Suppose $u = \langle l, \beta_0 \beta_1 \dots \beta_{r-1} \rangle$ is a vertex of $BF(k, r)$. Then,

$f_i^{ir}(u) = \langle l, (\beta_0 + ti)(\beta_1 + ti) \dots (\beta_{r-1} + ti) \rangle$, where $t > 0$ is an integer. The length of an i -cycle induced by u is computed as $\min\{tr \mid f_i^{ir}(u) = u\} = r \times \min\{t \mid t \equiv 0 \pmod{k}\}$.

Let $[i]$ represent the subset of integers which are congruent to i modulo k , i.e., $[i] = \{nk + i \mid \text{for all integers } n\}$, where $0 \leq i \leq k-1$. Then, $(\mathbb{Z}_k = \{[0], [1], \dots, [k-1]\}, \oplus)$ forms a commutative group, named the *congruence modulo group* [9], where $[x] \oplus [y] = [x + y \pmod{k}]$ for all $[x] \in \mathbb{Z}_k$ and $[y] \in \mathbb{Z}_k$. The *order* of $[i] \in \mathbb{Z}_k$, which was defined to be the minimal positive integer h with $[hx \pmod{k}] = [0]$, was equal to $k/\gcd(k, x)$, where $\gcd(k, x)$ represents the greatest common divisor of k and x . Since $\min\{t \mid t \equiv 0 \pmod{k}\}$ is equal to the order of $[i] \in \mathbb{Z}_k$, an i -cycle induced by u has length equal to $kr/\gcd(k, i) \leq kr$. When $i=1$, $kr/\gcd(k, i)$ has maximal value kr .

The following lemma can be shown similar to the proof of Lemma 1.

Lemma 10. Any two i -cycles in $BF(k, r)$ are either identical or disjoint, where $1 \leq i \leq k-1$.

Since an i -cycle induced by u has length equal to $kr/\gcd(k, i)$, the following lemma holds as an immediate consequence of Lemma 10.

Lemma 11. There are $\gcd(k, i) \times k^{r-1}$ disjoint i -cycles of each length $kr/\gcd(k, i)$ in $BF(k, r)$, where $1 \leq i \leq k-1$.

In the rest of this section, we concentrate our attention on 1-cycles. According to Lemma 11, there are k^{r-1} disjoint 1-cycles of each length kr in $BF(k, r)$. For convenience, we use $C_1(u)$ to denote the 1-cycle induced by u . Suppose $u = \langle l, \beta_0 \beta_1 \dots \beta_{r-1} \rangle$ and $f_1^h(u) = \langle l', \beta'_0 \beta'_1 \dots \beta'_{r-1} \rangle$, where $1 \leq h \leq kr$. If $r | h$, then $l' = l$ and $\beta'_j = \beta_j + s$ for all $0 \leq j \leq r-1$, where $1 \leq s \leq k-1$ is a constant. Otherwise, $l' \neq l$. Hence we have the following lemma.

Lemma 12. Suppose $u = \langle l, \beta_0 \beta_1 \dots \beta_{r-1} \rangle$ and $v = \langle l, \beta'_0 \beta'_1 \dots \beta'_{r-1} \rangle$ are two distinct vertices of $BF(k, r)$. Then, $C_1(u) = C_1(v)$ if and only if $\beta'_j = \beta_j + s$ for all $0 \leq j \leq r-1$, where $1 \leq s \leq k-1$ is a constant.

The following two lemmas are easy to verify.

Lemma 13. Suppose $C_1(u)$ and $C_1(v)$ are two adjacent 1-cycles in $BF(k, r)$, and $u_0 \in C_1(u)$ is adjacent to $v_0 \in C_1(v)$. Then, $f_1^{ir}(u_0) \in C_1(u)$ is adjacent to $f_1^{ir}(v_0) \in C_1(v)$ for all integers $i \geq 1$.

Lemma 14. Suppose $C_1(u)$ and $C_1(v)$ are two adjacent 1-cycles in $BF(k, r)$, and $u_0 \in C_1(u)$ is adjacent to $v_0 \in C_1(v)$. There are two adjacent vertices $u'_0 \in C_1(u)$ and $v'_0 \in C_1(v)$ so that u'_0 is adjacent to u_0 and v'_0 is adjacent to v_0 .

Lemma 15. Suppose $C_1(u)$ and $C_1(v)$ are two adjacent 1-cycles in $BF(k, r)$. There are $2k$ edges connecting $C_1(u)$ and $C_1(v)$.

