A New Approach to Constructing Optimal Prefix Circuits with Small Depth

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Abstract

Prefix computation has many applications, and should be implemented as a primitive operation. Many combinational circuits for performing the prefix operation in parallel, called parallel prefix circuits, have been designed and studied. The size of a prefix circuit D , $s(D)$, is the number of operation nodes in D , and the depth of D , $d(D)$, is the maximum level of operation nodes in D . Smaller depth implies faster computation, while smaller size implies less power consumption and smaller area in VLSI implementation and thus less cost. D is depth-size optimal if $d(D) + s(D) = 2n - 2$. Another circuit parameter is fan-out. A circuit having a smaller fan-out is faster and smaller in VLSI implementation. Thus, a circuit should have a small fan-out for it to be of practical use. In this paper, we take a new approach to designing a depth-size optimal parallel prefix circuit, WE4, with fan-out 4 and small depth. In many cases of n , WE4 has the smallest depth among all known prefix circuits.

1. Introduction

Given n values $x_1, x_2, \dots, x_n$ and an associative binary operation $\otimes$, the prefix operation is to compute the n values $x_1 \otimes x_2 \otimes \dots \otimes x_i$, $1 \leq i \leq n$. Prefix computation has many applications, such as sorting, job scheduling, loop parallelization, and processor allocation [1, 2, 7-9, 17]. Because of its usefulness, it should be implemented as a primitive operation [3]. Many combinational circuits for performing the prefix operation in parallel, called parallel prefix circuits, have been designed and studied [2, 4, 5, 8-11, 13-16].

In this paper, we assume that the number of inputs is n , unless otherwise stated. We represent a prefix circuit with a directed acyclic graph containing n input nodes, n output nodes, at least $n - 1$ operation nodes, and at least one duplication node. For ease of presentation, in this paper, all the directed edges are assumed to be downward; thus, the arrows need not be

shown. A serial prefix circuit $S(n)$ is shown in Fig. 1, in which vertical edges from left to right are named line 1, line 2, ..., line n , respectively. Input nodes are on the top of a circuit and have indegree 0 and outdegree 1, representing input items. Output nodes are at the bottom and have indegree 1 and outdegree 0, representing outputs. An operation node, represented by a black dot, performs the $\otimes$ operation on its two inputs, having indegree 2 and outdegree 1 or more. For any operation node on line i at level j , its left input is from a node at level $j - 1$, while its right input is always from line i . A duplication node has indegree 1 and outdegree 2 or more; it outputs multiple copies of its single input. In Fig. 1, a duplication node is on line 1 at level 0. The numbers at the left of a prefix circuit denote the depth levels of the nodes to the right. The input node on line i represents input x_i . When no confusion is caused, we may use simply D for any prefix circuit $D(p)$, where p may be the number of inputs or a parameter with another meaning.

Fig. 1. Serial prefix circuit $S(n)$.

The size of a prefix circuit D , $s(D)$, is the number of operation nodes in D , and the depth of D , $d(D)$, is the maximum level of operation nodes in D . Smaller depth implies faster computation. Smaller size implies less power consumption and smaller area in VLSI implementation and thus less cost. Therefore, size and depth are important parameters of prefix circuits. For any prefix circuit D , $d(D) + s(D) \geq 2n - 2$ [16]. Thus, D is depth-size optimal, or optimal for short, if $d(D) + s(D) = 2n - 2$. For

example, Fig. 1 shows that $s(S) = d(S) = n - 1$; thus, S is optimal.

A third important parameter is fan-out. The fan-out of a node is its outdegree. A node having a smaller fan-out is faster and smaller in VLSI implementation [18]. A node has unbounded fan-out if the fan-out is not fixed and is a function of n ; otherwise, the fan-out of the node is a constant, or is bounded. The fan-out of prefix circuit D , $fo(D)$, is the maximum fan-out of all nodes in D . For example, $fo(S) = 2$. A circuit has unbounded fan-out if one of its nodes has unbounded fan-out. A circuit should have a small bounded fan-out for it to be of practical use.

