Crisp-type fuzzy logic controller using Dubois and Prade's parametric t -norm-sum-gravity inference methods

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Abstract: The input/output relationship of a class of crisp-type fuzzy logic controllers (FLCs) using parametric t -norm-sum-gravity inference method is examined. The explicit mathematical form of reasoning surface using Dubois and Prade's parametric t -norm is addressed. Reasoning surfaces of crisp-type FLCs are proved to be composed of a two-dimensional multilevel relay and a local position-dependent nonlinear compensator. The influence of the α value, the parameter of Dubois and Prade's parametric t -norm, on the reasoning surfaces is also investigated.

1 Introduction

Much emphasis has been placed on the field of fuzzy logic controllers (FLC) since the introduction of fuzzy sets [1]. Owing to the intrinsic parallels between reasoning procedures of FLCs and those of the human operators, FLCs stand out as a viable alternative to those processes that are difficult to control.

Since the advance of the first FLC proposed by Mamdani [2], there have been numerous applications of FLCs suggested in almost every field from water quality control to automobile transmission control, and from fuzzy memory devices to fuzzy computers (see [3]). Most of the applications are, however, based on intuitive implementation of domain experts' experiences rather than systematic derivation of the operators used in FLCs. This approach complies with the very concept 'fuzziness' of fuzzy logic, but some important features of FLCs are thus concealed and cannot be gleaned by the designers. Furthermore, theoretical analysis of the basic properties of fuzzy control systems are usually considered as impossible or impractical. This prevailing notion makes the theoretical analysis of FLCs appear to be trivial work for designing FLCs and is not reversed until recently.

A thorough review of the design of FLCs and a solid collection of mathematical analyses for FLCs from a theoretical perspective are given in [3–6]. In [4, 7–9], significant efforts are made in performing the stability analysis and the design principles of fuzzy control systems. Meanwhile, some appealing FLC analyses are also reported in turn, such as [10–15]. These mathematical

analyses clearly describe input/output relations of FLCs under some specific restrictions of membership functions, rule base and inference methods. However, the significance of these investigations is downgraded by some impractical premises proposed in them. For example, all the membership functions adopted in [10, 13–15] are equally spaced triangles. However, it has been shown that the control ability of an FLC equipped with uniformly distributed triangular membership functions is inadequate for complex systems [10, 11]. On the other hand, in [11, 12], the restriction of membership functions has been chosen to include unequally spaced triangles. However, the inference operator is based only on the so-called product-sum-gravity method, and the analyses have been focused on the similarity and difference between the FLC and the linear PI controller. Recently, Chen *et al.* [4, 16] have extended the FLC analyses to adopt arbitrarily distributed triangular fuzzy partitions and to apply four frequently used t -norms for inference.

This article aims to generalise the theoretical analyses given in previous related researches [4, 10, 12–16]. Therein, the crisp-type output membership functions and the triangular-type input fuzzy partitions can be nonuniformly spaced, and the Dubois and Prade's parametric t -norm is applied, as an illustration, in evaluating true values of the control rules. The parametric t -norms, although not widely implemented in practical control procedures, possess the merits that they can be turned into nearly all types of nonparametric t -norms used in applications such as the standard fuzzy intersection (the minimum) and/or the algebraic product by only modifying their parameters. This feature is advantageous to both the FLC designer and field operators, because they can alter the behaviour of the FLC without first changing its operators. The explicit input/output relations for the FLCs using the Dubois and Prade's parametric t -norm-sum-gravity inference methods will be derived in this paper. The influence of applying various α values in the Dubois and Prade's parametric t -norm for inference on output patterns will be studied in this paper. It is expected that these analyses can provide some insight for effective FLC design and tuning.

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IEE Proceedings online no. 20000180

DOI: 10.1049/ip-cta:20000180

Paper first received 31st March and in revised form 25th November 1999

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2 Architecture of the fuzzy logic controllers

As the architecture of the FLC used in the following derivation is the same as that of [16], only an essential description is included here.

For an FLC with two inputs (error e and change in error r) and one output (change in controller output u), the associative universes of discourse for e , r , u are all set to be $[-1, 1]$ for convenience, and therefore e_m and r_m , the sensed values, are converted to e^* and r^* , the actual input values of the FLC, by scaling factors GE and GR. Fig. 1 shows the structure of an FLC and its detailed computational steps [16].

Three term sets, $T_e = \{E_i | i \in I_{m_e}\}$, $T_r = \{R_j | j \in I_{m_r}\}$ and $T_u = \{U_k | k \in I_{m_u}\}$, consisting of sets of linguistic terms, E_i , R_j and U_k , individually, are used to denote the possible linguistic values for e , r and u ; $I_{m_e} = \{-m_*, \dots, -1, 0, 1, \dots, m_*\}$ is the index set.

