Embed Longest Rings onto Star Graphs with Vertex Faults

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Abstract

The star graph has been recognized as an attractive alternative to the hypercube. Let F_e and F_v be the sets of vertex faults and edge faults, respectively. Previously, Tseng et al. showed that an n -dimensional star graph can embed a ring of length $n!$ if $|F_e| \leq n-3$ ($|F_v|=0$), and a ring of length at least $n!/4|F_v|$ if $|F_v| \leq n-3$ ($|F_e|=0$). Since an n -dimensional star graph is regular of degree $n-1$ and is bipartite with two partite sets of equal size, our result achieves optimality in the worst case.

1 Introduction

One crucial step on designing a large-scale multiprocessor system is to determine the topology of the interconnection network (network for short). In recent decades, a lot of networks have been proposed. The interested readers may refer to [4, 17] for extensive references. Among them, the star graph [1], which belongs to the class of Cayley graphs [2], has been recognized as an attractive alternative to the hypercube. It possesses many nice topological properties such as recursiveness, symmetry, maximal fault tolerance, sublogarithmic degree and diameter, and strong resilience, which are all desirable when we are building interconnection topologies for parallel and distributed systems. Besides, the star graph can embed rings [25], meshes [27], trees [3], and hypercubes [23]. Efficient communication algorithms for shortest-path routing [1], multiple-path routing [12], multicasting [16], broadcasting [31], gossiping [5], and scattering [16] have been proposed. Efficient algorithms for sorting [22], merging [25], selection [24], prefix sums [25], ranking [30], Fourier

transform [15], and computational geometry [26] have also been designed.

Since processor and/or link faults may happen when a network is put in use, it is practically meaningful to consider faulty networks. The problems of diameter [28], routing [14, 34], multicasting [21], broadcasting [33], gossiping [13], embedding [11, 29, 35], and fault-tolerant graphs [6-9, 19] have been solved on a variety of faulty networks. This paper is concerned with the problem of embedding a ring onto a faulty star graph. Throughout this paper, we use network and graph, processor and vertex, and link and edge, interchangeably.

Previously, embedding rings onto faulty star graphs has been studied in [20, 32]. In [20], Latifi and Bagherzadeh showed that an n -dimensional star graph with vertex faults can embed a ring of length $n!-m!$, where all vertex faults belong to an m -dimensional star graph and $m \leq n$ is minimal. Letting F_v and F_e denote the sets of vertex faults and edge faults, respectively, Tseng et al. [32] showed that an n -dimensional star graph can embed a ring of length $n!$ if $|F_e| \leq n-3$ ($|F_v|=0$), and a ring of length at least $n!/4|F_v|$ if $|F_v| \leq n-3$ ($|F_e|=0$). Since the n -dimensional star graph is regular of $n-1$ and contains $n!$ vertices, the former is optimal in the worst case. However, the latter is not optimal.

In this paper, we improve Tseng et al.'s latter result by lengthening the embedded ring from $n!/4|F_v|$ to $n!/2|F_v|$. We note that the star graph is bipartite and has two partite sets of equal size [18]. When $|F_v|$ vertex faults fall into the same partite set, an embedded ring has length not more than $n!/2|F_v|$. Hence, our result achieves optimality in the worst case.

2 Preliminaries

In this section the structure of the star graph is formally described. Some necessary notations and basic operations are also introduced. We use S_n to denote an n -dimensional star graph.

Definition 1. The vertex set of S_n is denoted by $\{a_1a_2...a_n \mid a_1a_2...a_n \text{ is a permutation of } 1, 2, \dots, n\}$. Vertex adjacency is defined as follows: $a_1a_2...a_n$ is adjacent to $a_ia_2...a_{i-1}a_{i+1}...a_n$ for all $2 \leq i \leq n$.

The vertices of S_n are $n!$ permutations of $1, 2, \dots, n$, and there is an edge between two vertices of S_n if and only if they can be obtained from each other by swapping the leftmost number with one of the other $n-1$ numbers. We refer to the position of a_i in $a_1a_2...a_n$ as the i th dimension, and the edge between $a_1a_2...a_n$ and $a_ia_2...a_{i-1}a_{i+1}...a_n$ as the i th-dimensional edge. Clearly S_n is regular of degree $n-1$. Since S_1 is a vertex, S_2 is an edge, and S_3 forms a cycle of length 6, we consider S_n with $n \geq 4$ throughout this paper.

There are embedded S_r 's contained in S_n , where $1 \leq r \leq n$. An embedded S_r can be conveniently represented by $\langle s_1s_2...s_n \rangle_r$, where $s_1 = *$, $s_i \in \{*, 1, 2, \dots, n\}$ for all $2 \leq i \leq n$, and exactly r of $s_1, s_2, \dots, s_n$ are $*$ ($*$ denotes a "don't care" symbol). For example, $\langle ***3 \rangle_3$, which represents an embedded S_3 in S_4 , contains six nodes 1243, 1423, 2143, 2413, 4123, and 4213. In terms of graph, $\langle ***3 \rangle_3$ is a subgraph of S_4 induced by $\{1243, 1423, 2143, 2413, 4123, 4213\}$. When $r=n$, $\langle s_1s_2...s_n \rangle_n$ represents S_n . Two basic operations on S_n are defined as follows.

Definition 2. An i -partition on $\langle s_1s_2...s_n \rangle_r$ partitions $\langle s_1s_2...s_n \rangle_r$ into r embedded S_{r-1} 's, denoted by $\langle s_1s_2...s_{i-1}qs_{i+1}...s_n \rangle_{r-1}$, where $2 \leq i \leq n$, $s_i = *$, and $q \in \{1, 2, \dots, n\} - \{s_1, s_2, \dots, s_n\}$.

For example, executing a 3-partition on $\langle ***15 \rangle_3$ produces three embedded S_2 's, i.e., $\langle **215 \rangle_2$, $\langle **315 \rangle_2$, and $\langle **415 \rangle_2$.

