Nonlinear Optimal Sliding Mode Midcourse Controller with Thrust Vector Control

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Abstract

In this paper, a nonlinear midcourse missile controller with Thrust Vector Control (TVC) inputs for the interception of a theater ballistic missile is presented. First, an optimal midcourse guidance law is designed to minimize the control effort and the distance between the missile and the target. Then, a quaternion-based sliding-mode attitude controller is proposed to track the attitude command and to cope with the effects from variations of missile's inertia, aerodynamic force, and wind gusts. The exponential stability of the overall system is analyzed thoroughly via Lyapunov stability theory. Extensive simulations are conducted to validate the effectiveness of the proposed guidance law and the associated TVC.

Keywords: Sliding Mode, Optimal Control, Target Tracking, Nonlinear System, Quaternion

1. Introduction

Based on the concept of the PN guidance law, the constant bearing guidance is often employed on the Bank-to-Turn (BTT) missiles [1], whereas a different kind of guidance law, namely, zero-sliding guidance law aims at eliminating the sliding velocity between the missile and the target in the direction normal to LOS [2]. Ha and Chong derived a new command to line-of-sight (CLOS) guidance law for short-range surface to air missile via feedback linearization [3] and its modified version [4] with improved performance.

In order to utilize the prior information on the future target maneuvers or on the autopilot lags, a new optimal guidance law without estimation of the interception time is proposed to deal with situation where the accurate time-to-go is unavailable [8]. Attitude control is another important issue to be addressed for successful missile operation. It is quite often that quaternion representation has been adopted to describe the attitude of a spacecraft [9]. To account for the non-ideal factors of the spacecraft under attitude control and to strengthen the robust property of the controller, the sliding mode control has been employed by Chen and Lo [11], which is then followed by a smooth version [12] incorporating a sliding layer as has been proposed by [6] to avoid the

chattering phenomenon, but at the price of slightly degrading the accuracy of the tracking system.

A missile equipped with TVC can effectively control its acceleration direction [2,14,15] when the missile built with fins fails, which in turn implies that the maneuverability and controllability of the missile can be greatly enhanced during the stage when the speed of the missile is slow and/or the air density around the missile is low. Thus, the midcourse guidance employs the TVC is common, but subject to a limitation that the control force is then constrained by the TVC mechanical structure, which further complicates the controller design.

2. Equations of Motion

The motion of a missile can be described in two parts as follows:

Translation:

$$\dot{\mathbf{r}}_M = \mathbf{a}_M + \mathbf{g}_M, \quad \dot{\mathbf{r}}_M = \mathbf{v}_M \quad (1)$$

Rotating:

$$\dot{\mathbf{J}}\boldsymbol{\omega} = -\mathbf{J}\boldsymbol{\omega} - \boldsymbol{\omega} \times (\mathbf{J}\boldsymbol{\omega}) + \bar{\mathbf{T}}_b + \mathbf{d} \quad (2)$$

where all the variables are defined in Nomenclature.

After referring to Fig. 1, the force and torque exerted on the missile can be expressed in the body coordinate frame, respectively, as

$$\bar{\mathbf{F}}_b = N [\cos d_p \cos d_y, \cos d_p \sin d_y, \sin d_p]^T \quad (3)$$

and

$$\begin{aligned} \bar{\mathbf{T}}_b &= L_s \times \bar{\mathbf{F}}_b + \mathbf{M}_s \\ &= lN [M_{xx}/lN \sin d_p, -\cos d_p \sin d_y]^T \end{aligned} \quad (4)$$

where $\mathbf{M}_s = [M_{xx} \ 0 \ 0]^T$ is an external moment command in the axial direction of the missile.

Fig.1. Two Angles of TVC in Body Coordinate

3. Guidance Law Design

Fig. 2. Relative Acceleration Along the LOS

A modified optimal guidance law without estimation of time-to-go is derived in the following based on the component of the relative velocity normal to the LOS, i.e., $v_p = v - (v^T \hat{r}) \hat{r}$. To proceed, we first derive the equation of the relative motion perpendicular to the LOS as follows: (referring to Fig. 2)

$$\dot{v}_p(t) = -a_p - \frac{1}{|r|} |v_p|^2 \hat{r} - \frac{v^T \hat{r}}{|r|} v_p, \quad (5)$$