Proof. Suppose $u_0 \in C_1(u)$ is adjacent to $v_0 \in C_1(v)$. According to Lemma 14, there are two adjacent vertices $u'_0 \in C_1(u)$ and $v'_0 \in C_1(v)$ so that u'_0 is adjacent to u_0 and v'_0

is adjacent to v_0 . According to Lemma 13, $f_1^{ir}(u_0)$ and $f_1^{ir}(u_0')$ are adjacent to $f_1^{ir}(v_0)$ and $f_1^{ir}(v_0')$, respectively, for all integers $1 \leq i \leq k-1$. Hence, there are $2k$ edges connecting $C_1(u)$ and $C_1(v)$.

For each 1-cycle we assign the vertex of level 0 whose leftmost symbol is 0 to be the *cycle leader*. In subsequent discussion, u is the cycle leader when $C_1(u)$ is mentioned. Moreover, we use $C_1(\beta_1\beta_2\cdots\beta_{r-1})$ instead of $C_1(<0, 0\beta_1\beta_2\cdots\beta_{r-1}>)$ to simplify the notation.

Lemma 16. Suppose $C_1(\beta_1\beta_2\cdots\beta_{r-1})$ is an arbitrary 1-cycle in $BF(k, r)$. Then $C_1(\beta_1\beta_2\cdots\beta_{r-1})$ is adjacent to $r(k-1)$ disjoint 1-cycles, i.e., $C_1((\beta_1+s)(\beta_2+s)\cdots(\beta_{r-1}+s))$ and $C_1(\beta_1\beta_2\cdots\beta_{j-1}(\beta_j+t)\beta_{j+1}\cdots\beta_{r-1})$ for all $1 \leq s \leq k-1$, $1 \leq j \leq r-1$, and $1 \leq t \leq k-1$.

Proof. Suppose u is an arbitrary vertex of $C_1(\beta_1\beta_2\cdots\beta_{r-1})$. There are $2k-2$ vertices of $BF(k, r)$ that are adjacent to u but not contained in $C_1(\beta_1\beta_2\cdots\beta_{r-1})$. These $2k-2$ vertices can induce $2k-2$ 1-cycles which are all adjacent to $C_1(\beta_1\beta_2\cdots\beta_{r-1})$. We note that u is with the same level as $f_1^{ir}(u)$, where $1 \leq i \leq k-1$. According to Lemma 12, these $2k-2$ 1-cycles are identical with those $2k-2$ 1-cycles induced by the $2k-2$ vertices that are adjacent to $f_1^{ir}(u)$ but not contained in $C_1(\beta_1\beta_2\cdots\beta_{r-1})$. So, for our purpose of finding disjoint 1-cycles, it suffices to consider only r consecutive vertices of $C_1(\beta_1\beta_2\cdots\beta_{r-1})$, say $f_1^j(\epsilon)$ for all $0 \leq j \leq r-1$, where $\epsilon = <0, 0\beta_1\beta_2\cdots\beta_{r-1}>$ is the cycle leader of $C_1(\beta_1\beta_2\cdots\beta_{r-1})$.

Similar to the proof of Lemma 3, there are only $r(k-1)$ disjoint 1-cycles among the $r(2k-2)$ 1-cycles that are adjacent to $C_1(\beta_1\beta_2\cdots\beta_{r-1})$. They are $C_1((\beta_1+s)(\beta_2+s)\cdots(\beta_{r-1}+s))$ and $C_1(\beta_1\beta_2\cdots\beta_{j-1}(\beta_j+t)\beta_{j+1}\cdots\beta_{r-1})$ for all $1 \leq s \leq k-1$, $1 \leq j \leq r-1$, and $1 \leq t \leq k-1$.

Similar to $CBF_0(k, r)$, we can define $CBF_1(k, r)$ based on 1-cycles, which is another condensed version of $BF(k, r)$. Each 1-cycle in $BF(k, r)$, say $C_1(<\beta_1\beta_2\cdots\beta_{r-1}>)$, corresponds to a vertex $\beta_1\beta_2\cdots\beta_{r-1}$ of $CBF_1(k, r)$. Two vertices $\beta_1\beta_2\cdots\beta_{r-1}$ and $\beta'_1\beta'_2\cdots\beta'_{r-1}$ of $CBF_1(k, r)$ are adjacent if and only if $C_1(\beta_1\beta_2\cdots\beta_{r-1})$ and $C_1(\beta'_1\beta'_2\cdots\beta'_{r-1})$ are adjacent. By the aid of Lemma 16, $CBF_1(k, r)$ can be defined as follows.