For ease of presentation, let $i:j$ represent the result of computing $x_i \otimes x_{i+1} \otimes \dots \otimes x_j$, where $i \leq j$. We use $ia(D) = a$ to denote that line 1 of prefix circuit D has a duplication node at level a and has no duplication nodes at any level less than a . In addition, we use $l(D) = b$ to denote that line n of D obtains $1:n$ at level b .

Many previous parallel prefix circuits are briefly reviewed in [11]. Some of them with smaller depth are in the following. *LYD* is an optimal prefix circuit with unbounded fan-out, whose depth is between $2 \lceil \lg n \rceil - 6$ and $2 \lceil \lg n \rceil - 3$ [9]. *M* is also an optimal prefix circuit with unbounded fan-out, whose depth is between $2 \lceil \lg n \rceil - 5$ and $2 \lceil \lg n \rceil - 3$ [12]. *H4* is an optimal prefix circuit with fan-out 4, whose depth is between $2 \lceil \lg n \rceil - 5$ and $2 \lceil \lg n \rceil - 3$ [11]. *Z4* is an optimal prefix circuit with fan-out 4, whose depth is between $2 \lceil \lg n \rceil - 6$ and $2 \lceil \lg n \rceil - 3$ [6].

In this paper, we introduce a new optimal parallel prefix circuit, denoted as *WE4*, with fan-out 4. In many cases of n , it has the smallest depth among all known prefix circuits. Section 2 briefly reviews previous results. Sections 3 and 4 present prefix circuits *WL* and *WV*, respectively; these circuits will be used in Section 5 to construct *WE4*. Section 6 compares *WE4* with the optimal parallel prefix circuits mentioned in the previous paragraph in some detail. Section 7 concludes this paper.

2. Brief review of previous results

A prefix circuit D can be defined with sets of operation nodes at level i , $i = 1, 2, \dots, d(D)$ [16]:

$G_i = \{(x, y) \mid \text{at level } i \text{ on line } y \text{ there is an operation node whose left input is the output of a node on line } x \text{ at level } i - 1\}$.

If $(x, y) \in G_i$, the corresponding operation node can be denoted as $(x, y)_i$.

2.1. Prefix circuits Q and W

Prefix circuit $Q(n)$ with fan-out at most 4 has been presented in [14].

Property 1 [14]: The depth of $Q(n)$ is

$$\begin{aligned} & \lceil \lg n \rceil && \text{if } n \leq 7, \\ & 2t - 4 && \text{if } t \geq 4 \text{ and } 2^{t-1} \leq n < 3 \times 2^{t-2}, \\ & 2t - 3 && \text{if } t \geq 4 \text{ and } 3 \times 2^{t-2} \leq n < 2^t. \end{aligned}$$

$W(n)$ is a prefix circuit defined with the following operation nodes [11]:

$$G_i = \{(i + 1, i + 2)\}, \quad i = 1, 2, \dots, n - 2;$$

$$G_{n-1} = \{(1, i) \mid i = 2, 3, \dots, n\}.$$

For example, $W(3)$ and $W(4)$ are shown in Fig. 2. By definition, $d(W) = n - 1$, $s(W) = 2n - 3$, $ia(W) = n - 2$, and $l(W) = n - 1$.

Fig. 2. $W(3)$, $W(4)$, and $W(3) \cdot W(4)$.

2.2. Composition of prefix circuits, size optimality, and other properties

Assume that $A(n_1)$ and $B(n_2)$ are two prefix circuits with n_1 and n_2 inputs, respectively. $A(n_1)$ and $B(n_2)$ can be composed into a prefix circuit with $n_1 + n_2 - 1$ inputs by merging the operation node that produces $1:n_1$ on line n_1 of $A(n_1)$ with the first, if not unique, duplication node on line 1 of $B(n_2)$; the resulting circuit is denoted by $A(n_1) \cdot B(n_2)$ [16]. For example, Fig. 2 also gives $W(3) \cdot W(4)$. Three or more prefix circuits can also be composed into a single prefix circuit, and the composition operation is associative.

Definition 2 [11]: For any prefix circuit A with n inputs, if $ia(A) = a$, $l(A) = b$, and $s(A) = 2n - 2 + a - b$, then A is size optimal; we say that A is $SOPC(n, a, b)$.