Definition 3. An $(i_1, i_2, \dots, i_m)$ -partition on $\langle s_1s_2...s_n \rangle_r$ performs an i_1 -partition, an i_2 -partition, ..., an i_m -partition, sequentially, on $\langle s_1s_2...s_n \rangle_r$, where $i_1i_2...i_m$ is a permutation of m elements from $\{2, 3, \dots, n\}$ and $s_{i_1} = s_{i_2} = \dots = s_{i_m} = *$. After executing an $(i_1, i_2, \dots, i_m)$ -partition, $\langle s_1s_2...s_n \rangle_r$ is partitioned into $r(r-1)\dots(r-m+1)$ embedded S_{r-m} 's.

For example, when a (3, 2)-partition is applied to $\langle ***15 \rangle_3$, a 3-partition is first executed on $\langle ***15 \rangle_3$ to produce three embedded S_2 's, i.e., $\langle **215 \rangle_2$, $\langle **315 \rangle_2$, and $\langle **415 \rangle_2$. Then, a 2-partition is executed on each of

these three S_2 's to produce six embedded S_1 's, i.e., $\langle *3215 \rangle_1$, $\langle *4215 \rangle_1$, $\langle *2315 \rangle_1$, $\langle *4315 \rangle_1$, $\langle *2415 \rangle_1$, and $\langle *3415 \rangle_1$.

Two embedded S_r 's $\langle s_1s_2...s_n \rangle_r$ and $\langle t_1t_2...t_n \rangle_r$ are said to be adjacent if $s_j \neq *$, $t_j \neq *$, and $s_j \neq t_j$ for some $2 \leq j \leq n$, and $s_i = t_i$ for all $1 \leq i \leq n$ and $i \neq j$. Moreover, the position j is denoted by $\text{dif}(\langle s_1s_2...s_n \rangle_r, \langle t_1t_2...t_n \rangle_r)$. For example, $\langle **23 \rangle_2$ is adjacent to $\langle **13 \rangle_2$, and $\text{dif}(\langle **23 \rangle_2, \langle **13 \rangle_2) = 3$.

In the rest of this paper, an embedded S_r is named an r -vertex when it is regarded as a supervertex. Imaginarily, there is a superedge, named an r -edge, between two adjacent r -vertices. Really, an r -edge in S_n comprises $(r-1)!$ edges. A 1-vertex in S_n is a vertex of S_n , and a 1-edge in S_n is an edge of S_n .

Definition 4. Suppose $A_0, A_1, \dots, A_{n(n-1)(n-2)\dots(r+1)-1}$ are $n(n-1)(n-2)\dots(r+1)$ r -vertices that result from executing an $(i_1, i_2, \dots, i_{n-r})$ -partition on S_n , where $1 \leq r \leq n-1$. They form an r -ring, denoted by $R_r = [A_0, A_1, \dots, A_{n(n-1)(n-2)\dots(r+1)-1}]$, if A_i is adjacent to $A_{(i-1) \bmod n(n-1)(n-2)\dots(r+1)}$ and $A_{(i+1) \bmod n(n-1)(n-2)\dots(r+1)}$ for all $0 \leq i \leq n(n-1)(n-2)\dots(r+1)-1$.

Definition 5. An i -partition on an $R_r = [A_0, A_1, \dots, A_{n(n-1)(n-2)\dots(r+1)-1}]$ performs an i -partition on $A_0, A_1, \dots, A_{n(n-1)(n-2)\dots(r+1)-1}$, where $2 \leq i \leq n$ and $r \geq 2$. An i -partition on an R_r is abbreviated to a partition on an R_r if the position i is "don't care".

After executing an i -partition on an $R_r = [A_0, A_1, \dots, A_{n(n-1)(n-2)\dots(r+1)-1}]$, each A_j is partitioned into r $(r-1)$ -vertices, where $0 \leq j \leq n(n-1)(n-2)\dots(r+1)-1$. Since every two of the r $(r-1)$ -vertices are adjacent, each A_j can be viewed as a complete graph of r $(r-1)$ -vertices. Throughout this paper, we use K_r^{r-1} to denote the complete graph.

Suppose $A_k = \langle s_1...s_{i-1}s_is_{i+1}...s_{j-1}xs_{j+1}...s_n \rangle_r$ and $A_{(k+1) \bmod n(n-1)(n-2)\dots(r+1)} = \langle s_1...s_{i-1}s_is_{i+1}...s_{j-1}ys_{j+1}...s_n \rangle_r$ are two neighboring r -vertices in an R_r , where $r \geq 2$, $x \neq y$, and $s_i = *$. After an i -partition, they each are partitioned into r $(r-1)$ -vertices. Among them, $r-1$ from A_k are adjacent to $r-1$ from $A_{(k+1) \bmod n(n-1)(n-2)\dots(r+1)}$. The other two that are not adjacent are $\langle s_1...s_{i-1}s_{i+1}...s_{j-1}xs_{j+1}...s_n \rangle_{r-1}$ and $\langle s_1...s_{i-1}s_{i+1}...s_{j-1}ys_{j+1}...s_n \rangle_{r-1}$.

Suppose an $R_k = [A_0, A_1, \dots, A_{n(n-1)(n-2)\dots(k+1)-1}]$ is obtained in S_n , where $k \geq 4$. After executing a partition on the R_k , each A_i forms a K_k^{k-1} , where $0 \leq i \leq n(n-1)(n-2)\dots(k+1)-1$. There exist two $(k-1)$ -vertices, say X_i and Y_i , in each A_i so that X_i and Y_i are adjacent to $Y_{(i-1) \bmod n(n-1)(n-2)\dots(k+1)}$ and $X_{(i+1) \bmod n(n-1)(n-2)\dots(k+1)}$, respectively. We refer to X_i and Y_i as the entry $(k-1)$ -vertex and exit $(k-1)$ -vertex of A_i , respectively, in subsequent discussion.