The input membership functions, $E_i(e)$ and $R_j(r)$, are triangular fuzzy partitions, i.e. $\sum_{\forall i} E_i(e) = 1$, $\forall e \in U_e$ and $\sum_{\forall j} R_j(r) = 1$, $\forall r \in U_r$. For $E_i(e)$, the explicit functions are defined as

$$E_i(e) = \Lambda(e; E_{i-1}^*, E_i^*, E_{i+1}^*) = \begin{cases} \frac{e - E_{i-1}^*}{E_i^* - E_{i-1}^*} & \text{for } E_{i-1}^* \leq e \leq E_i^* \\ \frac{E_{i+1}^* - e}{E_{i+1}^* - E_i^*} & \text{for } E_i^* \leq e \leq E_{i+1}^* \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i \in I_{m_e}$$

where E_i^* are principal values with the property that $-1 = E_{-m_e}^* < \dots < E_{-1}^* < E_0^* = 0 < E_1^* < \dots < E_{m_e}^* = 1$. R_j^* and U_k^* are defined in similar ways.

As for the output membership functions $U_k(u)$, the crisp-type membership functions (singleton) are considered. Let $-1 = U_{-m_u}^* < \dots < U_{-1}^* < U_0^* = 0 < U_1^* < \dots < U_{m_u}^* = 1$, then

$$U_k(u) = \begin{cases} 1 & \text{for } u = U_k^* \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

The assignment of principal values can be performed arbitrarily as long as the given conditions are observed. This type of membership functions (for either triangular fuzzy partitions or fuzzy singletons) is denoted as unequal-span membership functions (USMFs). In addition to the USMFs, two types of systematically allocated membership functions are also used in this research to derive some important properties. The first type is the shrinking-span

membership function (SSMF) [17], in which the principal values are defined as

$$E_i^* = \frac{i}{m_e} s_e^{m_e - |i|}, \quad R_j^* = \frac{j}{m_r} s_r^{m_r - |j|}, \quad U_k^* = \frac{k}{m_u} s_u^{m_u - |k|}$$

where $s_e, s_r, s_u \in (0, 1]$ are the shrinking factors for e, r and u , respectively. If the shrinking factors are all set to unity, then equal-span membership functions (ESMFs) are obtained and the distance between any two neighbouring principal values of the ESMFs is equal to the reciprocal of the respective m . The notations USMFs-, SSMFs-, ESMFs-FLC are used to indicate the types of membership functions used in the FLC.

The control rule in the rule base has the form

$$R_{i,j}: \text{IF } e \text{ is } E_i \text{ AND } r \text{ is } R_j \text{ THEN } u \text{ is } U_{k=f(i,j)} \quad \forall i \in I_{m_e}, j \in I_{m_r} \quad (3)$$

For simple rule base, $f(i, j) = i + j$.

In the decision-making step of FLC, under the assumption that $m_e = m_r = m$, given input pattern e^* and r^* , a series of fuzzy operators, $\mathbb{F}$, $\mathbb{I}$, $\mathbb{A}$ and $\mathbb{D}$ are used to obtain the final output action [17]:

$$u^* = \mathbb{D} \left(\mathbb{A}_{\forall i,j \in I_m} \{ \mathbb{I}[\mathbb{F}(E_i(e^*), R_j(r^*)), U_{f(i,j)}(u)] \} \right)$$

The $\mathbb{F}$ calculates the firing level, $\phi_{i,j} = \mathbb{F}(E_i(e^*), R_j(r^*))$, by using t -norm operator for each rule in the rule base; $\mathbb{I}$ and $\mathbb{A}$ perform the implication and aggregation of the FLC, respectively; and $\mathbb{D}$ converts the aggregated output fuzzy set $U(u)$ into a single crisp output value u^* . In this article, the authors have chosen the summation for $\mathbb{A}$, the centre of area (COA) method for $\mathbb{D}$, and the algebraic product for $\mathbb{I}$. However, as the output membership functions are fuzzy singletons, the choice of $\mathbb{I}$ is irrelevant to the final inference result. As for the $\mathbb{F}$ operator, the Dubois and Prade's parametric t -norm described in Section 3 is employed to find the firing levels. This type of fuzzy controllers is named the crisp-type FLCs using parametric t -norm-sum-gravity inference methods by the authors. The final crisp output u_m for such crisp-type FLCs is [12, 13]:

$$u_m = GU \times u^* = GU \times \frac{\sum_{\phi_{i,j} \neq 0} \phi_{i,j} U_{f(i,j)}^*}{\sum_{\phi_{i,j} \neq 0} \phi_{i,j}} \quad (4)$$