Property 3 [11]: $Q(n)$ is $SOPC(n, 0, \lceil \lg n \rceil)$.

Property 4 [11]: $W(n)$ is $SOPC(n, n - 2, n - 1)$ with fan-out n .

Property 5 [11]: If A is $SOPC(n, 0, b)$ and $d(A) = b$, then A is optimal.

Property 6 [11]: If $A(n_1)$ and $B(n_2)$ are $SOPC(n_1, a, b)$ and $SOPC(n_2, c, d)$, respectively, where $b \geq c$, then $A(n_1) \cdot B(n_2)$ is

$SOPC(n_1 + n_2 - 1, a, b - c + d)$ with depth $\max\{d(A(n_1)), d(B(n_2)) + b - c\}$.

3. Size optimal prefix circuit WL

Prefix circuit $WL(5)$ is defined with the following operation nodes:

$$G_1 = \{(2, 3), (6, 7), (10, 11)\},$$

$$G_2 = \{(3, 4), (7, 8), (11, 12)\},$$

$$G_3 = \{(4, 5), (8, 9), (12, 13)\},$$

$$G_4 = \{(5, 9)\},$$

$$G_5 = \{(9, 13)\},$$

$$G_6 = \{(1, 5), (1, 9), (1, 13)\},$$

$$G_7 = \{(1, 2), (1, 3), (1, 4), (5, 6), (5, 7), (5, 8), (9, 10), (9, 11), (9, 12)\}.$$

Fig. 3 shows that $d(WL(5)) = 7$, $fo(WL(5)) = 4$, $ia(WL(5)) = 5$, and $l(WL(5)) = 6$. In addition, $s(WL(5)) = 23 = 2 \times 13 - 2 + 5 - 6$. Therefore, by Definition 2, $WL(5)$ is $SOPC(13, 5, 6)$.

Fig. 3. $WL(5)$.

We can move the operation node $(1, 13)_6$ of $WL(5)$ downward by 1 level to be $(1, 13)_7$, and move the other operation nodes at level 6 and level 7 downward by 2 levels, resulting in a new prefix circuit $WA(5)$. Note that $d(WA(5)) = 9$, $fo(WA(5)) = 4$, and $WA(5)$ is $SOPC(13, 6, 7)$. We can also obtain $WB(5)$ by moving down all the nodes at levels 6 and 7 of $WL(5)$ by 2 levels. Clearly, $d(WB(5)) = 9$, $fo(WB(5)) = 4$, and $WB(5)$ is $SOPC(13, 7, 8)$.

By Property 6, $WA(5) \bullet WB(5)$ is $SOPC(25, 6, 8)$. We can delete the operation node $(13, 25)_8$ of $WA(5) \bullet WB(5)$, and then add operation nodes $(13, 25)_6$ and $(1, 25)_7$, obtaining $WL(6)$ as shown in Fig. 4. Clearly, $d(WL(6)) = 9$, $fo(WL(6)) = 4$ and $ia(WL(6)) = 6$. Since only operation nodes on line 25 are modified and $WL(6)$ obtains 1:25 at level 7, clearly $WL(6)$ is a prefix circuit and $l(WL(6)) = 7$.

Fig. 4. $WL(6)$.

As already mentioned, $WA(5) \bullet WB(5)$ is $SOPC(25, 6, 8)$, thus $s(WA(5) \bullet WB(5)) = 2 \times 25 - 2 + 6 - 8 = 46$. Clearly, $s(WL(6)) = 47$. That is, $s(WL(6)) = 2 \times 25 - 2 + 6 - 7$. Since $ia(WL(6)) = 6$ and $l(WL(6)) = 7$, by Definition 2, $WL(6)$ is $SOPC(25, 6, 7)$.

The above method of obtaining $WL(6)$ from $WL(5)$ can be generalized to derive $WL(t+1)$ from $WL(t)$, $t \geq 5$.

Algorithm A: Let N be the set of nodes at levels $t+1$ through $2t-3$ of $WL(t)$, where $t \geq 5$. We can obtain $WL(t+1)$ by the following procedures.