3 Embed a longest ring onto S_n with vertex faults

In this section we assume that S_n has $|F_v| \leq n-3$ vertex faults. We say that an i -vertex is *healthy* if it has no vertex fault, and *faulty* otherwise, where $1 \leq i \leq n$. Similarly, a ring (or a path) is *healthy* if it has no vertex fault. In this section we show that S_n with $n \geq 4$ can embed a healthy ring of length $n!-2|F_v|$. Since S_n is bipartite with two partite sets of equal size, the length $n!-2|F_v|$ is maximum in the worst case. Moreover, $|F_v|=n-3$ is also maximum in the worst case in order to embed a healthy ring of maximum length, because S_n is regular of $n-1$. Our method is first to determine an R_4 in S_n that owns the following three properties:

- (P1) each 4-vertex of the R_4 has at most one vertex fault;
- (P2) for every three consecutive 4-vertices $U=\langle u_1u_2\dots u_n \rangle_4$, $V=\langle v_1v_2\dots v_n \rangle_4$, and $W=\langle w_1w_2\dots w_n \rangle_4$ in the R_4 , $u_{\text{diff}(U,V)} \neq w_{\text{diff}(V,W)}$ holds;
- (P3) both every two consecutive 4-vertices in the R_4 are not faulty.

Then, a healthy ring of length $n!-2|F_v|$ can be generated from the R_4 .

Lemma 1. Suppose $U=\langle u_1u_2\dots u_n \rangle_r$, $V=\langle v_1v_2\dots v_n \rangle_r$, and $W=\langle w_1w_2\dots w_n \rangle_r$ are arbitrary three consecutive r -vertices in an R_r , where $r \geq 2$. Let $p=\text{diff}(U, V)$ and $q=\text{diff}(V, W)$. If $u_p \neq w_q$, then after executing a partition on the R_r , each $(r-1)$ -vertex of V is connected to U or W .

Proof. Without loss of generality, we assume that a j -partition is executed on the R_r , where $2 \leq j \leq n$. Hence, $u_j=v_j=w_j=*$. Since $p=\text{diff}(U, V) \neq 1$ and $q=\text{diff}(V, W) \neq 1$, we have $u_p \neq v_p$, $v_q \neq w_q$, $u_i=v_i$ for all $1 \leq i \leq n$ and $i \neq p$, and $v_i=w_i$ for all $1 \leq i \leq n$ and $i \neq q$. Suppose conversely that $u_p \neq w_q$ and there exists an $(r-1)$ -vertex, say $V_1=\langle v_1v_2\dots v_{j-1}zv_{j+1}\dots v_n \rangle_{r-1}$, of V , which is connected to neither of U and W . Thus, $z=u_p$, for otherwise V_1 is adjacent to some $(r-1)$ -vertex of U . Similarly, $z=w_q$. Hence, $u_p=z=w_q$, which contradicts our assumption. Q.E.D.

The following lemma was shown in [32].

Lemma 2. [32] Suppose $|F_v| \leq n-3$. There exists a sequence $a_1, a_2, \dots, a_{n-4}$ of positions so that after executing an $(a_1, a_2, \dots, a_{n-4})$ -partition on S_n , each resulting 4-vertex contains at most one vertex fault.

The positions $a_1, a_2, \dots, a_{n-4}$ in Lemma 2 can be easily determined as follows. We let a_1 be a position where at least two vertices in F_v differ. For example, if $F_v=\{123456,$

$123654\}$, a_1 is set to 4 or 6. An a_1 -partition is then executed on S_n to produce n $(n-1)$ -vertices, and F_v is partitioned accordingly. That is, two vertices in F_v fall into the same subset if and only if they belong to the same $(n-1)$ -vertex. The position a_2 is determined similarly. We simply let a_2 be a position where at least two vertices in some subset differ. Then an a_2 -partition is executed on the n $(n-1)$ -vertices to produce $n(n-1)$ $(n-2)$ -vertices, and every non-empty subset of F_v is partitioned accordingly. The process is repeated until every non-empty subset contains one vertex, when the remaining positions are determined arbitrarily.

A path in a graph G is said to be a *hamiltonian path* if it contains every vertex of G exactly once [10]. In the following discussion, a path between two vertices, say X and Y , are abbreviated to an X - Y path.

Lemma 3. Suppose $n \geq 6$ and $|F_v| \leq n-3$. An R_4 with properties (P1), (P2), and (P3) can be generated in S_n .

Proof. Suppose $a_1, a_2, \dots, a_{n-4}$ are a sequence of positions that satisfy Lemma 2. We first construct an R_6 in S_n . When $n=6$, S_n is an R_6 with only one 6-vertex. For $n>6$, an R_6 can be obtained by executing an $(a_1, a_2, \dots, a_{n-6})$ -partition on S_n as follows. Initially, an a_1 -partition is applied to S_n , and so a K_n^{n-1} results. An R_{n-1} can be easily generated from the K_n^{n-1} . Then, for $j=2, 3, \dots, n-6$, an R_{n-j} can be generated from an R_{n-j+1} as explained below.

Suppose the $R_{n-j+1}=[A_{n-j+1,0}, A_{n-j+1,1}, \dots, A_{n-j+1,n(n-1)\dots(n-j+2)-1}]$. Each $A_{n-j+1,k}$ forms a K_{n-j+1}^{n-j} after an a_j -partition is executed on the R_{n-j+1} , where $0 \leq k \leq n(n-1)\dots(n-j+2)-1$. Let X_k and Y_k denote a pair of distinct entry and exit $(n-j)$ -vertices of $A_{n-j+1,k}$. It is easy to establish a hamiltonian X_k - Y_k path in each K_{n-j+1}^{n-j} formed by $A_{n-j+1,k}$. Then an R_{n-j} can be generated if all the hamiltonian paths are interleaved with $(n-j)$ -edges $(Y_0, X_1), (Y_1, X_2), \dots, (Y_{n(n-1)\dots(n-j+2)-1}, X_0)$. When $j=n-6$, an $R_6=[A_{6,0}, A_{6,1}, \dots, A_{6,n(n-1)\dots7-1}]$ is obtained.