where $a_p = a_M - (a_M^T \hat{r}) \hat{r}$ denotes acceleration of the missile perpendicular to the LOS. Referring to (5), the perpendicular acceleration component a_p is specified as

$$a_p = -u - \frac{v^T \hat{r}}{|r|} v_p, \quad (6)$$

where u is another exogenous design input calculated by minimizing the quadratic cost function defined as

$$J = \frac{1}{2} \gamma(T) v_p^T(T) v_p(T) + \frac{1}{2} \int_{t_0}^T (\sigma v_p^T(t) v_p(t) + \rho u^T(t) u(t)) dt \quad (7)$$

where $\gamma(T) \geq 0$, $\sigma > 0$, $\rho > 0$, and $[t_0, T]$ is the time interval over which we are interested in the behavior of the missile interception. By optimal control theory [5], the acceleration component perpendicular to the LOS in (6) is

$$a_p(t) = \left[\sqrt{\frac{\sigma}{\rho}} \left(1 + \frac{2}{e^{\frac{\gamma}{\rho}(T-t)}} \right) - \frac{v^T \hat{r}}{|r|} \right] v_p(t) \quad (8)$$

so that the equation of the relative motion in (5) becomes

$$\dot{v}_p = -\sqrt{\frac{\sigma}{\rho}} \left(1 + \frac{2}{e^{\frac{\gamma}{\rho}(T-t)}} \right) v_p - \frac{1}{|r|} |v_p|^2 \hat{r} \quad (9)$$

which is well defined except at $r=0$. The resulting dynamics in above will testify that the presented guidance law will ensure target interception under some appropriate condition. This is stated in Lemma 1 as follows.

Lemma 1: Let the equation of the relative motion perpendicular to the LOS and the modified

optimal guidance law be given by (5) and (8), respectively. If v has no component on the normal direction of the LOS, and $v^T r \leq \beta < 0$ with v being bounded away from zero, then the ideal midcourse guidance system will ensure target reaching.

Proof: See Appendix A.

3. Autopilot System Design

The dynamic model of a missile, treated as a rigid body, can be derived as

$$\dot{\bar{q}}_e = \frac{1}{2} (\bar{q}_e \times) \omega_e + \frac{1}{2} q_{e4} \omega_e$$

$$\dot{q}_{e4} = -\frac{1}{2} \omega_e^T \bar{q}_e$$

$$J \dot{\omega} = -J \dot{\omega} - \omega \times (J \omega) + \bar{T}_b + d \quad (10)$$

where $\omega_e = \omega - \omega_d$ is the error between angular velocities at the present attitude and the desired attitude, respectively.

Step 1: Let us choose the sliding manifold as

$$S_a = P \bar{q}_e + \omega_e \quad (11)$$

where $P = \text{diag}[p_1 \ p_2 \ p_3]$ is a positive definite diagonal matrix. If the sliding mode condition is kept, then the system dynamics are then constrained by the following differential equations

$$\dot{\bar{q}}_e = -\frac{1}{2} (\bar{q}_e \times) P \bar{q}_e - \frac{1}{2} q_{e4} P \bar{q}_e$$

$$\dot{q}_{e4} = \frac{1}{2} \bar{q}_e^T P \bar{q}_e \quad (12)$$

It has been shown that [2], the system origin $(\bar{q}_e, \omega_e) = (0_{3 \times 1}, 0_{3 \times 1})$ of the ideal system (12) is indeed exponentially stable.

Step 2: Assume that J is symmetric and positive definite, and let the candidate of a Lyapunov function be set as

$$V_s = \frac{1}{2} S_a^T J S_a, \quad (13)$$

where $S_a = [S_{a1} \ S_{a2} \ S_{a3}]^T$.

Let the control law be proposed as

$$\bar{T}_b = -J_0 P \left(\frac{1}{2} (\bar{q}_e \times) \omega_e + \frac{1}{2} q_{e4} \omega_e \right) + \omega \times (J_0 \omega) + J_0 \dot{\omega}_d + \tau \quad (14)$$

where $\tau = [\tau_1 \ \tau_2 \ \tau_3]^T$, $\tau_i = -k_i(q, \omega, q_d, \dot{q}_d, \ddot{q}_d) \text{sgn}(S_{ai})$, $\text{sgn}(\cdot)$ is a sign function, and properly chooses $k_i(\cdot)$, $i=1,2,3$, then the time derivative of V_s can be evaluated as:

$$\dot{V}_s = S_a^T [\dot{\delta} + \tau] \leq -\sum_{i=1}^3 |S_{ai}| [k_i - \delta_i^{\max}] < 0 \quad (15)$$

where

$$\delta = -j\omega - \omega \times (\Delta J \omega) + d - \Delta J \dot{\omega}_d + \frac{1}{2} J S_a + \Delta J P \left(\frac{1}{2} (\bar{q}_e \times) \omega_e + \frac{1}{2} q_{e4} \omega_e \right) \quad (16)$$

and $k_i(\cdot)$, $i=1,2,3$, are chosen sufficiently large as will be commented below. Let the external disturbances d and the magnitudes of j and ΔJ be all bounded, then the bounding function on $|\delta_i|$, which obviously is a function of q , ω , q_d , $\dot{q}_d$ and $\ddot{q}_d$, can be found and represented as $\delta_i^{\max}(q, \omega, q_d, \dot{q}_d, \ddot{q}_d) \geq |\delta_i|$. It is evident that if we choose $k_i(q, \omega, q_d, \dot{q}_d, \ddot{q}_d) > \delta_i^{\max}(q, \omega, q_d, \dot{q}_d, \ddot{q}_d)$ for $i=1,2,3$, the reaching and sliding conditions of the sliding mode $S_a = 0$ are guaranteed.

In other words, if the desired acceleration is $a_p + \bar{r}$, and here we assume that the flying direction will be in alignment with the axial direction of the missile, then the relationship between the actual acceleration a_M applied on the missile and the desired acceleration $a_p + \bar{r}$ is the following:

$$a_M = B_b(q) B^T(q_e) B_b^T(q) (a_p + \bar{r}) \quad (17)$$

Remark: Recall that $B_b(q)$ is the transformation from the current body coordinate to the inertial coordinate, and $B(q_e)$ is the transformation from the current body coordinate to the desired body coordinate. Since the actual acceleration exerted on the missile is a_M , the component of the actual acceleration perpendicular to the LOS is

$$a_{Mp} = \frac{\left\{ \left[B_b(q) B^T(q_e) B_b^T(q) (a_p + \bar{r}) \right]^T a_p \right\} a_p}{|a_p|^2},$$

where a_p , in (8), is the desired acceleration perpendicular to the LOS as has been mentioned before. As a result, we get a new state equation from (5) as follows:

$$\dot{v}_p = - \frac{\left\{ \left[B_b(q) B^T(q_e) B_b^T(q) (a_p + \bar{r}) \right]^T a_p \right\} a_p}{|a_p|^2} - \frac{1}{|r|} |v_p|^2 \hat{r} - \frac{v_p^T \hat{r}}{|r|} v_p \quad (18)$$

To evaluate the stability of the relative velocity component v_p , we define a Lyapunov function V_G as

$$V_G = \frac{1}{2} v_p^T v_p.$$

Then, the time derivative of the Lyapunov function can be derived as

$$\dot{V}_G = - \left[\frac{\sigma}{\rho} \left(1 + \frac{2}{e^{\frac{1}{\sqrt{\rho}} (T-t)} - 1} \right) - \frac{v_p^T \hat{r}}{|r|} \right] v_p^T v_p + \left[\frac{2 \bar{B}^T(q_e) F_b^T \bar{a}_p}{|a_p|^2} \right] - \frac{v_p^T \hat{r}}{|r|} v_p^T v_p \quad (19)$$

where a_p is given as in (8), $F_b = B_b^T(a_p + \bar{r})$, $B_b^T = B_b^{-1}$, $\bar{a}_p = B_b^T a_p$, $\bar{B}(q_e) = (\bar{q}_e \times) \langle \bar{q}_e, \times \rangle$. Note that we use the fact that $v_p^T \hat{r} = \bar{r}^T a_p = 0$, and $B(q_e) = I_{3 \times 3} + 2(\bar{q}_e \times) \langle \bar{q}_e, \times \rangle + 2q_{e4} \langle \bar{q}_e, \times \rangle$.