1. Move down the operation node $(1, 3 \times 2^{t-3} + 1)_{t+1}$ of $WL(t)$ by 1 level to become $(1, 3 \times 2^{t-3} + 1)_{t+2}$, and move down all the other operation nodes in N by 2 levels, resulting in $WA(t)$. (If the left input of an operation node comes from a duplication node, then moving the operation node implies that the duplication node should also be moved accordingly.)
2. Move down all the operation nodes in N by 2 levels to obtain $WB(t)$.
3. Delete node $(3 \times 2^{t-3} + 1, 3 \times 2^{t-2} + 1)_{t+3}$ of the composed prefix circuit $WA(t) \bullet WB(t)$, and add 2 nodes $(3 \times 2^{t-3} + 1, 3 \times 2^{t-2} + 1)_{t+1}$ and $(1, 3 \times 2^{t-2} + 1)_{t+2}$, resulting in $WL(t+1)$.

Theorem 7: $WL(t)$ is $SOPC(3 \times 2^{t-3} + 1, t, t + 1)$ with depth $2t - 3$ and fan-out 4, for $t \geq 5$.

Proof: This theorem can be proved by induction on t . The proof is omitted because of space limitation.

4. Size optimal prefix circuit WV

Prefix circuit $WV(5)$ is defined with the following operation nodes:

$$G_1 = \{(2, 3), (5, 6), (8, 9)\},$$

$$G_2 = \{(3, 4), (6, 7), (9, 10)\},$$

$$G_3 = \{(4, 7)\},$$

$$G_4 = \{(7, 10)\},$$

$$G_5 = \{(1, 4), (1, 7), (1, 10)\},$$

$$G_6 = \{(1, 2), (1, 3), (4, 5), (4, 6), (7, 8), (7, 9)\}.$$

Fig. 5 shows that $d(WV(5)) = 6$, $fo(WV(5)) = 4$, $ia(WV(5)) = 4$, and $l(WV(5)) = 5$. In addition, $s(WV(5)) = 17 = 2 \times 10 - 2 + 4 - 5$. Therefore, by Definition 2, $WV(5)$ is $SOPC(10, 4, 5)$.

Fig. 5. $WV(5)$.

We can move down the operation node $(1, 10)_5$ of $WV(5)$ by 1 level to be $(1, 10)_6$, and move the other operation nodes at level 5 and level 6 downward by 2 levels, resulting in a new prefix circuit $VA(5)$. Note that $d(VA(5)) = 8$, $fo(VA(5)) = 3$, and $VA(5)$ is $SOPC(10, 5, 6)$. We can also obtain $VB(5)$ by moving down all the nodes at levels 5 and 6 of $WV(5)$ by 2 levels. Clearly, $d(VB(5)) = 8$, $fo(VB(5)) = 4$, and $VB(5)$ is $SOPC(10, 6, 7)$.

By Property 6, $VA(5) \cdot VB(5)$ is $SOPC(19, 5, 7)$. We can delete the operation node $(10, 19)_7$ of $VA(5) \cdot VB(5)$, and then add operation nodes $(10, 19)_5$ and $(1, 19)_6$, obtaining $WV(6)$ as shown in Fig. 6. Clearly, $d(WV(6)) = 8$, $fo(WV(6)) = 3$, and $ia(WV(6)) = 5$. Since only operation nodes on line 19 are modified and $WV(6)$ obtains 1:19 at level 6, clearly $WV(6)$ is a prefix circuit and $l(WV(6)) = 6$.

Fig. 6. $WV(6)$.

Since $VA(5) \cdot VB(5)$ is $SOPC(19, 5, 7)$, $s(VA(5) \cdot VB(5)) = 2 \times 19 - 2 + 5 - 7 = 34$. Thus, $s(WV(6)) = s(VA(5) \cdot VB(5)) - 1 + 2 = 2 \times 19 - 2 + 5 - 6$. Furthermore, since $ia(WV(6)) = 5$ and $l(WV(6)) = 6$,

by Definition 2, $WV(6)$ is $SOPC(19, 5, 6)$.

The above method of obtaining $WV(6)$ from $WV(5)$ can be generalized to derive $WV(t+1)$ from $WV(t)$, $t \geq 5$.