According to the properties of triangular fuzzy partitions used in the USMFs-FLCs, at most four control rules are fired in the i 'th block, i.e. for input pattern $e^* \in [E_i^*, E_{i+1}^*]$ and $r^* \in [R_j^*, R_{j+1}^*]$ ($i, j \in I_m - \{m\}$), there are at most four

Fig. 1 Detailed computational steps of fuzzy logic controller

rules having nonzero firing levels [4, 13, 16]. These four rules are as follows:

- (i) if e is E_{i+1} and r is R_{j+1} then u is $U_{f(i+1,j+1)}$;
- (ii) if e is E_{i+1} and r is R_j then u is $U_{f(i+1,j)}$;
- (iii) if e is E_i and r is R_{j+1} then u is $U_{f(i,j+1)}$;
- (iv) if e is E_i and r is R_j then u is $U_{f(i,j)}$.

Hence, it is straightforward, from eqn. 4, that u^* is bounded by the maximal and minimum of the four centroids, and, for simple rule base, it is

$$U_{i+j}^* \leq u^* \leq U_{i+j+2}^* \quad (5)$$

The details of the input/output relations of these FLCs will be described in the following Section.

3 Mathematical input-output relation of the FLC

According to eqn. 4, the final control output cannot be obtained without first acquiring the firing levels. Generally speaking, the t -norms used to find the firing levels in practical applications are, for their simplicity, either fuzzy intersection (min) or algebraic product. However, the parametric t -norms, although not widely implemented in practical control procedures, possess the merits that they can be turned into nearly all types of nonparametric t -norms used in industries by only adjusting their parameters.

Many classes of parametric t -norms have been proposed to perform fuzzy intersections of various strengths, such as Hamacher, Frank, Dubois and Prade, Yager, and Weber, to name just a few (see [18] for a list of main families of parametric t -norms with detailed discussions). Without loss of generality, the authors have chosen the Dubois and Prade's parametric t -norm for subsequent analysis because of its simplicity (it has simple functional form with bounded parameter range) and flexibility (it covers the whole useful range of t -norms with algebraic product and standard intersection as its lower bound and upper bound, respectively). Notably, it is possible to conduct similar examination by using other simple parametric t -norms such as Weber or Hamacher families.

This Section provides a clear derivation and some important properties for the USMFs-FLCs using the Dubois and Prade's parametric t -norm-sum-gravity inference methods.

Because some common features are shared by all ij th blocks, it is more convenient to defined a normalised ij th block, a unit square, by setting the normalised input values $\varepsilon = (e^* - E_i^*) / (E_{i+1}^* - E_i^*)$ and $\mathcal{R} = (r^* - R_j^*) / (R_{j+1}^* - R_j^*)$ than using e^* , r^* alone. And hence, for input pattern $(e^*, r^*) \in ij$ th block, the four nonzero membership values are $E_{i+1}(e^*) = \varepsilon$, $E_i(e^*) = 1 - \varepsilon$, $R_{j+1}(r^*) = \mathcal{R}$, and $R_j(r^*) = 1 - \mathcal{R}$.

In order to find the firing levels listed in eqn. 4, a t -norm operator must be choose to perform this task. Here, for reasons stated in the beginning of this Section, the authors use the Dubois and Prade's parametric t -norm defined below as the fuzzy intersection operator (see [18]):

$$\begin{aligned} \mathbb{T}_{D\&P}(a, b, \alpha) &= \frac{a \cdot b}{\max(a, b, \alpha)} \quad \alpha \in [0, 1] \\ &= \begin{cases} \mathbb{T}_{\min}(a, b) = \min(a, b) & \text{for } \alpha = 0 \text{ (standard intersection)} \\ \mathbb{T}_{ap}(a, b) = a \cdot b & \text{for } \alpha = 1 \text{ (algebraic product)} \end{cases} \end{aligned}$$

Notably, $\mathbb{T}_{ap}(a, b) \leq \mathbb{T}_{D\&P}(a, b) \leq \mathbb{T}_{\min}(a, b)$. The firing levels of the four rules listed in the preceding Section are, therefore:

$$\begin{aligned} \phi_{i+1,j+1} &= \mathbb{T}_{D\&P}(\varepsilon, \mathcal{R}, \alpha) & \phi_{i+1,j} &= \mathbb{T}_{D\&P}(\varepsilon, 1 - \mathcal{R}, \alpha) \\ \phi_{i,j+1} &= \mathbb{T}_{D\&P}(1 - \varepsilon, \mathcal{R}, \alpha) & \phi_{i,j} &= \mathbb{T}_{D\&P}(1 - \varepsilon, 1 - \mathcal{R}, \alpha) \end{aligned}$$

Owing to the max operator shown in the equation for $\mathbb{T}_{D\&P}$, the relative magnitudes of ε , $\mathcal{R}$ and α determine the inference results and, therefore, some partitions are made in the three-dimensional space constituted by ε , $\mathcal{R}$ and α to emphasise the relationship among them.