To verify the stability of the overall system, we define a Lyapunov function candidate of the overall system as the sum of the previously defined functions V_s and V_G , i.e.,

$$V = V_s + V_G, \quad (20)$$

The time derivative of the Lyapunov function can be derived as

$$\dot{V} = S_a^T [-j\omega - \omega \times (J\omega) + \bar{T}_b + d - J\dot{\omega}_d + J P \left(\frac{1}{2} (\bar{q}_e \times) \omega_e + \frac{1}{2} q_{e4} \omega_e \right) + \frac{1}{2} J S_a] - K_1(v_p, T, t) - K_2(v_p, v_r, T, t) \frac{2 \bar{B}^T(q_e) F_b^T \bar{a}_p}{|a_p|^2} \quad (21)$$

referring to Eqs. (15) and (19), where

$$K_1(v_p, T, t) = \sqrt{\frac{\sigma}{\rho}} \left(1 + \frac{2}{e^{\frac{1}{\sqrt{\rho}} (T-t)} - 1} \right) v_p^T v_p = \bar{K}_1(T, t) v_p^T v_p > \sqrt{\frac{\sigma}{\rho}} v_p^T v_p \geq 0$$

$$K_2(v_p, v_r, T, t) = \left[\sqrt{\frac{\sigma}{\rho}} \left(1 + \frac{2}{e^{\frac{1}{\sqrt{\rho}} (T-t)} - 1} \right) - \frac{v_p^T \hat{r}}{|r|} \right] v_p^T v_p = \bar{K}_2(v_p, v_r, T, t) v_p^T v_p > \sqrt{\frac{\sigma}{\rho}} v_p^T v_p \geq 0 \quad (22)$$

so that

$$\dot{V} = S_a^T [-j\omega - \omega \times (J\omega) + \bar{T}_b + d - J\dot{\omega}_d + J P \left(\frac{1}{2} (\bar{q}_e \times) \omega_e + \frac{1}{2} q_{e4} \omega_e \right) + \frac{1}{2} J S_a] + \frac{2 K_3 \langle \bar{q}_e, \times \rangle (\langle \bar{q}_e, \times \rangle - q_{e4} J_{33}) F_b^T}{|a_p|^2} - K_3 v_p^T v_p \quad (23)$$

where $K_3 = K_3(v_p, v_r, q_e, \omega_e, a_p, T, t)$ is defined as:

$$K_3 = \bar{K}_3 + \frac{2 \bar{K}_2 F_b^T \langle \bar{q}_e, \times \rangle + q_{e4} J_{33} \langle \bar{q}_e, \times \rangle}{|a_p|^2} \omega_e \text{ and the matrix}$$

P is chosen to be $P = p \times I_{3 \times 3}$.

Theorem 1: Let the modified optimal guidance law be proposed as in (8), (18) and the torque input of the autopilot is given as follows:

$$\bar{T}_b = -J_a p \left[\frac{1}{2} (\bar{q}_e \times) \omega_e + \frac{1}{2} q_{e4} \omega_e \right] + \omega \times (J_a \omega) + J_a \dot{\omega}_d + \tau' \quad (24)$$

where $r' = [\dot{r}_1 \ \dot{r}_2 \ \dot{r}_3]^T$, $\dot{r}_i = -\bar{k}_i(q, \omega, q_d, \dot{q}_d, \ddot{q}_d, v_p, v, r, T, t)$ $\text{sgn}(S_{\omega_i})$ for some appropriate functional gains $\bar{k}_i$, $i = 1, 2, 3$, and p is chosen to be large enough. If v is such that $v^T \dot{r}(t_0) < 0$, where t_0 is the starting time and v is bounded away from zero, then the integrated overall midcourse guidance and autopilot system will be stable and the target reaching property is achieved.

Proof: After substitution of the torque input (24) by hypothesis, the expression of $\dot{v}$ can be readily simplified as:

$$\dot{v} = S_{\omega}^T [\delta + \tau] - K_3 v_p^T v_p \quad (25)$$

where

$$\begin{aligned} \delta = & -j\omega - \omega \times (\Delta J \omega) + d - \Delta J \dot{\omega}_d + p \Delta J \left(\frac{1}{2} \langle \bar{q}_e \times \rangle \omega_e \right. \\ & \left. + \frac{1}{2} q_{e4} \omega_e \right) + \frac{1}{2} j S_{\omega} + \frac{2K_2}{p} \frac{\langle \bar{q}_p \times \rangle \langle \bar{q}_e \times \rangle - q_{e4} I_{3 \times 3} F_b}{|a_p|^2} \end{aligned} \quad (26)$$