Algorithm B: Let N be the set of nodes at levels t through $2t-4$ of $WV(t)$, where $t \geq 5$. We can obtain $WV(t+1)$ by the following procedures.

1. Move down the operation node $(1, 9 \times 2^{t-5} + 1)_t$ of $WV(t)$ by 1 level to become $(1, 9 \times 2^{t-5} + 1)_{t+1}$, and move down all the other operation nodes in N by 2 levels, resulting in $VA(t)$.
2. Move down all the operation nodes in N by 2 levels to obtain $VB(t)$.
3. Delete the node $(9 \times 2^{t-5} + 1, 9 \times 2^{t-4} + 1)_{t+2}$ of the composed prefix circuit $VA(t) \cdot VB(t)$, and add operation nodes $(9 \times 2^{t-5} + 1, 9 \times 2^{t-4} + 1)_t$ and $(1, 9 \times 2^{t-4} + 1)_{t+1}$, resulting in $WV(t+1)$.

Theorem 8: $WV(t)$ is $SOPC(9 \times 2^{t-5} + 1, t-1, t)$ with depth $2t-4$ and fan-out 3, for $t \geq 6$.

Proof: This theorem can be proved by induction on t . The proof is omitted because of space limitation.

5. Depth-size optimal prefix circuit WE4

For ease of presentation, let $WG(t) = WL(t) \cdot WL(t-1) \cdot \dots \cdot WL(5) \cdot W(4)$, for $t \geq 5$, and $WG(4) = W(4)$.

Lemma 9: $WG(t)$ is $SOPC(3 \times 2^{t-2} - 8, t, 2t-3)$ with depth $2t-3$, for $t \geq 5$.

Proof: By Theorem 7, $WL(t)$ is $SOPC(3 \times 2^{t-3} + 1, t, t+1)$ with depth $2t-3$. By Property 6, $WL(t) \cdot WL(t-1)$ is $SOPC(3 \times 2^{t-3} + 3 \times 2^{t-4} + 1, t, t+2)$ with depth $2t-3$. Using Property 6 repeatedly, we can obtain that $WL(t) \cdot WL(t-1) \cdot \dots \cdot WL(5)$ is $SOPC(3 \times 2^{t-3} + 3 \times 2^{t-4} + \dots + 3 \times 2^2 + 1, t, 2t-4)$, i.e., $SOPC(3 \times 2^{t-2} - 11, t, 2t-4)$, with depth $2t-3$. Since $W(4)$ is $SOPC(4, 2, 3)$ with depth 3, by Property 6, $WG(t) = WL(t) \cdot WL(t-1) \cdot \dots \cdot WL(5) \cdot W(4)$ is $SOPC(3 \times 2^{t-2} - 8, t, 2t-3)$ with depth $2t-3$. Q.E.D.

Assume that $n \geq 21$ and $t = \lceil \lg n \rceil$, we define n -input prefix circuit $WE4(n)$ as follows:

1. When $21 \leq n \leq 28$ or $63 \leq n \leq 64$,

$$WE4(n) = Q(n - 21 \times 2^{t-5} + 9) \cdot WV(t) \cdot WG(t-1).$$

2. When $29 \leq n \leq 32$;

$$37 \times 2^{t-5} - 9 \leq n \leq 5 \times 2^{t-2} - 8, t \geq 6; \text{ or}$$

$$n = 7 \times 2^{t-2} - 9, t \geq 5,$$

$$WE4(n) = Q(n - 3 \times 2^{t-2} + 8) \cdot S(2) \cdot WG(t).$$

3. When $48 \leq n \leq 62$;

$$5 \times 2^{t-3} - 7 \leq n \leq 7 \times 2^{t-3} - 10, t \geq 6;$$

$$7 \times 2^{t-3} - 8 \leq n \leq 2^t, t \geq 7; \text{ or}$$

$$2^t < n \leq 9 \times 2^{t-3} - 10, t \geq 7,$$

$$WE4(n) = Q(n - 3 \times 2^{t-3} + 9) \cdot WG(t - 1).$$

4. When $9 \times 2^{t-4} - 9 \leq n \leq 37 \times 2^{t-6} - 10, t \geq 8$,

$$WE4(n) = Q(n - 21 \times 2^{t-5} + 9) \cdot WV(t - 1) \cdot WG(t - 2).$$

As an example, Fig. 7 shows $WE4(27)$.