For variables $(\varepsilon, \mathcal{R})$, it has been shown that [4, 16] there are four different combinations of the magnitudes of $(\varepsilon, \mathcal{R})$, and these four regions are designated as regions 1~4 as shown in Fig. 2. The same partitions can be applied to $(\mathcal{R}, \alpha)$ and (ε, α) , and the relative magnitude of ε , $\mathcal{R}$ and α in these regions are all shown in Table 1.

Because the range of the three variables are all of $[0, 1]$, a unit cube is formed by applying the three variables as three co-ordinates. Slicing the unit cube along the two diagonal lines on ε - $\mathcal{R}$ -0 plane, 0- $\mathcal{R}$ - α plane and ε -0- α plane, respectively, as can be shown in Fig. 3, 24 triangular cones are obtained.

All points in any one of the 24 cones have the same relative magnitude among ε , $\mathcal{R}$ and α , and therefore all input values located in a specific cone would have the same

Fig. 2 Block diagram showing the four designated regions shown in Table 1

a ε - $\mathcal{R}$ block
b $\mathcal{R}$ - α block
c ε - α block

Table 1: Relative magnitude of ε , $\mathcal{R}$ and α on ε - $\mathcal{R}$ -0 plane, 0- $\mathcal{R}$ - α plane, and ε -0- α plane, respectively

Region	ε - $\mathcal{R}$ block	$\mathcal{R}$ - α block	ε - α block
1	$\varepsilon \geq \mathcal{R}$ $(1 - \mathcal{R} \geq 1 - \varepsilon)$	$\mathcal{R} \geq \alpha$	$\varepsilon \geq \alpha$
2	$\varepsilon \geq 1 - \mathcal{R}$ $(\mathcal{R} \geq 1 - \varepsilon)$	$\alpha \geq 1 - \mathcal{R}$	$\alpha \geq 1 - \varepsilon$
3	$\mathcal{R} \geq \varepsilon$ $(1 - \varepsilon \geq 1 - \mathcal{R})$	$\alpha \geq \mathcal{R}$	$\alpha \geq \varepsilon$
4	$\varepsilon \geq 1 - \mathcal{R}$ $(\mathcal{R} \geq 1 - \varepsilon)$	$\alpha \geq 1 - \mathcal{R}$	$\alpha \geq 1 - \varepsilon$
1	$\mathcal{R} \geq \varepsilon$ $(1 - \varepsilon \geq 1 - \mathcal{R})$	$1 - \mathcal{R} \geq \alpha$ $(1 - \varepsilon \geq \mathcal{R})$	$1 - \varepsilon \geq \alpha$
2	$\varepsilon \geq \mathcal{R}$ $(1 - \mathcal{R} \geq 1 - \varepsilon)$	$\mathcal{R} \geq \alpha$	$\varepsilon \geq \alpha$
3	$\varepsilon \geq 1 - \mathcal{R}$ $(\mathcal{R} \geq 1 - \varepsilon)$	$1 - \mathcal{R} \geq \alpha$	$1 - \varepsilon \geq \alpha$

inference results for all four rules fired. Fig. 4 depicts the four cones with base on the ε - $\mathcal{R}$ -0 plane, denoted as c_{i44} , $i \in \{1, 2, 3, 4\}$. Subscripts i, j and k of the cone c_{ijk} mean that the projection of the points of the cone would fall on region i of the ε - $\mathcal{R}$ -0 plane, region j of the 0- $\mathcal{R}$ - α plane and region k of the ε -0- α plane, respectively.

For the normalised inputs ε and $\mathcal{R}$, the firing levels of the four rules within each of these 24 cones are shown in Table 2. By applying the firing levels listed in Table 2, the explicit mathematical form of final output $\text{GU} \times u^*$ of the USMFs-FLC, under the assumption of simple rule base, can be expressed as the sum of a global part, $\text{GU} \times u_G^*$, and a local part, $\text{GU} \times u_L^*$.

The value of the global part of the crisp output, regardless of the α value, is determined by the specific ij th block in which (e^*, r^*) is located and is a constant across the four regions within the ij th block. However, as can be shown in Theorem 1, the local part varies with the values of e^* , r^* and α .