Since the external disturbance d and the uncertainties j and ΔJ are all bounded by assumption, we conclude that the upper bounds $\bar{\delta}_i^{\max}$, $\bar{\delta}_i^{\max} \geq |\bar{\delta}_i|$, $i = 1, 2, 3$, are functions of q , ω , q_d , $\dot{q}_d$, $\ddot{q}_d$, v_p , v , r , T and t . It is evident that if the functional gain $\bar{k}_i(q, \omega, q_d, \dot{q}_d, \ddot{q}_d, v_p, v, r, T, t) > \max\{\bar{\delta}_i^{\max}, \bar{\delta}_i^{\max}\} + \eta$, $i = 1, 2, 3$, for $\eta_i > 0$, referring to Eqs. (16) and (26), can be chosen, then (15) apparently holds and the expression (25) can be further explored as:

$$\dot{v} = -\sum_{i=1}^3 S_{\omega_i} [\bar{k}_i - \bar{\delta}_i \text{sgn}(S_{\omega_i})] - K_3 v_p^T v_p \quad (27)$$

Working Lemma 2: Through the entire midcourse phase, if $v^T \dot{r} < 0$, with v being bounded away from zero, then we can always find appropriate gain p and the adjustable convergent time parameter $T > t$ such that

$$K_3 = \bar{K}_1 + \frac{2\bar{K}_2}{p} \frac{F_b^T \langle \bar{q}_e \times \rangle + q_{e4} I_{3 \times 3} \langle \bar{q}_p \times \rangle}{|a_p|^2} \omega_e \geq K_{30} > 0 \quad (28)$$

for all $t \geq 0$.

Proof: See Appendix A.

As a result, (28) can be expressed as:

$$\dot{v} \leq -\sum_{i=1}^3 \eta_i |S_{\omega_i}| - K_{30} v_p^T v_p$$

which means that $-\dot{v}$ is positive definite, and hence $S_{\omega} \rightarrow 0$, $v_p \rightarrow 0$ as $t \rightarrow \infty$ via use of Lyapunov stability theory.

Finally, to show that Lemma 1 is satisfied, i.e., target reaching, we need to show that $v^T \dot{r} \leq \beta < 0$ at all time. First, v_p has to be verified to be exponentially decaying. Although $\dot{v}$ and $\dot{v}_s$ can be shown negative definite via Theorem 1 via torque input (24), by definition $\dot{v} = \dot{v}_G + \dot{v}_s$ in (21) $\dot{v}_G$ can not be proved negative definite directly. To establish such a fact, we derive $\dot{v}_G$ explicitly from Eqs. (19), and (21)-(23), i.e.,

$$\dot{v}_G = -[\bar{K}_1 - \frac{2\bar{K}_2}{p} \frac{F_b^T \langle \bar{q}_e \times \rangle + q_{e4} I_{3 \times 3} \langle \bar{q}_p \times \rangle}{|a_p|^2} (S_{\omega} - \omega_e)] v_p^T v_p \quad (29)$$

By use of the proof Lemma 2, the maximum value $\bar{P}_{\max}$ in (29) can also be obtained as

$$\bar{P}_{\max} \geq \frac{\left| F_b^T \langle \bar{q}_e \times \rangle + q_{e4} I_{3 \times 3} \langle \bar{q}_p \times \rangle (S_{\omega} - \omega_e) \right|}{|a_p|^2} \quad (30)$$

where we already showed that $S_{\omega} \rightarrow 0$ and $\omega_e \rightarrow 0$ as $t \rightarrow \infty$ due to the autopilot system design in section 4 referring to (11) and (15). Thus, if the inequality (A4) in Lemma 2 can be modified as

$$P > 2 \max\{P_{\max}, \bar{P}_{\max}\} \cdot \bar{K}^{\max} \quad (31)$$

then $\dot{v}_G$, $\dot{v}$, and $\dot{v}_s$ are all negative definite, which means v_p will be attenuated exponentially at all time due to the definition of

$$v_G = \frac{1}{2} v_p^T v_p$$

which is always positive definite.

By defining the angle, θ , between v and v_r , referring to Fig. 5 as

$$\theta = \tan^{-1} \left(\frac{|v_r|}{|v_s|} \right),$$

thus we can conclude that $|v_r|$ will decay exponentially referring to (29)-(31), and $|v_r| = |v - v_p| \geq |v| - |v_p|$ due to $v = v_p + v_r$ referring to Fig. 3. Therefore, $|v_r|$ will be an increasing function with respect to time, which implies, in turn, the angle θ will be monotonically decreasing as time goes on. Hence, we can conclude that $v^T \dot{r} < 0$ for all $t \geq 0$ since $v^T \dot{r}(t_0) < 0$ which justifies our assumption in Lemma 1. Therefore, the target tracking objective can be achieved as claimed by the aforementioned theorem.