Definition 7. The vertex set of $CBF_1(k, r)$ is $\{\beta_1\beta_2\cdots\beta_{r-1} \mid \beta_i \in \{0, 1, \dots, k-1\} \text{ for all } 1 \leq i \leq r-1\}$. Two vertices $\beta_1\beta_2\cdots\beta_{r-1}$ and $\beta'_1\beta'_2\cdots\beta'_{r-1}$ are adjacent if and only if $\beta'_1\beta'_2\cdots\beta'_{r-1} = \beta_1\beta_2\cdots\beta_{r-1}(\beta_1+t)\beta_{r-1}$ or $(\beta_1+s)(\beta_2+s)\cdots(\beta_{r-1}+s)$ for some $1 \leq i \leq r-1$, $1 \leq t \leq k-1$, and $1 \leq s \leq k-1$.

Lemma 17. A path of length $m-1$ in $CBF_1(k, r)$ can induce a cycle of length mkr in $BF(k, r)$.

Proof. Suppose $(x_1, x_2, \dots, x_m)$ is a path of length $m-1$ in $CBF_1(k, r)$. According to Lemma 14, there exist two adjacent vertices u_i, u_i' of $C_1(x_i)$ and two adjacent vertices

u_{i-1}, u_{i-1}' of $C_1(x_{i-1})$ so that u_i and u_i' are adjacent to u_{i-1} and u_{i-1}' , respectively, where $2 \leq i \leq m-1$. Similarly, there exist two adjacent vertices v_i, v_i' of $C_1(x_i)$ and two adjacent vertices v_{i+1}, v_{i+1}' of $C_1(x_{i+1})$ so that v_i and v_i' are adjacent to v_{i+1} and v_{i+1}' , respectively. According to Lemma 15, $v_i \notin \{u_i, u_i'\}$ and $v_i' \notin \{u_i, u_i'\}$ surely exist. Hence, a cycle of length mkr in $BF(k, r)$ can be constructed as shown in Figure 3.

Figure 3. Constructing a cycle of length mkr in $BF(k, r)$.

According to Lemma 17, an mkr -uniform cycle partition of $BF(k, r)$ can be obtained if there exist disjoint paths of each length $m-1$ which contain all the vertices of $CBF_1(k, r)$. For this purpose, we first review the generalized hypercube graph [4]. An n -dimensional generalized hypercube of size $m_0 \times m_1 \times \dots \times m_{n-1}$ has vertex set $\{x_0x_1\cdots x_{n-1} \mid x_i \in \{0, 1, \dots, m_i-1\} \text{ for all } 0 \leq i \leq n-1\}$, and edge set $\{(x_0x_1\cdots x_{n-1}, x_0\cdots x_{i-1}x_{i+1}\cdots x_{n-1}) \mid x_i' \in \{0, 1, \dots, m_i-1\} - \{x_i\} \text{ for all } 0 \leq i \leq n-1\}$. Conveniently we use $GH_{n,d}$ to denote the n -dimensional generalized hypercube with $m_0=m_1=\dots=m_{n-1}=d$. It is not difficult to see that the vertex set of $GH_{r-1,k}$ is identical to the vertex set of $CBF_1(k, r)$, and each edge of $GH_{r-1,k}$ corresponds to an edge of $CBF_1(k, r)$. Hence, the following lemma holds.

Lemma 18. $GH_{r-1,k}$ is a spanning subgraph of $CBF_1(k, r)$.

A path (cycle) in a graph G is said to be a *hamiltonian path (cycle)* if it contains every vertex of G exactly once [5]. In the following we show that $GH_{r-1,k}$ contains a hamiltonian path.

Lemma 19. $GH_{r-1,k}$ contains a hamiltonian path between 0^{r-1} and $(k-1)0^{r-2}$ if k is even, and between 0^{r-1} and $(k-1)^{r-1}$ if k is odd, where $r \geq 3$, 0^{r-1} denotes $r-1$ consecutive 0s, $(k-1)^{r-1}$ denotes $r-1$ consecutive $(k-1)$ s, and so on.