We continue to generate an R_5 from the R_6 . When $n=6$, it is easy to generate an R_5 so that for arbitrary three consecutive 5-vertices $U=\langle u_1u_2\dots u_n \rangle_5$, $V=\langle v_1v_2\dots v_n \rangle_5$, and $W=\langle w_1w_2\dots w_n \rangle_5$ in the R_5 , $u_{\text{diff}(U,V)} \neq w_{\text{diff}(V,W)}$ holds. When $n>6$, each $A_{6,r}$ forms a K_6^5 after an a_{n-5} -partition is executed on the R_6 , where $0 \leq r \leq n(n-1)\dots7-1$. Let X_r and Y_r denote a pair of distinct entry and exit 5-vertices of $A_{6,r}$. It is easy to establish a hamiltonian X_r - Y_r path in the K_6^5 formed by $A_{6,r}$ whose first (last) two 5-vertices are connected to $A_{6,(r-1) \bmod n(n-1)\dots7}$ ($A_{6,(r+1) \bmod n(n-1)\dots7}$). All the hamiltonian paths interleaved with 5-edges $(Y_0, X_1), (Y_1,$

X_2), ..., $(Y_{n(n-1)\dots 7-1}, X_0)$ form an R_5 . Besides, for arbitrary three consecutive 5-vertices $U=\langle u_1u_2\dots u_n \rangle_5$, $V=\langle v_1v_2\dots v_n \rangle_5$, and $W=\langle w_1w_2\dots w_n \rangle_5$ in the R_5 , $u_{\text{diff}(U,V)} \neq w_{\text{diff}(V,W)}$ holds. The reason is explained below.

Let $p=\text{diff}(U, V)$ and $q=\text{diff}(V, W)$. If $U=Y_i$ for some $0 \leq i \leq n(n-1)(n-2)\dots 7-1$, then $V=X_{(i+1) \bmod n(n-1)\dots 7}$ and W is the second 5-vertex in the hamiltonian $X_{(i+1) \bmod n(n-1)\dots 7}-Y_{(i+1) \bmod n(n-1)\dots 7}$ path, which cause $p \neq a_{n-5}$ and $q=a_{n-5}$. Moreover, W is connected to $A_{6,i}$ according to the construction above. Recall that the 5-vertex of $A_{6,i}$ that are not connected to $A_{6,(i+1) \bmod n(n-1)\dots 7}$ is $\langle x_1\dots x_{q-1}y_px_{q+1}\dots x_n \rangle_5$ and the 5-vertex of $A_{6,(i+1) \bmod n(n-1)\dots 7}$ that are not connected to $A_{6,i}$ is $\langle y_1\dots y_{q-1}x_py_{q+1}\dots y_n \rangle_5$, where $x_q=y_p \neq x_p=y_q$ and $x_i=y_i$ for all $1 \leq i \leq n$ and $i \notin \{p, q\}$. Hence, if $w_q=u_p$, then W is not connected to $A_{6,i}$, which is a contradiction. If $W=X_i$ for some $0 \leq i \leq n(n-1)(n-2)\dots 7-1$, then $u_p \neq w_q$ can be shown similarly. Otherwise, if U, V , and W belong to the same 6-vertex, then $u_p \neq w_q$ because $p=q=a_{n-5}$.

We note that at most one 5-vertex in the R_5 contains two vertex faults and the others each contain at most one vertex fault. If there are two 5-vertices in the R_5 that each contain two or more vertex faults, then at most $n-5$ of the 5-vertices in the R_5 are faulty. This is a contradiction because the R_5 , which results from executing an $(a_1, a_2, \dots, a_{n-5})$ -partition on S_n , contains at least $n-4$ faulty 5-vertices (see the paragraph after Lemma 2).

Next, we generate a desired R_4 from the R_5 . Suppose the $R_5=[A_{5,0}, A_{5,1}, \dots, A_{5,n(n-1)\dots 6-1}]$. Without loss of generality, we assume that one 5-vertex, say $A_{5,k}$ ($0 \leq k \leq n(n-1)(n-2)\dots 6-1$), of the R_5 contains two vertex faults and the others each contain at most one vertex fault. An a_{n-4} -partition is executed on the R_5 , and so each $A_{5,j}$ forms a K_5^4 , where $0 \leq j \leq n(n-1)\dots 6-1$. According to Lemma 1, each 4-vertex of $A_{5,k}$ is connected to $A_{5,(k-1) \bmod n(n-1)\dots 6}$ or $A_{5,(k+1) \bmod n(n-1)\dots 6}$. We further assume, without loss of generality, that $A_{5,k}$ contains two faulty 4-vertices, say C and E . A hamiltonian path in the K_5^4 formed by $A_{5,k}$ can be established according to the following three cases.

Case 1. Both of C and E are connected to both $A_{5,(k-1) \bmod n(n-1)\dots 6}$ and $A_{5,(k+1) \bmod n(n-1)\dots 6}$. Let B (F) be the 4-vertex in $A_{5,k}$ that is not connected to $A_{5,(k+1) \bmod n(n-1)\dots 6}$ ($A_{5,(k-1) \bmod n(n-1)\dots 6}$). A hamiltonian path is established as (B, C, D, E, F) , where D is the other 4-vertex of $A_{5,k}$.