Fig. 3. Relative Velocity between the Target and the Missile

Q.E.D.

5. Simulations

We assume the target is launched from somewhere 600 km far away from the missile base. The missile has a sampling period of 10ms. The bandwidth of the TVC is 5Hz and the two angular displacements are both limited to 5° . Here, we

consider the variation of the moment of inertia of the missile. Thus, the inertia matrix including the nominal part J_0 and the uncertain part ΔJ used here is as follows:

$$J = J_0 + \Delta J (\text{kg} \cdot \text{m}^2)$$

where

$$J_0 = \begin{bmatrix} 1000 & 100 & 200 \\ 100 & 2000 & 200 \\ 200 & 200 & 2000 \end{bmatrix}, \quad \Delta J = \begin{bmatrix} 100 & 100 & 200 \\ 100 & 200 & 200 \\ 200 & 200 & 200 \end{bmatrix}$$

and the variation of the inertial matrix is as

$$\dot{J} = \begin{bmatrix} -1 & -1 & -2 \\ -1 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}$$

The initial conditions are set as $q = [0 \ 0.707 \ 0 \ 0.707]^T$, $\omega(0) = [0 \ 0 \ 0]^T$, and the variation of missile's mass is as $\dot{m} = -4$ (kg/sec) for the initial mass $m = 600$ (kg). Further, we also consider the aerodynamic force and wind gusts exerted on the missile by $d_i(t) = \sin(t) + 10(u(t-20) - u(t-21))$ (Nt-m) for $i = 1, 2, 3$, where $u(t)$ is the step function. Our objective here is to drive the missile close to the target within the range of 9 km. The simulation time is 93.13 sec. The feasibility of the presented approach is satisfactorily demonstrated by the simulation results in Fig. 4.

Fig. 4. Simulation Results

6. Conclusions

In this paper, we focus on the midcourse phase of the interception, which is a period of time lasting until the missile is close enough to the target such that the sensor located on the missile can lock on the target.

We prove the stability of the individual guidance system, the autopilot system, and the overall system, respectively, via Lyapunov stability theory. Computer simulations have been conducted to verify the feasibility of the integrated midcourse guidance and control system with TVC.

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Appendix A

Proof of Lemma 1: Taking

$$V_G = \frac{1}{2} v_p^T r v_p$$

as a Lyapunov function candidate, it can be easily seen that

$$\frac{1}{4} v_p^T r v_p \leq V_G \leq v_p^T r v_p$$

Hence, V_G is positive definite, decreasent, and radially unbounded. The time derivative of V_G along the trajectories of the system is given by

$$\dot{V}_G = -v_p^T r \left[\sqrt{\frac{\sigma}{\rho}} \left(1 + \frac{2}{e^{\frac{2}{\sqrt{\sigma}(\tau-t)} - 1}} \right) v_p - \frac{1}{|r|} |v_p|^2 v_p^T \hat{r} \right] \leq -\sqrt{\frac{\sigma}{\rho}} v_p^T r v_p$$

where $\sqrt{\hat{r}}=0$ and $\sqrt{\frac{\sigma}{\rho}}$ is a positive constant.

Thus, $\dot{V}_G$ is apparently negative definite, and hence via Lyapunov stability theory, we can conclude that the origin of v_p is globally exponentially stable.

We take $V_r = \frac{1}{2} r^T r$ as another Lyapunov function

candidate. Thus, it can be easily seen that V_r is positive definite, decreasent, and radially unbounded, then the time derivative is as follows:

$$\dot{V}_r = v_r^T r = (v - v_p)^T r = v^T r < 0$$

where $v_r = v - v_p$, so that $\dot{V}_r$ is negative definite. Therefore, via the Lyapunov stability theory and constant bearing condition [2], the ideal midcourse guidance will render the origin of the missile interceptions system globally exponentially stable. ■