Proof. We first define generalized Gray code for $GH_{r-1,k}$, denoted by $G(r-1, k)$, as follows. Initially, let $G(1, k) = (0, 1, \dots, k-1)$. For $2 \leq i \leq r$, $G(i, k) = (0G(i-1, k), 1G^R(i-1, k), 2G(i-1, k), 3G^R(i-1, k), \dots, (k-1)G(i-1, k))$ if k is odd, and $(0G(i-1, k), 1G^R(i-1, k), 2G(i-1, k), 3G^R(i-1, k), \dots, (k-1)G^R(i-1, k))$ if k is even, where $0G(i-1, k)$ denotes the sequence obtained by prefixing 0 to $G(i-1, k)$, $1G^R(i-1, k)$ denotes the sequence obtained by prefixing 1 to the reverse of $G(i-1, k)$, and so on.

It is not difficult to see that $G(r-1, k)$ contains all the vertices of $GH_{r-1,k}$ and every two adjacent codewords of $G(r-1, k)$ correspond to an edge of $GH_{r-1,k}$. Besides, the first codeword of $G(r-1, k)$ is 0^{r-1} and the last codeword of $G(r-1, k)$ is $(k-1)0^{r-2}$ if k is even and $(k-1)^{r-1}$ if k is odd.

This completes the proof.

Since 0^{r-1} is adjacent to both $(k-1)0^{r-2}$ and $(k-1)^{r-1}$ in $CBF_1(k, r)$, the following lemma holds as a consequence of Lemma 18 and Lemma 19.

Lemma 20. $CBF_1(k, r)$ contains a hamiltonian cycle.

Theorem 3. $BF(k, r)$ has kr -complete uniform cycle partitions.

Proof. We show that $BF(k, r)$ has mkr -uniform cycle partitions for all $1 \leq m \leq k^{r-1}$ and $m|k^{r-1}$. According to Lemma 20, $CBF_1(k, r)$ contains a cycle of length k^{r-1} , where k^{r-1} is the number of vertices in $CBF_1(k, r)$. The cycle can be split into k^{r-1}/m disjoint paths of each length $m-1$, where $1 \leq m \leq k^{r-1}$ and $m|k^{r-1}$. The latter can further induce k^{r-1}/m disjoint cycles of each length mkr in $BF(k, r)$ according to Lemma 17. This completes the proof.

The following corollary is obtained from Theorem 3 when $m=k^{r-1}$.

Corollary 1. $BF(k, r)$ contains a hamiltonian cycle.

Remarks. A hamiltonian cycle of $BF(k, r)$ was originally obtained in [18] using a recursive method. Alternatively, Corollary 1 provides a nonrecursive method for the same purpose.

5. Pancyclic properties of $BF(2, r)$

In this section we intend to explore the pancyclic properties of $BF(2, r)$. A graph $G=(V, E)$ is *pancyclic* if G contains all cycles of lengths ranging from 3 to $|V|$, and *even pancyclic* if G contains all cycles of even lengths ranging from 4 to $|V|-1$ (if $|V|$ is odd) or $|V|$ (if $|V|$ is even). For example, a bipartite graph may be even pancyclic, but may not be pancyclic because it contains no cycles of odd lengths. Since $BF(2, 2)$ is clearly even pancyclic, we consider $BF(2, r)$ with $r \geq 3$ in the rest of this section.

Recall that in the proof of Lemma 5 we have shown that $C_0(x_1)$ and $C_0(x_2)$ can be merged into a cycle of length $2r$ in $BF(k, r)$ which contains the crossing pair $(u_1, v_2), (v_1, u_2)$ and all the edges of $C_0(x_1)$ and $C_0(x_2)$ but (u_1, v_1) and (u_2, v_2) . If only (u_1, v_1) is removed instead, a cycle of length $r+2$ can result which contains (u_2, v_2) , the crossing pair, and all the edges of $C_0(x_1)$ but (u_1, v_1) . It remains true for any two adjacent cycles in $BF(k, r)$, provided there is a crossing pair between them. Hence we have the following lemma.

Lemma 21. Suppose C and C' are two adjacent cycles in $BF(2, r)$. If there is a crossing pair between C and C' , the length of C can increase by two by adding two adjacent vertices of C' . The two adjacent vertices of C' added to C are incident vertices of the crossing pair.

Since $BF(2, r)$ with even r is bipartite, it contains cycles only of even lengths. The following lemma shows that $BF(2, r)$ with even r contains all cycles of even lengths, but those whose lengths are smaller than r .

Lemma 22. $BF(2, r)$ with even r contains cycles of even lengths ranging from r to $r \times 2^r$.