Case 2. One of C and E is connected to both $A_{5,(k-1) \bmod n(n-1)\dots 6}$ and $A_{5,(k+1) \bmod n(n-1)\dots 6}$, and the other is connected to one of $A_{5,(k-1) \bmod n(n-1)\dots 6}$ and $A_{5,(k+1) \bmod n(n-1)\dots 6}$. Without loss of generality, we assume C is connected to both $A_{5,(k-1) \bmod n(n-1)\dots 6}$ and $A_{5,(k+1) \bmod n(n-1)\dots 6}$. If E is connected to $A_{5,(k+1) \bmod n(n-1)\dots 6}$, a hamiltonian path is established as (B, C, D, E, F) ,

where B is the 4-vertex of $A_{5,k}$ that is not connected to $A_{5,(k+1) \bmod n(n-1)\dots 6}$, and D and F are the other two 4-vertices of $A_{5,k}$. If E is connected to $A_{5,(k-1) \bmod n(n-1)\dots 6}$, a hamiltonian path is established as (D, E, F, C, B) , where B is the 4-vertex of $A_{5,k}$ that is not connected to $A_{5,(k-1) \bmod n(n-1)\dots 6}$, and D and F are the other two 4-vertices of $A_{5,k}$.

Case 3. One of C and E is not connected to $A_{5,(k-1) \bmod n(n-1)\dots 6}$, and the other is not connected to $A_{5,(k+1) \bmod n(n-1)\dots 6}$ (it is impossible that C and E are not connected to $A_{5,(k-1) \bmod n(n-1)\dots 6}$ or $A_{5,(k+1) \bmod n(n-1)\dots 6}$). Without loss of generality, we assume C is not connected to $A_{5,(k+1) \bmod n(n-1)\dots 6}$ and E is not connected to $A_{5,(k-1) \bmod n(n-1)\dots 6}$. A hamiltonian path is established as (B, C, D, E, F) , where B, D , and F are the other three 4-vertices of $A_{5,k}$.

The first (last) 4-vertex in the hamiltonian paths above is the entry (exit) 4-vertex of $A_{5,k}$. Besides, the first (last) two 4-vertices are connected to $A_{5,(k-1) \bmod n(n-1)\dots 6}$ ($A_{5,(k+1) \bmod n(n-1)\dots 6}$). Then we determine the entry and exit 4-vertices of $A_{5,j}$, denoted by $X_{5,j}$ and $Y_{5,j}$, respectively, for all $0 \leq j \leq n(n-1)(n-2)\dots 5-1$ and $j \neq k$ so that both $Y_{5,(j-1) \bmod n(n-1)\dots 6}$ and $X_{5,j}$ are not faulty and both $Y_{5,j}$ and $X_{5,(j+1) \bmod n(n-1)\dots 6}$ are not faulty. It is not difficult to establish a hamiltonian $X_{5,j}-Y_{5,j}$ path in the K_5^4 formed by each $A_{5,j}$ whose first (last) two 4-vertices are connected to $A_{5,(j-1) \bmod n(n-1)\dots 6}$ ($A_{5,(j+1) \bmod n(n-1)\dots 6}$).

All the hamiltonian paths interleaved with 4-edges constitute an R_4 . Clearly the R_4 has properties (P1) and (P3). Recall that for arbitrary three consecutive 5-vertices $U=\langle u_1u_2\dots u_n \rangle_5$, $V=\langle v_1v_2\dots v_n \rangle_5$, and $W=\langle w_1w_2\dots w_n \rangle_5$ in the R_5 , $u_{\text{diff}(U,V)} \neq w_{\text{diff}(V,W)}$ holds. With similar arguments, the R_4 has property (P2). This completes the proof. Q.E.D.

Lemma 4. Suppose $|F_v| \leq 1$. There is a healthy path of maximal length $4!-3$ between arbitrary two adjacent healthy vertices of S_4 .

Proof. Suppose u and v are arbitrary two adjacent healthy vertices of S_4 , and let $f=f_1f_2f_3f_4$ denote the faulty vertex. Since the star graph is edge symmetric [2], we assume $u=u_1u_2u_3u_4=1234$ and $v=v_1v_2v_3v_4=3214$ for illustration. We have $f_2 \neq u_2=v_2$ or $f_4 \neq u_4=v_4$, for otherwise $f=u$ or $f=v$, which is a contradiction. We assume $f_4 \neq u_4=v_4=4$, without loss of generality. After executing a 4-partition on S_4 , u and v belong to $\langle **** \rangle_3$ and f belongs to $\langle ***1 \rangle_3$ or $\langle ***2 \rangle_3$ or $\langle ***3 \rangle_3$.

If $f \in \langle ***1 \rangle_3$, a healthy $u-v$ path of length $4!-3$ can be constructed as follows:

$(u=1234, 2134, 3124, 1324, 4321, 3421, 2431, 4231, 3241, 1243, 2143, 4123, 1423, 2413, 3412, 1432,$

4132, 3142, 1342, 4312, 2314, 3214= v) if $f=2341$;

($u=1234, 4231, 3241, 2341, 4321, 3421, 1423, 2413, 4213, 1243, 2143, 3142, 1342, 4312, 3412, 1432, 4132, 2134, 3124, 1324, 2314, 3214=v$) if $f=2431$;

($u=1234, 4231, 2431, 3421, 4321, 2341, 1342, 4312, 3412, 1432, 4132, 3142, 2143, 1243, 4213, 2413, 1423, 4123, 3124, 1324, 2314, 3214=v$) if $f=3241$;

($u=1234, 2134, 3124, 4123, 1423, 2413, 4213, 1243, 2143, 3142, 1342, 4312, 3412, 1432, 2431, 4231, 3241, 2341, 4321, 1324, 2314, 3214=v$) if $f=3421$;

($u=1234, 2134, 4132, 3142, 1342, 4312, 3412, 1432, 2431, 3421, 4321, 2431, 3241, 1243, 4213, 2413, 1423, 4123, 3124, 1324, 2314, 3214=v$) if $f=4231$;

($u=1234, 2134, 3124, 4123, 2143, 1243, 4213, 2413, 1423, 3421, 2431, 4231, 3241, 2341, 1342, 3142, 4132, 1432, 3412, 4312, 2314, 3214=v$) if $f=2341$.