Theorem 1: For the crisp-type USMFs-FLC using the Dubois and Prade's parametric t -norm-sum-gravity inference method and simple rule base, let $\varepsilon = (e^* - E_i^*) / (E_{i+1}^* - E_i^*)$ and $\mathcal{R} = (r^* - R_j^*) / (R_{j+1}^* - R_j^*)$ denote the normalised input values for any given input pattern in the ij th block, i.e. $e^* \in [E_i^*, E_{i+1}^*]$ and $r^* \in [R_j^*, R_{j+1}^*]$, the explicit input/output relation is the sum of a global two-

Fig. 3 Slicing of ε - $\mathcal{R}$ - α unit cube

Fig. 4 The four cones with base on the ε - $\mathcal{R}$ -0 plane

a The ε - $\mathcal{R}$ -0 plane

b c_{144}

c c_{244}

d c_{344}

e c_{444}

Table 2: Firing levels of each rule in the 24 cones

Based on	Region	Firing level of each rule				Summation of firing levels
		rule 1	rule 2	rule 3	rule 4	
		$\phi_{i+1,j+1}$	$\phi_{i+1,j}$	$\phi_{i,j+1}$	$\phi_{i,j}$	$\sum_{\phi_{i,j} \neq 0} \phi_{i,j}$
ε - $\mathcal{R}$ -0	c_{144}	$\mathcal{R}$	$1 - \mathcal{R}$	$1 - \varepsilon$	$1 - \varepsilon$	$1 + 2(1 - \varepsilon)$
	c_{244}	ε	$1 - \mathcal{R}$	$1 - \varepsilon$	$1 - \mathcal{R}$	$1 + 2(1 - \mathcal{R})$
	c_{344}	ε	ε	$\mathcal{R}$	$1 - \mathcal{R}$	$1 + 2\varepsilon$
	c_{444}	$\mathcal{R}$	ε	$\mathcal{R}$	$1 - \varepsilon$	$1 + 2\mathcal{R}$
ε - $\mathcal{R}$ -1	c_{122}	$\varepsilon\mathcal{R}/\alpha$	$\varepsilon(1 - \mathcal{R})/\alpha$	$(1 - \varepsilon)\mathcal{R}/\alpha$	$(1 - \varepsilon)(1 - \mathcal{R})/\alpha$	$1/\alpha$
	c_{222}	$\varepsilon\mathcal{R}/\alpha$	$\varepsilon(1 - \mathcal{R})/\alpha$	$(1 - \varepsilon)\mathcal{R}/\alpha$	$(1 - \varepsilon)(1 - \mathcal{R})/\alpha$	$1/\alpha$
	c_{322}	$\varepsilon\mathcal{R}/\alpha$	$\varepsilon(1 - \mathcal{R})/\alpha$	$(1 - \varepsilon)\mathcal{R}/\alpha$	$(1 - \varepsilon)(1 - \mathcal{R})/\alpha$	$1/\alpha$
	c_{422}	$\varepsilon\mathcal{R}/\alpha$	$\varepsilon(1 - \mathcal{R})/\alpha$	$(1 - \varepsilon)\mathcal{R}/\alpha$	$(1 - \varepsilon)(1 - \mathcal{R})/\alpha$	$1/\alpha$