Proof. We show that $BF(2, r)$ with even r contains cycles of lengths $mr+n$ ($\leq r \times 2^r$) for all $1 \leq m \leq 2^r$, even n , and $0 \leq n \leq r-2$. We first consider $m > 1$, and assume $2^p < m \leq 2^{p+1}$ for some integer $0 \leq p \leq r-1$. Let $x_1, x_2, \dots, x_m$ be the first m codewords of $G(r)$. Then $(x_1, x_2, \dots, x_m)$ forms a feasible path of length $m-1$ in $CBF_0(2, r)$ and their leftmost $r-p-1$ bits are all zeros. Similar to the proof of Lemma 5, it can be shown that $C_0(x_1), C_0(x_2), \dots, C_0(x_m)$ can be merged into a cycle of length mr in $BF(2, r)$. We denote the cycle by C . The crossing pair between $C_0(x_i)$ and $C_0(x_{i+1})$ is with level $\text{dif}(x_i, x_{i+1})$, where $1 \leq i \leq m-1$. Clearly, $r-p-1 \leq \text{dif}(x_i, x_{i+1}) \leq r-1$.

Let (u_i', v_{i+1}') , (v_i', u_{i+1}') be the crossing pair between $C_0(x_i)$ and $C_0(x_{i+1})$, where $1 \leq i \leq m-1$, $u_i', v_i' \in C_0(x_i)$, and $v_{i+1}', u_{i+1}' \in C_0(x_{i+1})$. Also, let (v_i, u_{i-1}') , (u_i, v_{i-1}') be the crossing pair between $C_0(x_i)$ and $C_0(x_{i-1})$, where $2 \leq i \leq m$, $u_i, v_i \in C_0(x_i)$ and $u_{i-1}', v_{i-1}' \in C_0(x_{i-1})$. Since $\text{dif}(x_i, x_{i-1}) \neq \text{dif}(x_i, x_{i+1})$, $\{u_i, v_i\} \neq \{u_i', v_i'\}$. Two cases need to be discussed below, according to the values of p .

Case 1. $0 \leq p \leq r-2$. Let $x_i' = x_i \oplus 100 \dots 0$, where $1 \leq i \leq m$ and $\oplus$ performs an XOR operation. Then, $\text{dif}(x_i, x_i') = 0$ and $C_0(x_i')$, $C_0(x_2'), \dots, C_0(x_m')$ are all disjoint. According to Lemma 1, each $C_0(x_i')$ is disjoint with $C_0(x_1), C_0(x_2), \dots, C_0(x_m)$. According to Lemma 4, there is a unique crossing pair with level $\text{dif}(x_i', x_i)$ between $C_0(x_i')$ and $C_0(x_i)$, which is also a crossing pair between $C_0(x_i')$ and C . We use (g_i', h_i) , (h_i', g_i) to denote the crossing pair, where $g_i', h_i' \in C_0(x_i')$ and $h_i, g_i \in C_0(x_i)$. Since $0 = \text{dif}(x_i, x_i') \neq \text{dif}(x_i, x_{i+1}) \geq r-p-1 \geq 1$, $\{g_i, h_i\} \neq \{u_i', v_i'\}$, where $1 \leq i \leq m-1$. Similarly, $\{g_i, h_i\} \neq \{u_i, v_i\}$, where $2 \leq i \leq m$. According to Lemma 21, the length of C will increase by two if two adjacent vertices (i.e., g_i' and h_i') of $C_0(x_i')$ are added to C , where $1 \leq i \leq m$. Consequently, the length of C can increase by 2 or 4 or ... or $2m$. This implies that there are cycles of lengths $mr, mr+2, mr+4, \dots, mr+(r-2)$ in $BF(2, r)$ if $m \geq (r-2)/2$.

If $m < (r-2)/2$, we define $x_1^{(j)}$ to be the codeword of $G(r)$ with $\text{dif}(x_1^{(j)}, x_1) = j$, where $0 \leq j \leq r-p-2$. Since $x_1^{(j)} \notin \{x_1, x_2, \dots, x_m\}$, $C_0(x_1^{(j)})$ is disjoint with $C_0(x_1), C_0(x_2), \dots, C_0(x_m)$ according to Lemma 1. Moreover, since $C_0(x_1^{(j)})$ is adjacent to $C_0(x_1)$, there is a unique crossing pair with level j between them according to Lemma 4, which is also a crossing pair between $C_0(x_1^{(j)})$ and C . With similar arguments, the length of C will increase by two if two adjacent vertices of $C_0(x_1^{(j)})$ are added to C . The length of C can thus increase by 2 or 4 or ... or $2(r-p-1)$. Since $m < (r-2)/2$, $p < \log_2 m < \log_2 (r-2)/2 < (r-2)/2$, which implies $r-p-1 > r/2$. Hence, there are cycles of lengths $mr, mr+2, mr+4, \dots, mr+(r-2)$ in $BF(2, r)$.