If $f \in \langle ***2 \rangle_3$ or $\langle ***3 \rangle_3$, a healthy $u-v$ path of length 4!-3 can be obtained similarly. Since the star graph is bipartite, the $u-v$ path is the longest in the worst case. Q.E.D.

We note that S_3 forms a ring of length six. The following two lemmas were shown in [32].

Lemma 5.[32] Suppose U and V are two adjacent 3-vertices in S_n , and let $(c_0, c_1, \dots, c_5)$ denote the ring formed by U . Then, the vertices of U that are connected to V are c_j and $c_{(j+3) \bmod 6}$ for some $0 \leq j \leq 5$.

Lemma 6.[32] Suppose $U = \langle u_1 u_2 \dots u_n \rangle_3$, $V = \langle v_1 v_2 \dots v_n \rangle_3$, and $W = \langle w_1 w_2 \dots w_n \rangle_3$ are three 3-vertices in S_n , and V is adjacent to both U and W . If $u_{\text{diff}(U, V)} \neq w_{\text{diff}(V, W)}$, then the two vertices of V that are connected to U are disjoint from the two vertices of V that are connected to W .

Lemma 7. Suppose $n \geq 5$ and $|F_v| \leq n-3$. A healthy ring of maximal length $n!-2|F_v|$ can be generated from an R_4 in S_n that owns properties (P1), (P2), and (P3).

Proof. Suppose an $R_4 = [A_0, A_1, \dots, A_{n(n-1)\dots 5-1}]$ with properties (P1), (P2), and (P3) is obtained after executing an $(a_1, a_2, \dots, a_{n-4})$ -partition on S_n . An a_{n-3} -partition is then executed on the R_4 , and so each A_i is partitioned into four 3-vertices that form a K_4^3 , where $a_{n-3} \in \{2, 3, \dots, n\} - \{a_1, a_2, \dots, a_{n-4}\}$ and $0 \leq i \leq n(n-1)\dots 5-1$. According to Lemma 1, each 3-vertex of A_i is connected to $A_{(i-1) \bmod n(n-1)\dots 5}$ or $A_{(i+1) \bmod n(n-1)\dots 5}$ because of property (P2). Since three 3-vertices

of A_i are connected to $A_{(i-1) \bmod n(n-1)\dots 5}$ and three 3-vertices of A_i are connected to $A_{(i+1) \bmod n(n-1)\dots 5}$, two 3-vertices of A_i are connected to both $A_{(i-1) \bmod n(n-1)\dots 5}$ and $A_{(i+1) \bmod n(n-1)\dots 5}$.

We then determine the entry and exit 3-vertices of each A_i , denoted by X_i and Y_i , respectively, as follows. For each faulty A_i , we let $X_i = Y_i = Q$, where Q is a healthy 3-vertex of A_i that is connected to both $A_{(i-1) \bmod n(n-1)\dots 5}$ and $A_{(i+1) \bmod n(n-1)\dots 5}$. We note that A_i contains one vertex fault (property P(1)). Since $A_{(i-1) \bmod n(n-1)\dots 5}$ and $A_{(i+1) \bmod n(n-1)\dots 5}$ are healthy (property (P3)), $Y_{(i-1) \bmod n(n-1)\dots 5}$ and $X_{(i+1) \bmod n(n-1)\dots 5}$ are determined as the 3-vertex of $A_{(i-1) \bmod n(n-1)\dots 5}$ and the 3-vertex of $A_{(i+1) \bmod n(n-1)\dots 5}$, respectively, that are adjacent to Q . For every two consecutive healthy A_i and $A_{(i+1) \bmod n(n-1)\dots 5}$, Y_i and $X_{(i+1) \bmod n(n-1)\dots 5}$ are determined as two adjacent 3-vertices that belong to A_i and $A_{(i+1) \bmod n(n-1)\dots 5}$, respectively.

All X_i and Y_i thus determined are healthy. Moreover, when $X_i = Y_i$, $u_{\text{diff}(Y_{(i-1) \bmod n(n-1)\dots 5}, X_i)} \neq w_{\text{diff}(X_i, X_{(i+1) \bmod n(n-1)\dots 5})}$ holds, where $Y_{(i-1) \bmod n(n-1)\dots 5} = \langle u_1 u_2 \dots u_n \rangle_4$ and $X_{(i+1) \bmod n(n-1)\dots 5} = \langle w_1 w_2 \dots w_n \rangle_4$ are assumed. The reason is explained as follows. Suppose conversely

$u_{\text{diff}(Y_{(i-1) \bmod n(n-1)\dots 5}, X_i)} = w_{\text{diff}(X_i, X_{(i+1) \bmod n(n-1)\dots 5})}$. If $\text{diff}(Y_{(i-1) \bmod n(n-1)\dots 5}, X_i) = \text{diff}(X_i, X_{(i+1) \bmod n(n-1)\dots 5})$, then $Y_{(i-1) \bmod n(n-1)\dots 5} = X_{(i+1) \bmod n(n-1)\dots 5}$, which is a contradiction. If $\text{diff}(Y_{(i-1) \bmod n(n-1)\dots 5}, X_i) \neq \text{diff}(X_i, X_{(i+1) \bmod n(n-1)\dots 5})$, then there is a 3-vertex of A_i which is connected to neither of $A_{(i-1) \bmod n(n-1)\dots 5}$ and $A_{(i+1) \bmod n(n-1)\dots 5}$. This is again a contradiction because each 3-vertex of A_i is connected to $A_{(i-1) \bmod n(n-1)\dots 5}$ or $A_{(i+1) \bmod n(n-1)\dots 5}$.