0- $\mathcal{R}$ - α	c_{313}	ε	$\varepsilon(1 - \mathcal{R})/\alpha$	$\mathcal{R}$	$1 - \mathcal{R}$	$1 + \varepsilon + \varepsilon(1 - \mathcal{R})/\alpha$
	c_{323}	$\varepsilon\mathcal{R}/\alpha$	$\varepsilon(1 - \mathcal{R})/\alpha$	$\mathcal{R}$	$1 - \mathcal{R}$	$1 + \varepsilon/\alpha$
	c_{333}	$\varepsilon\mathcal{R}/\alpha$	ε	$\mathcal{R}$	$1 - \mathcal{R}$	$1 + \varepsilon + \varepsilon\mathcal{R}/\alpha$
	c_{343}	ε	ε	$\mathcal{R}$	$1 - \mathcal{R}$	$1 + 2\varepsilon$
1- $\mathcal{R}$ - α	c_{111}	$\mathcal{R}$	$1 - \mathcal{R}$	$1 - \varepsilon$	$(1 - \varepsilon)(1 - \mathcal{R})/\alpha$	$1 + (1 - \varepsilon) + (1 - \varepsilon)(1 - \mathcal{R})/\alpha$
	c_{121}	$\mathcal{R}$	$1 - \mathcal{R}$	$(1 - \varepsilon)\mathcal{R}/\alpha$	$(1 - \varepsilon)(1 - \mathcal{R})/\alpha$	$1 + (1 - \varepsilon)/\alpha$
	c_{131}	$\mathcal{R}$	$1 - \mathcal{R}$	$(1 - \varepsilon)\mathcal{R}/\alpha$	$1 - \varepsilon$	$1 + (1 - \varepsilon) + (1 - \varepsilon)\mathcal{R}/\alpha$
	c_{141}	$\mathcal{R}$	$1 - \mathcal{R}$	$1 - \varepsilon$	$1 - \varepsilon$	$1 + 2(1 - \varepsilon)$
ε -0- α	c_{431}	$\mathcal{R}$	ε	$(1 - \varepsilon)\mathcal{R}/\alpha$	$1 - \varepsilon$	$1 + \mathcal{R} + (1 - \varepsilon)\mathcal{R}/\alpha$
	c_{432}	$\varepsilon\mathcal{R}/\alpha$	ε	$(1 - \varepsilon)\mathcal{R}/\alpha$	$1 - \varepsilon$	$1 + \mathcal{R}/\alpha$
	c_{433}	$\varepsilon\mathcal{R}/\alpha$	ε	$\mathcal{R}$	$1 - \varepsilon$	$1 + \mathcal{R} + \varepsilon\mathcal{R}/\alpha$
	c_{434}	$\mathcal{R}$	ε	$\mathcal{R}$	$1 - \varepsilon$	$1 + 2\mathcal{R}$
ε -1- α	c_{211}	ε	$1 - \mathcal{R}$	$1 - \varepsilon$	$(1 - \varepsilon)(1 - \mathcal{R})/\alpha$	$1 + (1 - \mathcal{R}) + (1 - \varepsilon)(1 - \mathcal{R})/\alpha$
	c_{212}	ε	$\varepsilon(1 - \mathcal{R})/\alpha$	$1 - \varepsilon$	$(1 - \varepsilon)(1 - \mathcal{R})/\alpha$	$1 + (1 - \mathcal{R})/\alpha$
	c_{213}	ε	$\varepsilon(1 - \mathcal{R})/\alpha$	$1 - \varepsilon$	$1 - \mathcal{R}$	$1 + (1 - \mathcal{R}) + \varepsilon(1 - \mathcal{R})/\alpha$
	c_{214}	ε	$1 - \mathcal{R}$	$1 - \varepsilon$	$1 - \mathcal{R}$	$1 + 2(1 - \mathcal{R})$