Case 2. $p=r-1$. In this case, $2^{r-1} < m \leq 2^r$. When $m=2^r$, $n=0$. According to Theorem 2, there exists a cycle of length $r \times 2^r$ in $BF(2, r)$. So, we only need to consider $2^{r-1} < m < 2^r$. Let x_{m+1} be the $(m+1)$ th codeword of $G(r)$. We show an iterative procedure to expand C , which contains $(r/2)-1$ iterations. Let $C_0=C$, C_t be the resulting C after the t th iteration, and $x_{m+1}^{(2^{t-1})}$ be the codeword of $G(r)$ with $\text{dif}(x_{m+1}^{(2^{t-1})}, x_{m+1})=2t-1$, where $1 \leq t \leq (r/2)-1$. The execution of the t th iteration is described below.

If $x_{m+1}^{(2^{t-1})}$ proceeds x_{m+1} in $G(r)$, then $x_{m+1}^{(2^{t-1})}=x_s$ for some $1 \leq s \leq m$. Since $C_0(x_{m+1})$ is adjacent to $C_0(x_s)$, there is a unique crossing pair between them with level $\text{dif}(x_{m+1}, x_s)=\text{dif}(x_{m+1}, x_{m+1}^{(2^{t-1})})=2t-1$ according to Lemma 4, which is also a crossing pair between $C_0(x_{m+1})$ and C_{t-1} . We assume $1 \leq s < m$, without loss of generality. Since $x_{m+1} \notin \{x_{s-1}, x_{s+1}\}$, $\text{dif}(x_s, x_{m+1}) \notin \{\text{dif}(x_s, x_{s-1}), \text{dif}(x_s, x_{s+1})\}$. Hence the length of C_{t-1} can increase by two by adding two adjacent vertices of $C_0(x_{m+1})$ according to Lemma 21.

Otherwise, if $x_{m+1}^{(2^{t-1})}$ succeeds x_{m+1} in $G(r)$, then $x_{m+1}^{(2^{t-1})} \in \{x_{m+2}, x_{m+3}, \dots, x_{2^r}\}$. Let $y_i = x_{m+1}^{(2^{t-1})} \oplus 100 \dots 0$. Since $m > 2^{r-1}$, $y_i \in \{x_1, x_2, \dots, x_{2^{r-1}}\}$. Since $C_0(x_{m+1}^{(2^{t-1})})$ is adjacent to $C_0(y_i)$, there is a unique crossing pair between them with level $\text{dif}(x_{m+1}^{(2^{t-1})}, y_i)=0$ according to Lemma 4, which is also a crossing pair between $C_0(x_{m+1}^{(2^{t-1})})$ and C_{t-1} . Similarly, the length of C_{t-1} can increase by two by adding two adjacent vertices of $C_0(x_{m+1}^{(2^{t-1})})$.

It is easy to see that C_t has length $mr+2t$, where $1 \leq t \leq (r/2)-1$. Hence there are cycles of lengths $mr, mr+2, mr+4, \dots, mr+(r-2)$ in $BF(2, r)$. Finally, we consider $m=1$. Let $p=0$ and $C=C_0(x_1)$. Similar to the discussion of Case 1, there are cycles of lengths $r, r+2, r+4, \dots, r+(r-2)$ in $BF(2, r)$.

The following lemma can be proved by substituting $r-1$ for $r-2$ and substituting $(r-1)/2$ for $(r-2)/2$, $r/2$, and $(r/2)-1$ in the proof above.

Lemma 23. $BF(2, r)$ with odd r contains cycles of lengths $mr+n$ ($\leq r \times 2^r$) for all $1 \leq m \leq 2^r$, even n , and $0 \leq n \leq r-1$.

Lemma 24. $BF(2, r)$ contains cycles of lengths $4c$ for all $1 \leq c \leq r$.