Next we generate a healthy ring of length $n!-2|F_v|$ by establishing healthy paths in $A_0, A_1, \dots, A_{n(n-1)\dots 5-1}$ in this sequence. First we construct a healthy path in A_0 as follows. Let x_0 be a healthy vertex of X_0 which is adjacent to a healthy vertex of $Y_{n(n-1)\dots 5-1}$. When $X_0 = Y_0$, $u_{\text{diff}(Y_{n(n-1)\dots 5-1}, X_0)} \neq w_{\text{diff}(X_0, X_1)}$ holds for three consecutive 3-vertices $Y_{n(n-1)\dots 5-1} = \langle u_1 u_2 \dots u_n \rangle_4$, $X_0 (= Y_0)$, and $X_1 = \langle w_1 w_2 \dots w_n \rangle_4$. According to Lemmas 5 and 6, there is a healthy vertex y_0 of $Y_0 (= X_0)$ which is adjacent to both x_0 and x_1 , where x_1 is a healthy vertex of X_1 . If A_0 is healthy, it is not difficult to establish a healthy hamiltonian x_0 - y_0 path in A_0 . If A_0 is faulty, a healthy x_0 - y_0 path of maximal length 4!-3 in A_0 can be established according to Lemma 4.

We then consider the situation of $X_0 \neq Y_0$. Let Q_0 (L_0) be the 3-vertex of A_0 that is not connected to A_1 ($A_{n(n-1)\dots 5-1}$), where $Q_0 \neq Y_0$ ($L_0 \neq X_0$). A healthy hamiltonian X_0 - Y_0 path for the K_4^3 formed by A_0 is established as (X_0, B, C, Y_0) if

$Q_0=X_0$ and $L_0=Y_0$, (X_0, Q_0, C, Y_0) if $Q_0 \neq X_0$ and $L_0=Y_0$, (X_0, B, L_0, Y_0) if $Q_0=X_0$ and $L_0 \neq Y_0$, and (X_0, Q_0, L_0, Y_0) if $Q_0 \neq X_0$ and $L_0 \neq Y_0$, where B and C are the other 3-vertices of A_0 .

Without loss of generality, we assume (X_0, Q_0, L_0, Y_0) is the healthy hamiltonian X_0 - Y_0 path. With the same arguments as the proof of Lemma 3, every three consecutive 3-vertices in $\{Y_{n(n-1) \dots 5-1}, X_0, Q_0, L_0, Y_0, X_1\}$ satisfy property (P2). Since A_0 is healthy, according to Lemmas 5 and 6 there are four pairs of distinct healthy vertices, denoted by $\{x_0, p\}$, $\{q, r\}$, $\{s, t\}$, and $\{z, y_0\}$, of X_0, Q_0, L_0 , and Y_0 , respectively, so that every two consecutive vertices in $\{x_0, p, q, r, s, t, z, y_0, x_1\}$ are adjacent, where x_1 is a healthy vertex of X_1 . Clearly, there are healthy hamiltonian x_0 - p , q - r , s - t , and z - y_0 paths in X_0, Q_0, L_0 , and Y_0 , respectively. These healthy hamiltonian paths interleaved with edges (p, q) , (r, s) , and (t, z) constitute a healthy hamiltonian x_0 - y_0 path in A_0 .

Then we construct healthy paths in $A_1, A_2, \dots, A_{n(n-1) \dots 5-2}$, sequentially. Suppose a healthy x_{j-1} - y_{j-1} path in A_{j-1} is obtained, where $1 \leq j \leq n(n-1) \dots 5-2$. Very similar to the situation of A_0 , a healthy path in A_j can be constructed as follows. Let x_j be the (healthy) vertex of X_j that is adjacent to y_{j-1} . When $X_j=Y_j$, $u_{\text{dif}(Y_{j-1}, X_j)} \neq w_{\text{dif}(X_j, X_{j+1})}$ holds for three consecutive 3-vertices $Y_{j-1}=\langle u_1 u_2 \dots u_n \rangle$, $X_j(=Y_j)$, and $X_{j+1}=\langle w_1 w_2 \dots w_n \rangle$. According to Lemmas 5 and 6, there is a healthy vertex y_j of $Y_j(=X_j)$ which is adjacent to both x_j and a healthy vertex of X_{j+1} . If A_j is healthy, there is a healthy hamiltonian x_j - y_j path in A_j . If A_j is faulty, a healthy x_j - y_j path of maximal length $4!-3$ in A_j can be established according to Lemma 4. When $X_j \neq Y_j$, a hamiltonian X_j - Y_j path for the K_4^3 formed by A_j is first established. Then a healthy hamiltonian x_j - y_j path in A_j can be obtained by concatenating healthy hamiltonian paths for the four 3-vertices of A_j , where y_j is a healthy vertex of Y_j and is adjacent to a healthy vertex of X_{j+1} .

Thus far we have established healthy paths in $A_0, A_1, \dots, A_{n(n-1) \dots 5-2}$. We continue to establish a healthy path in $A_{n(n-1) \dots 5-1}$. Let $x_{n(n-1) \dots 5-1}$ be the (healthy) vertex of $X_{n(n-1) \dots 5-1}$ that is adjacent to $y_{n(n-1) \dots 5-2}$, and $y_{n(n-1) \dots 5-1}$ be the (healthy) vertex of $Y_{n(n-1) \dots 5-1}$ that is adjacent to x_0 . Without loss of generality, we use $(c_0, c_1, c_2, c_3, c_4, c_5)$ to represent $Y_{n(n-1) \dots 5-1}$, where $c_0=y_{n(n-1) \dots 5-1}$ is assumed. When $X_{n(n-1) \dots 5-1}=Y_{n(n-1) \dots 5-1}$, $u_{\text{dif}(Y_{n(n-1) \dots 5-2}, X_{n(n-1) \dots 5-1})} \neq w_{\text{dif}(X_{n(n-1) \dots 5-1}, X_0)}$ holds for three consecutive 3-vertices $Y_{n(n-1) \dots 5-2}=\langle u_1 u_2 \dots u_n \rangle$, $X_{n(n-1) \dots 5-1}(=Y_{n(n-1) \dots 5-1})$, and $X_0=\langle w_1 w_2 \dots w_n \rangle$. According to Lemmas 5 and 6, $x_{n(n-1) \dots 5-1}$ cannot be c_0 or c_3 . Actually, $x_{n(n-1) \dots 5-1}=c_1$ or c_5 (i.e., $x_{n(n-1) \dots 5-1}$ and $y_{n(n-1) \dots 5-1}$ are adjacent) as explained below.