Fig. 7. $WE4(27) = Q(15) \cdot WV(5) \cdot W(4)$.

Theorem 10: $WE4(n)$ is an optimal prefix circuit with fan-out 4 and depth

$$\begin{aligned} &2 \lceil \lg n \rceil - 3 \quad \text{when } 29 \leq n \leq 32; \\ &2 \lceil \lg n \rceil - 4 \quad \text{when } 21 \leq n \leq 28, 47 \leq n \leq 64, \text{ or} \\ &\quad 7 \times 2^{t-3} - 9 \leq n \leq 2^t, \text{ where } t \geq 7; \\ &2 \lceil \lg n \rceil - 5 \quad \text{when } 33 \leq n \leq 46 \text{ or } 37 \times 2^{t-6} - 9 \\ &\quad \leq n \leq 7 \times 2^{t-3} - 10, \text{ where } t \geq 7; \\ &2 \lceil \lg n \rceil - 6 \quad \text{when } 2^{t-1} < n \leq 37 \times 2^{t-6} - 10, \\ &\quad \text{where } t \geq 8. \end{aligned}$$

Proof: The proof is through distinguishing 10 cases. Because the processes for the cases each are very similar, we will give details for the first case only and give tips for the others.

Case 1: When $21 \leq n \leq 28$, let $m = n - 12$; thus $9 \leq m \leq 16$ and $\lceil \lg m \rceil = 4$. By Property 1, we have $4 \leq d(Q(m)) \leq 6$. By Property 3, $Q(m)$ is $SOPC(m, 0, 4)$. As already mentioned, $WV(5)$ is $SOPC(10, 4, 5)$ with depth 6. Thus, by Property 6, $Q(m) \cdot WV(5)$ is $SOPC(m + 9, 0, 5)$. Since $W(4)$ is $SOPC(4, 2, 3)$, by Property 6, $WE4(n) = Q(m) \cdot WV(5) \cdot W(4)$ is $SOPC(m + 12, 0, 6)$ with depth $6 = 2 \lceil \lg n \rceil - 4$. Finally, by Property 5, $WE4(n)$ is optimal.

Case 2: When $29 \leq n \leq 32$, let $m = n - 16$.

Case 3: When $48 \leq n \leq 62$, let $m = n - 15$.

Case 4: When $63 \leq n \leq 64$, let $m = n - 33$.

Case 5: When $37 \times 2^{t-6} - 9 \leq n \leq 5 \times 2^{t-3} - 8$, where $t \geq 7$, let $m = n - 3 \times 2^{t-3} + 8$.

Case 6: When $5 \times 2^{t-3} - 7 \leq n \leq 7 \times 2^{t-3} - 10$, where $t \geq 6$, let $m = n - 3 \times 2^{t-3} + 9$.

Case 7: When $n = 7 \times 2^{t-3} - 9$, where $t \geq 6$, let $m = n - 3 \times 2^{t-3} + 8$.

Case 8: When $7 \times 2^{t-3} - 8 \leq n \leq 2^t$, where $t \geq 7$, let $m = n - 3 \times 2^{t-3} + 9$.

Case 9: When $2^{t-1} < n \leq 9 \times 2^{t-4} - 10$, where $t \geq 8$, let $m = n - 3 \times 2^{t-4} + 9$.

Case 10: When $9 \times 2^{t-4} - 9 \leq n \leq 37 \times 2^{t-6} - 10$, where $t \geq 8$, let $m = n - 21 \times 2^{t-6} + 9$.