Proof. Suppose p, q are arbitrary two integers satisfying $1 \leq p \leq r-1$, $1 \leq q \leq r-1$, and $p+q \leq r$. A cycle of length $4p+4q \geq 8$ can be constructed as follows: $\langle 0, 0^r \rangle, \langle 1, 0^r \rangle, \dots, \langle p, 0^r \rangle, \langle p-1, 0^{p-1}10^{r-p} \rangle, \langle p-2, 0^{p-1}10^{r-p} \rangle, \dots, \langle 0, 0^{p-1}10^{r-p} \rangle, \langle r-1, 0^{p-1}10^{r-p} \rangle, \langle r-2, 0^{p-1}10^{r-p} \rangle, \dots, \langle r-q, 0^{p-1}10^{r-p} \rangle, \langle r-q+1, 0^{p-1}10^{r-p-q}10^{q-1} \rangle, \langle r-q+2, 0^{p-1}10^{r-p-q}10^{q-1} \rangle, \dots, \langle r-1, 0^{p-1}10^{r-p-q}10^{q-1} \rangle, \langle 0, 0^{p-1}10^{r-p-q}10^{q-1} \rangle, \langle 1, 0^{p-1}10^{r-p-q}10^{q-1} \rangle, \dots, \langle p, 0^{p-1}10^{r-p-q}10^{q-1} \rangle, \langle p-1, 0^{r-q}10^{q-1} \rangle, \langle p-2, 0^{r-q}10^{q-1} \rangle, \dots, \langle 0, 0^{r-q}10^{q-1} \rangle, \langle r-1,$

$0^{r-q}10^{q-1} \rangle, \langle r-2, 0^{r-q}10^{q-1} \rangle, \dots, \langle r-q, 0^{r-q}10^{q-1} \rangle, \langle r-q+1, 0^r \rangle, \langle r-q+2, 0^r \rangle, \dots, \langle r-1, 0^r \rangle, \langle 0, 0^r \rangle$.

Besides, a cycle of length 4 in $BF(2, r)$ can be constructed as $\langle 0, 0^r \rangle, \langle 1, 0^r \rangle, \langle 0, 10^{r-1} \rangle, \langle 1, 10^{r-1} \rangle, \langle 0, 0^r \rangle$.

Lemma 25. $BF(2, r)$ contains cycles of lengths $4c+2$ for all $3 \leq c \leq r$.

Proof. A cycle of length $4c+2$ can be constructed as follows: $\langle 0, 0^r \rangle, \langle r-1, 0^r \rangle, \langle 0, 0^{r-1}1 \rangle, \langle 1, 0^{r-1}1 \rangle, \dots, \langle c-1, 0^{r-1}1 \rangle, \langle c-2, 0^{c-2}10^{r-c}1 \rangle, \langle c-3, 0^{c-2}10^{r-c}1 \rangle, \dots, \langle 0, 0^{c-2}10^{r-c}1 \rangle, \langle r-1, 0^{c-2}10^{r-c}1 \rangle, \langle 0, 0^{c-2}10^{r-c+1} \rangle, \langle 1, 0^{c-2}10^{r-c+1} \rangle, \dots, \langle c-1, 0^{c-2}10^{r-c+1} \rangle, \langle c-2, 0^r \rangle, \langle c-3, 0^r \rangle, \dots, \langle 1, 0^r \rangle, \langle 0, 10^{r-1} \rangle, \langle 1, 10^{r-1} \rangle, \langle 0, 0^r \rangle$, where $3 \leq c \leq r$.

The following lemma can be proved similar to Lemma 22.

Lemma 26. $BF(2, r)$ with odd r contains all cycles of lengths ranging from r to $r \times 2^r$, but cycles of length 6 in $BF(2, 5)$ and length 10 in $BF(2, 7)$ and $BF(2, 9)$.

Now we summarize the results of Lemma 22, Lemma 24, Lemma 25, and Lemma 26 (Lemma 23 can be incorporated into Lemma 26) as follows. When r is even, $BF(2, r)$ contains (1) all cycles of even lengths ranging from r to $r \times 2^r$, (2) cycles of lengths 4 and 8, and (3) all cycles of even lengths ranging from 12 to $r-2$, as $r \geq 14$. When r is odd, $BF(2, r)$ contains (1) all cycles of lengths ranging from r to $r \times 2^r$, but cycles of length 6 in $BF(2, 5)$ and length 10 in $BF(2, 7)$ and $BF(2, 9)$, (2) cycles of lengths 4 and 8, and (3) all cycles of even lengths ranging from 12 to $r-1$, as $r \geq 13$. In what follows, we show that none of cycles with the other lengths can be found in $BF(2, r)$. The following lemma originally appeared in part (c) of Theorem 6.1 of [17].