Fig. 5 Effects of shrinking factors on the t -norm-independent global output surface of the crisp-type SSMFs-FLC ($m_e = m_r = 3$)

- a $s_e = s_r = 1.0$, $s_u = 1.0$
b $s_e = s_r = 0.6$, $s_u = 1.0$
c $s_e = s_r = 1.0$, $s_u = 0.6$
d $s_e = s_r = 0.6$, $s_u = 0.6$

dimensional multilevel relay and an α -value dependent local nonlinear compensator:

$$u_m = \text{GU} \times u^* = \text{GU} \times (u_G^* + u_L^*) = u_G + u_L$$

$$u_G^* = U_{k+1}^*$$

$$(u_L^*)_{\text{D&P}} = \begin{cases} \frac{\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(U_{k+1}^* - U_k^*)}{1 + 2(1 - \varepsilon)} & \text{for } c_{144} \\ \frac{\varepsilon(U_{k+2}^* - U_{k+1}^*) - (1 - \mathcal{R})(U_{k+1}^* - U_k^*)}{1 + 2(1 - \mathcal{R})} & \text{for } c_{244} \\ \frac{\varepsilon(U_{k+2}^* - U_{k+1}^*) - (1 - \mathcal{R})(U_{k+1}^* - U_k^*)}{1 + 2\varepsilon} & \text{for } c_{344} \\ \frac{\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(U_{k+1}^* - U_k^*)}{1 + 2\mathcal{R}} & \text{for } c_{444} \\ \varepsilon\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(1 - \mathcal{R})(U_{k+1}^* - U_k^*) & \text{for } c_{122} \\ \varepsilon\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(1 - \mathcal{R})(U_{k+1}^* - U_k^*) & \text{for } c_{222} \\ \varepsilon\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(1 - \mathcal{R})(U_{k+1}^* - U_k^*) & \text{for } c_{322} \\ \varepsilon\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(1 - \mathcal{R})(U_{k+1}^* - U_k^*) & \text{for } c_{422} \\ \frac{\varepsilon(U_{k+2}^* - U_{k+1}^*) - (1 - \mathcal{R})(U_{k+1}^* - U_k^*)}{1 + \varepsilon + \varepsilon(1 - \mathcal{R})/\alpha} & \text{for } c_{313} \\ \frac{\varepsilon\mathcal{R}/\alpha(U_{k+2}^* - U_{k+1}^*) - (1 - \mathcal{R})(U_{k+1}^* - U_k^*)}{1 + \varepsilon/\alpha} & \text{for } c_{323} \\ \frac{\varepsilon\mathcal{R}/\alpha(U_{k+2}^* - U_{k+1}^*) - (1 - \mathcal{R})(U_{k+1}^* - U_k^*)}{1 + \varepsilon + \varepsilon\mathcal{R}/\alpha} & \text{for } c_{333} \\ \frac{\varepsilon(U_{k+2}^* - U_{k+1}^*) - (1 - \mathcal{R})(U_{k+1}^* - U_k^*)}{1 + 2\varepsilon} & \text{for } c_{343} \\ \frac{\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(1 - \mathcal{R})/\alpha(U_{k+1}^* - U_k^*)}{1 + (1 - \varepsilon) + (1 - \varepsilon)(1 - \mathcal{R})/\alpha} & \text{for } c_{111} \\ \frac{\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(1 - \mathcal{R})/\alpha(U_{k+1}^* - U_k^*)}{1 + (1 - \varepsilon)/\alpha} & \text{for } c_{121} \\ \frac{\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(U_{k+1}^* - U_k^*)}{1 + (1 - \varepsilon) + (1 - \varepsilon)\mathcal{R}/\alpha} & \text{for } c_{131} \\ \frac{\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(U_{k+1}^* - U_k^*)}{1 + 2(1 - \varepsilon)} & \text{for } c_{141} \\ \frac{\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(U_{k+1}^* - U_k^*)}{1 + \mathcal{R} + (1 - \varepsilon)\mathcal{R}/\alpha} & \text{for } c_{431} \\ \frac{\varepsilon\mathcal{R}/\alpha(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(U_{k+1}^* - U_k^*)}{1 + \mathcal{R}/\alpha} & \text{for } c_{432} \\ \frac{\varepsilon\mathcal{R}/\alpha(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(U_{k+1}^* - U_k^*)}{1 + \mathcal{R} + \varepsilon\mathcal{R}/\alpha} & \text{for } c_{433} \\ \frac{\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(U_{k+1}^* - U_k^*)}{1 + 2\mathcal{R}} & \text{for } c_{434} \\ \frac{\varepsilon(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(1 - \mathcal{R})/\alpha(U_{k+1}^* - U_k^*)}{1 + (1 - \mathcal{R}) + (1 - \varepsilon)(1 - \mathcal{R})/\alpha} & \text{for } c_{211} \\ \frac{\varepsilon(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(1 - \mathcal{R})/\alpha(U_{k+1}^* - U_k^*)}{1 + (1 - \mathcal{R})/\alpha} & \text{for } c_{212} \\ \frac{\varepsilon(U_{k+2}^* - U_{k+1}^*) - (1 - \mathcal{R})(U_{k+1}^* - U_k^*)}{1 + (1 - \mathcal{R}) + \varepsilon(1 - \mathcal{R})/\alpha} & \text{for } c_{213} \\ \frac{\varepsilon(U_{k+2}^* - U_{k+1}^*) - (1 - \mathcal{R})(U_{k+1}^* - U_k^*)}{1 + 2(1 - \mathcal{R})} & \text{for } c_{214} \end{cases}$$

Proof: From eqn. 4,

$$\begin{aligned} u^* &= \frac{\sum_{\phi_{i,j} \neq 0} \phi_{i,j} U_{i+j}^*}{\sum_{\phi_{i,j} \neq 0} \phi_{i,j}} \\ &= \frac{\phi_{i+1,j+1} U_{i+j+2}^* + (\phi_{i+1,j} + \phi_{i,j+1}) U_{i+j+1}^* + \phi_{i,j} U_{i+j}^*}{\phi_{i+1,j+1} + (\phi_{i+1,j} + \phi_{i,j+1}) + \phi_{i,j}} \\ &= U_{i+j+1}^* \\ &\quad + \frac{\phi_{i+1,j+1} [U_{i+j+2}^* - U_{i+j+1}^*] - \phi_{i,j} [U_{i+j+1}^* - U_{i+j}^*]}{\phi_{i+1,j+1} + (\phi_{i+1,j} + \phi_{i,j+1}) + \phi_{i,j}} \end{aligned}$$

Fig. 6 Illustration for influence of t -norms on local outputs of the crisp-type SSMFs-FLCs ($m_e = m_r = 3$)

a $\alpha = 0$
b $\alpha = 0.25$
c $\alpha = 0.5$
d $\alpha = 0.75$
e $\alpha = 1$

Substituting the firing levels for all cones listed in Table 2 leads to the results.