Proof of Lemma 2: Since $\frac{a_p}{|a_p|}$, q_e , and ω_e are bounded,

and $\omega_e \rightarrow 0$ as $t \rightarrow \infty$, we have both $\frac{F_b^T}{|a_p|} = \frac{[B_b^T(a_p + \bar{r})]^T}{|a_p|}$

and $\frac{\langle \bar{a}_p \times \rangle}{|a_p|} = \frac{\langle (B_b^T a_p) \times \rangle}{|a_p|}$ are bounded, so that the value of

$$\frac{F_b^T (\langle \bar{q}_e \times \rangle + q_{e4} I_{3 \times 3}) \langle \bar{a}_p \times \rangle \omega_e}{|a_p|^2}$$

can be concluded to be bounded and converge to zero as $t \rightarrow \infty$. Therefore, the maximum value

$p_{\max}$ can be obtained as

$$p_{\max} \geq \left| \frac{F_b^T (\langle \bar{q}_e \times \rangle + q_{e4} I_{3 \times 3}) \langle \bar{a}_p \times \rangle \omega_e}{|a_p|^2} \right| \quad (A1)$$

Moreover, $\bar{K}_1$ and $\bar{K}_2$ are both positive, and

$$K_3 = \bar{K}_1 \left(1 + \frac{1}{p} \frac{2\bar{K}_2}{\bar{K}_1} \frac{F_b^T (\langle \bar{q}_e \times \rangle + q_{e4} I_{3 \times 3}) \langle \bar{a}_p \times \rangle \omega_e}{|a_p|^2} \right) \text{ due to the fact of}$$

(22). Hence, the ratio of $\bar{K}_2$ to $\bar{K}_1$ can be expressed as

$$\bar{K} = \frac{\bar{K}_2}{\bar{K}_1} = \frac{\sqrt{\frac{\sigma}{\rho}} \left(1 + \frac{2}{e^{\frac{2}{\sqrt{\sigma}(\tau-t)} - 1}} \right) - \frac{v^T \hat{r}}{|r|}}{\sqrt{\frac{\sigma}{\rho}} \left(1 + \frac{2}{e^{\frac{2}{\sqrt{\sigma}(\tau-t)} - 1}} \right)} \quad (A2)$$

where the assumptions that $v^T \hat{r} < 0$ and

$$-\frac{v^T \hat{r}}{|r|} = -\frac{1}{|r|} \frac{d}{dt} |r| = -\frac{d}{dt} \ln |r| \text{ is bounded for } |r| \neq 0 \text{ are}$$

satisfied. Therefore, the maximum $\bar{K}^{\max}$ of $\bar{K}$ in (A2) can be found and is denoted as $\bar{K} \leq \bar{K}^{\max}$. Then, (28) can be re-expressed as

$$\bar{K}_3 = \bar{K}_1 + \frac{2\bar{K}_2}{p} \frac{F_b^T (\langle \bar{q}_e \times \rangle + q_{e4} I_{3 \times 3}) \langle \bar{a}_p \times \rangle \omega_e}{|a_p|^2} = \bar{K}_1 \left(1 + \frac{2\bar{K}}{p} \frac{F_b^T (\langle \bar{q}_e \times \rangle + q_{e4} I_{3 \times 3}) \langle \bar{a}_p \times \rangle \omega_e}{|a_p|^2} \right) \geq \bar{K}_1 \left(1 - \frac{2\bar{K}^{\max}}{p} p_{\max} \right) \quad (A3)$$

Now, if we let

$$p > 2 p_{\max} \cdot \bar{K}^{\max}, \quad (A4)$$

which together with that fact that $\bar{K}_1$ is lower bounded by a positive constant immediately implies the inequality conclusion (28). ■

Nomenclature

a Acceleration vector	$ r $ The magnitude of r
d Disturbances vector	t The present time
d_p Pitch angle of propellant	τ The intercepting time
d_y Yaw angle of propellant	$t_g = \tau - t$ Time-to-go until intercept
$\bar{F}$ Thrust vector	T Adjustable time parameter
g Gravitational acceleration vector	$\bar{T}$ Torque
J Moment of inertial matrix	v Velocity vector
J_0 Nominal parts of J	ω Angular velocity vector
ΔJ Variation of J	Subscripts
ℓ Distance between nozzle and gravity center	b The body coordinate frame
$L_b = [-\ell \ 0 \ 0]^T$ Displacement vector	d Desired
m Mass of the missile	e Error
N Magnitude of thrust	i The inertial coordinate frame
q Quaternion	M Missile
r Position vector from missile to target	p Perpendicular to LOS
$\hat{r}$ The unit vector of r	T Target