Lemma 27 [17]. There is no cycle in $BF(2, r)$ whose length is odd and smaller than r .

The following lemma can be derived from the proof of Proposition 2.12 of [11].

Lemma 28 [11]. There is no cycle of length 6 in $BF(2, r)$ with $r=5$ or $r \geq 7$, and there is no cycle of length 10 in $BF(2, r)$ with $r=7$ or 9 or $r \geq 11$.

By Lemma 22, Lemma 24, Lemma 25, Lemma 26, Lemma 27, and Lemma 28. We note that $BF(2, r)$ with even r is bipartite.

Theorem 4. $BF(2, r)$ has the following pancyclic properties.

(1) $BF(2, 2)$ is even pancyclic. (2) $BF(2, 3)$ is pancyclic. (3) $BF(2, r)$ with odd r and $r \geq 5$ is almost pancyclic, except no cycles of length 6 in $BF(2, r)$ with $r \geq 5$, length 10 in $BF(2, r)$ with $r \geq 7$, and odd lengths $< r$. (4) $BF(2, 4)$ and $BF(2, 6)$ are even pancyclic. (5) $BF(2, r)$ with even r and $r \geq 8$ is almost even pancyclic, except no cycles of length 6 in $BF(2, r)$ with $r \geq 8$ and length 10 in $BF(2, r)$ with $r \geq 12$.

6. Conclusion and discussion

In this paper, we have solved the cycle partition problem and the pancycle problem on the butterfly graphs. For the cycle partition problem, we have constructed a lot of uniform cycle partitions for $BF(k, r)$, r -complete uniform cycle partitions for $BF(2, r)$, and kr -complete uniform cycle partitions for $BF(k, r)$.

When 0-cycles are adopted as fundamental cycles, at most $r+1$ different uniform cycle partitions of $BF(2, r)$ can be constructed whose cycle lengths are $r, r \times 2, r \times 2^2, r \times 2^3, \dots, r \times 2^r$, respectively. These $r+1$ uniform cycle partitions constitute r -complete uniform cycle partitions of $BF(2, r)$. In this paper, we have constructed r -complete uniform cycle partitions for $BF(2, r)$. On the other hand, adopting 1-cycles as fundamental cycles, we have constructed kr -complete uniform cycle partitions for $BF(k, r)$. Besides, a hamiltonian cycle in $BF(k, r)$ was also established as a by-product. Compared with Wong's recursive method [18], our method for establishing a hamiltonian cycle in $BF(k, r)$ is nonrecursive. It is still open if there are other λ -complete uniform cycle partitions for $BF(k, r)$, where $\lambda \neq kr$.

The pancycle problem on $BF(2, r)$ was well solved by Theorem 4. By its aid, it can be determined for any pair of n and r if there exists a cycle of length n in $BF(2, r)$. Besides, the cycle can be constructed if it exists. Previously, the pancycle problem on $BF(2, r)$ was partially solved in [10, 17]. In [17], Rosenberg showed that $BF(2, r)$ contains cycles whose lengths constitute the set $\{r\} \cup \{r \times 2^s - (r-2)t \mid 1 \leq s \leq r \text{ and } 0 \leq t \leq 2^s\}$. In [10], Germa, Heydemann, and Sotteau constructed cycles of various lengths in cube-connected cycles which were shown subgraphs of $BF(2, r)$. Their results can imply that $BF(2, r)$ contains cycles whose lengths constitute the set $\{r\} \cup \{c \mid 8 \leq c \leq r \times 2\}$ if $r=3$ or 4, $\{r\} \cup \{c \mid 8 \leq c \leq r \times 2^r \text{ and } c \text{ is even}\} - \{10, r \times 2^r - 2\}$ if $r \geq 5$, and $\{r \times 2^r - r + 2\} \cup \{c \mid r + 6 \leq c \leq r \times 2^r - r - 2 \text{ and } c \text{ is odd}\}$ if r is odd.

The pancycle problem on $BF(2, r)$ remained unsolved in [10, 17] for many pairs of n and r . For example, when $BF(2, 6)$ is considered, cycles of lengths 6 and $4n$ for all $1 \leq n \leq 96$ can be obtained from [17], and cycles whose lengths constitute the set $\{c \mid 6 \leq c \leq 384 \text{ and } c \text{ is even}\} - \{10, 382\}$ can be obtained from [10]. It was unknown in [10, 18] if there are cycles of the other lengths in $BF(2, 6)$. Solving the pancycle problem on $BF(k, r)$ is another further research topic.

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