Remarks:

(i) The role of the local output is to compensate the discrete-type global output to produce a continuous final reasoning surface. The Dubois and Prade's parametric t -norm would yield various compensation behaviours by adjusting the α value.

(ii) The general equations listed above can be simplified by assigning special values to α . If $\alpha = 0$, the Dubois and Prade's parametric t -norm is then a standard fuzzy intersection (min), and the local nonlinear compensator

becomes

$$(u_D^*)_{\min} = \begin{cases} \frac{\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \epsilon)(U_{k+1}^* - U_k^*)}{1 + 2(1 - \epsilon)} & \text{region I} \\ \frac{\epsilon(U_{k+2}^* - U_{k+1}^*) - (1 - \mathcal{R})(U_{k+1}^* - U_k^*)}{1 + 2(1 - \mathcal{R})} & \text{region II} \\ \frac{\epsilon(U_{k+2}^* - U_{k+1}^*) - (1 - \mathcal{R})(U_{k+1}^* - U_k^*)}{1 + 2\epsilon} & \text{region III} \\ \frac{\mathcal{R}(U_{k+2}^* - U_{k+1}^*) - (1 - \epsilon)(U_{k+1}^* - U_k^*)}{1 + 2\mathcal{R}} & \text{region IV} \end{cases} \quad (6)$$

Fig. 7 Effects of shrinking factors on final reasoning surface of the crisp-type SSMFs-FLC $m_e = m_r = 3$, $\alpha = 0.5$ in Dubois and Prade's parametric t -norm

- a $s_e = s_r = 1.0$, $s_u = 1.0$
b $s_e = s_r = 0.6$, $s_u = 1.0$
c $s_e = s_r = 1.0$, $s_u = 0.6$
d $s_e = s_r = 0.6$, $s_u = 0.6$

If $\alpha = 1$, the Dubois and Prade's parametric t -norm is then an algebraic product, and the local nonlinear compensator becomes

$$(u_D^*)_{\mp} = (\varepsilon)(\mathcal{R})(U_{k+2}^* - U_{k+1}^*) - (1 - \varepsilon)(1 - \mathcal{R})(U_{k+1}^* - U_k^*) \quad (7)$$

(iii) The effects of distributions of principal values for e , r and u on final reasoning surface, for both global and local parts, can be further investigated. Without a loss of generality, only the SSMFs-FLC and ESMFs-FLC are illustrated in the following. For input variables of the SSMFs-FLC, the values of s_e and s_r influence the location and the shape of the ij th block in the domain $U_e \times U_r$. In light of this, a specific set of (e^*, r^*) may be located in a different ij th block for different s_e and s_r values and, thus, may instigate distinct control rules in obtaining the final output. Regarding the output variable u , different effects are induced for u_G^* and u_L^* by changing the shrinking factor s_u . For u_G^* , modifying s_u only changes its magnitude. However, for u_L^* , both magnitude and nonlinearity are altered by varying the value of s_u .

Fig. 5 shows the t -norm-independent global response for the crisp-type SSMFs-FLCs with various shrinking factors. Figs. 6 illustrates the influence of t -norms on local outputs. Fig. 7 illustrates the effects of the shrinking factors for $\alpha = 0.5$ in the Dubois and Prade's parametric t -norm on the final reasoning surfaces.

4 Conclusion

This study has examined the input/output parametric relationship of a class of crisp-type fuzzy logic controllers using the Dubois and Prade's parametric t -norm-sum-gravity inference method. The topology of the FLCs being studied is defined as follows: (a) the input membership functions are composed of a series of unequally spaced triangles and the output membership functions consist of a set of singletons (the crisp type); (b) the Dubois and Prade's parametric t -norm is used in calculating the firing level and implication of each single control rule; (c) the inference results of all control rules are summed together to produce a final output fuzzy set; and (d) the centre-of-gravity method is used for defuzzification. The explicit mathematical expressions of reasoning surfaces for the FLCs, using different α values of the Dubois and Prade's parametric t -norm as the firing operator, are addressed. The reasoning outputs of these FLCs are decomposed into two parts: a two-dimensional multilevel relay which is independent of the t -norm used and a local nonlinear compensator with different output patterns according to the α value of the Dubois and Prade's parametric t -norm. Such theoretical analysis is significant for illustrating the inner characteristics of fuzzy logic controllers.

5 Acknowledgments

The authors would like to thank the National Science Council of the Republic of China for financially supporting this research under Contract NSC87-2214-E-002-014.

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