By Case 2, when $29 \leq n \leq 32$, we have $d(WE4) = 2 \lceil \lg n \rceil - 3$. By Cases 1, 3, 4, 7, and 8, when $21 \leq n \leq 28$, $47 \leq n \leq 64$, or $7 \times 2^{t-3} - 9 \leq n \leq 2^t$, where $t \geq 7$, we have $d(WE4) = 2 \lceil \lg n \rceil - 4$. By Case 6, when $t = 6$, $33 \leq n \leq 46$, we have $d(WE4) = 2 \lceil \lg n \rceil - 5$. By Cases 5 and 6, when $37 \times 2^{t-6} - 9 \leq n \leq 7 \times 2^{t-3} - 10$, where $t \geq 7$, we have $d(WE4) = 2 \lceil \lg n \rceil - 5$. Finally, by Cases 9 and 10, when $2^{t-1} < n \leq 37 \times 2^{t-6} - 10$, where $t \geq 8$, we have $d(WE4) = 2 \lceil \lg n \rceil - 6$.

As already mentioned, the fan-out of Q is at most 4 and the fan-out of $WV(5)$ is 4. By Theorem 8, the fan-out of $WV(t)$ is 3, $t \geq 6$. By Property 4 and Theorem 7, the fan-out of $W(4)$ and WL is 4. Hence, $WE4(n)$ has a fan-out of 4. Q.E.D.

6. Comparisons of optimal prefix circuits

For easy and exact comparisons, Table 1 gives the numbers of inputs that can be processed by representative optimal parallel prefix circuits with specific depths. From Table 1, for $n \geq 21$, $d(WE4) \leq d(H4)$; that is, $WE4$ is better than $H4$.

Also from Table 1, when $n = 29$, $65 \leq n \leq 67$, $139 \leq n \leq 145$, $287 \leq n \leq 303$, $583 \leq n \leq 621$, $1175 \leq n \leq 1259$, $2359 \leq n \leq 2537$, or $4727 \leq n \leq 5095$, we get $d(Z4) = d(WE4) - 1$; however, when $45 \leq n \leq 46$, $99 \leq n \leq 102$, $209 \leq n \leq 214$, $431 \leq n \leq 438$, $877 \leq n \leq 886$, $1771 \leq n \leq 1782$, or $3561 \leq n \leq 3574$, we have $d(WE4) = d(Z4) - 1$; otherwise, $d(WE4) = d(Z4)$.

We now compare $WE4$ with optimal prefix circuits with unbounded fan-out. It can be checked that $d(WE4) \leq d(M)$ except when $n = 29$. As for LYD , when $21 \leq n \leq 77$, $d(LYD) \leq d(WE4)$; when $78 \leq n \leq 446$, $d(WE4) \leq d(LYD)$; when $n \geq 447$, $d(WE4) < d(LYD)$. That is, although both M and LYD have unbounded fan-out, only when n is small, $d(M)$ and $d(LYD)$ have some chances to be smaller than $d(WE4)$ by 1. Specifically, when $n > 29$, M can be replaced by $WE4$; when $n > 77$, LYD can be replaced by $WE4$.

Table 1. The numbers of inputs that some optimal parallel prefix circuits with specific depths can process.

Depth	<i>WE4</i>	<i>Z4</i>	<i>H4</i>
6	21–28	19–29	19–26
7	29–46	30–44	27–41
8	47–64	45–67	42–57
9	65–102	68–98	58–87
10	103–138	99–145	88–119
11	139–214	146–208	120–179
12	215–286	209–303	180–243
13	287–438	304–430	244–363
14	439–582	431–621	364–491
15	583–886	622–876	492–731
16	887–1174	877–1259	732–987
17	1175–1782	1260–1770	988–1467
18	1783–2358	1771–2537	1468–1979
19	2359–3574	2538–3560	1980–2939
20	3575–4726	3561–5095	2940–3963

7. Conclusion

In this paper, we have presented Algorithm A and Algorithm B for constructing size optimal prefix circuits *WL* and *WV*; together with *Q* and *W*, these circuits can be used to compose depth-size optimal parallel prefix circuit *WE4*. With our new approach, tedious and lengthy proofs are not required.

WE4 can replace all the other optimal parallel prefix circuits with fan-out 4 except for *Z4*. We have seen that $d(WE4)$ may be greater than $d(Z4)$ by 1, $d(WE4)$ may be less than $d(Z4)$ by 1, and in most cases of n , $d(WE4) = d(Z4)$.

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