If $x_{n(n-1) \dots 5-1}=c_2$, a healthy path (c_2, c_1, c_0) in $Y_{n(n-1) \dots 5-1}$ together with all healthy paths that we have established in

$A_0, A_1, \dots, A_{n(n-1) \dots 5-2}$ form a ring of odd length in S_n . This is a contradiction because S_n is a bipartite graph in which every ring has even length. Similarly, there is a contradiction for $x_{n(n-1) \dots 5-1}=c_4$. Now that $x_{n(n-1) \dots 5-1}$ and $y_{n(n-1) \dots 5-1}$ are adjacent, there is a healthy hamiltonian $x_{n(n-1) \dots 5-1}$ - $y_{n(n-1) \dots 5-1}$ path in $A_{n(n-1) \dots 5-1}$ if $A_{n(n-1) \dots 5-1}$ is healthy. If $A_{n(n-1) \dots 5-1}$ is faulty, there is a healthy $x_{n(n-1) \dots 5-1}$ - $y_{n(n-1) \dots 5-1}$ path of maximal length $4!-3$ in $A_{n(n-1) \dots 5-1}$ according to Lemma 4.

When $X_{n(n-1) \dots 5-1} \neq Y_{n(n-1) \dots 5-1}$, a healthy hamiltonian $X_{n(n-1) \dots 5-1}$ - $Y_{n(n-1) \dots 5-1}$ path for the K_4^3 formed by $A_{n(n-1) \dots 5-1}$ is first established, similar to the situation of A_0 . Without loss of generality, we assume $(X_{n(n-1) \dots 5-1}, D, E, Y_{n(n-1) \dots 5-1})$ is the healthy hamiltonian $X_{n(n-1) \dots 5-1}$ - $Y_{n(n-1) \dots 5-1}$ path. There are three pairs of distinct healthy vertices, denoted by $\{x_{n(n-1) \dots 5-1}, a\}$, $\{b, c\}$, and $\{d, e\}$, in $X_{n(n-1) \dots 5-1}, D$, and E , respectively, so that every two consecutive vertices in $\{x_{n(n-1) \dots 5-1}, a, b, c, d, e, f\}$ are adjacent, where f is a healthy vertex of $Y_{n(n-1) \dots 5-1}$. Since $y_{n(n-1) \dots 5-1}=c_0$, f cannot be c_0 or c_3 according to Lemmas 5 and 6. Moreover, $f \in \{c_2, c_4\}$ will result in a ring of odd length in S_n , which is a contradiction. Hence, f is adjacent to $y_{n(n-1) \dots 5-1}$ (i.e., $f=c_1$ or c_5). It is easy to establish a healthy hamiltonian $x_{n(n-1) \dots 5-1}$ - $y_{n(n-1) \dots 5-1}$ path in $A_{n(n-1) \dots 5-1}$.

Clearly, all the healthy paths that we have established in $A_0, A_1, \dots, A_{n(n-1) \dots 5-1}$ together with edges $(y_0, x_1), (y_1, x_2), \dots, (y_{n(n-1) \dots 5-1}, x_0)$ constitute a healthy ring of length $n!-2|F_v|$. Since S_n is bipartite with two partite sets of equal size, the ring is the longest in the worst case. Q.E.D.

Theorem 1. Suppose $n \geq 4$ and $|F_v| \leq n-3$. S_n can embed a healthy ring of maximal length $n!-2|F_v|$.

Proof. We show that S_n contains a healthy ring of maximal length $n!-2|F_v|$ according to different values of n . When $n=4$, $|F_v| \leq 1$. By the aid of Lemma 4, there is a healthy ring of maximal length $4!-2|F_v|$ in S_4 .

When $n=5$, $|F_v| \leq 2$. Without loss of generality, we assume $|F_v|=2$. According to Lemma 2, the two vertex faults of S_5 will be distributed to two different 4-vertices after an a_1 -partition is executed, where $2 \leq a_1 \leq 5$. Then an R_4 is established so that the two faulty 4-vertices are not adjacent in the R_4 . Clearly the R_4 satisfies properties (P1) and (P3). Besides, it is not difficult to check that the R_4 also satisfies property (P2). By Lemma 7, a healthy ring of maximal length $5!-2|F_v|$ can be generated from the R_4 .

When $n \geq 6$, an R_4 with properties (P1), (P2), and (P3) is first obtained by Lemma 3. Then, a healthy ring of maximal length $n!-2|F_v|$ can be generated by Lemma 7. Q.E.D.

4 Concluding remarks

In this paper, we have shown that a healthy ring of length $n!-2|F_v|$ can be embedded onto an n -dimensional star graph with $|F_v| \leq n-3$ vertex faults. This improves Tseng *et al.*'s result by lengthening the embedded ring from $n!-4|F_v|$ to $n!-2|F_v|$. Since the star graph is regular of degree $n-1$ and is bipartite with two partite sets of equal size, our result achieves optimality in the worst case.

In [32], Tseng *et al.* also showed that an n -dimensional star graph with $|F_v|$ vertex faults and $|F_e|$ edge faults can embed a healthy ring of length at least $n!-4|F_v|$ if $|F_v|+|F_e| \leq n-3$. With our result, the embedded ring can be lengthened from $n!-4|F_v|$ to $n!-2|F_v|$ as well.

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