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Computation of Discrete Hilbert Transform Through Fast Hartley Transform

SOO-CHANG PEI AND SY-BEEN JAW

Abstract—A new fast algorithm is proposed to compute the discrete Hilbert transform via the fast Hartley transform (FHT). Instead of the conventional fast Fourier transform (FFT) approach, the processing is carried out entirely in the real domain, also many efficient fast algorithms exist for computing the discrete Hartley transform, the computation complexity can be greatly reduced from 2 complex FFT's into 2 real FHT's.

I. INTRODUCTION

Hilbert transforms have played a useful role in signal and network theory, and have also been of practical importance in various signal processing systems; analytic signal, bandpass sampling, minimum phase networks, and much of spectral analysis theory are based on Hilbert transform relations. Systems for performing Hilbert transform operations have been proven useful in such diverse fields as radar moving target indicators, analytic signal rooting, envelope detection, and generation of the phase of a spectrum given its amplitude. To be able to use digital signal processing techniques in these applications, Kak, Cizek, and Oppenheim [1]–[3] have developed the one-dimensional discrete Hilbert transform, analytic signal, minimum phase sequence, etc., for the time series of discrete samples.

The conventional approach for computing the discrete Hilbert transform is through the fast Fourier transform (FFT). It is performed in complex domain and needs many time consuming complex multiplications. Recently, the discrete Hartley transform (DHT) [4] has been considered as an interesting alternative to the FFT for spectral analysis and fast convolution of real data [5], and many efficient fast algorithms have been developed for computing the DHT [6]–[8].

In this paper, a new fast algorithm is proposed to compute the discrete Hilbert transform through the fast Hartley transform (FHT). Since the processing is performed entirely in real domain, the computation complexity can be greatly reduced from 2 complex FFT's into 2 real FHT's.

II. DISCRETE HILBERT TRANSFORM BY FFT

In the continuous case, the Fourier transform of the Hilbert transform $\hat{f}(t)$ of $f(t)$ is

$$F[\hat{f}(t)] = F[H(f(t))] = H(w) \cdot F(w) = j \cdot \text{sgn}(w) \cdot F(w) \quad (1)$$

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Fig. 1. Implementation of discrete Hilbert transform by FFT.

where

$$H(w) = j \text{sgn}(w) = \begin{cases} -j, & w > 0 \\ 0, & w = 0 \\ +j, & w < 0 \end{cases} \quad (2)$$

For the discrete case, Cizek [1] derived a discrete Hilbert transform $\hat{f}(n)$ for a periodic sequence $f(n)$ for which the above relationship still holds:

$$\begin{aligned} \text{DFT}[\hat{f}(n)] &= \text{DFT}\{\text{DHT}[f(n)]\} \\ &= H(k) \cdot F(k) = j \cdot \text{sgn}(k) \cdot F(k) \end{aligned} \quad (3)$$

where

$$H(k) = j \cdot \text{sgn}(k) = \begin{cases} -j, & k = 1, 2, \dots, \frac{N}{2} - 1 \\ 0, & k = 0, \frac{N}{2} \\ +j, & k = \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, N - 1 \end{cases} \quad (4)$$

So it is very easy to obtain the discrete Hilbert transform $\hat{f}_n$ of the sequence f_n via the FFT shown in Fig. 1.

III. DISCRETE HARTLEY TRANSFORM (DHT) AND CONVOLUTION THEOREM

The DHT of $f(n)$ is defined as [4]

$$H_f(k) = \sum_{n=0}^{N-1} f(n) \text{cas}\left(\frac{2nk}{N}\pi\right) \quad (5)$$

where $\text{cas}(x) = \sin(x) + \cos(x)$, the inverse relation is

$$f(n) = \frac{1}{N} \sum_{k=0}^{N-1} H_f(k) \text{cas}\left(\frac{2nk}{N}\pi\right). \quad (6)$$

The convolution theorem obeyed by the DHT is as follows: if $x(n)$ is the convolution of $x_1(n)$ with $x_2(n)$, i.e.,

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{N-1} x_1(k) \cdot x_2(n-k) \quad (7)$$

then

$$H_x(k) = H_{x_1}(k) \cdot \text{even}[H_{x_2}(k)] + H_{x_1}(-k) \cdot \text{odd}[H_{x_2}(k)] \quad (8)$$

where $H_x(k)$, $H_{x_1}(k)$, and $H_{x_2}(k)$ are the DHT's of $x(n)$, $x_1(n)$, and $x_2(n)$, respectively, and $H_{x_2}(k) = \text{even}[H_{x_2}(k)] + \text{odd}[H_{x_2}(k)]$, the sum of its even and odd parts. The proof of this Hartley transform convolution theorem is shown in the Appendix.

Fig. 2. Implementation of discrete Hilbert transform by FHT.

IV. DISCRETE HILBERT TRANSFORM BY FHT

The relation of the DHT and the DFT of a real sequence $f(n)$ is

$$H_f(k) = \text{even}[H_f(k)] + \text{odd}[H_f(k)] : \text{DHT} \quad (9)$$

$$F(k) = \text{even}[H_f(k)] - j \text{odd}[H_f(k)] : \text{DFT}. \quad (10)$$

From (3), the discrete Hilbert transform of a sequence $f(n)$ is equivalent to the convolution of $f(n)$ with the Hilbert transform kernel $h(n)$ as follows:

$$\hat{f}(n) = f(n) * h(n) \xrightarrow{\text{DFT}} \hat{F}(k) = F(k) \cdot j \cdot \text{sgn}(k). \quad (11)$$

Using the Hartley transform convolution theorem in (8), we obtain

$$H_f(k) = H_f(k) \cdot \text{even}[H_h(k)] + H_f(-k) \cdot \text{odd}[H_h(k)]. \quad (12)$$

Since the Hilbert transform kernel $h(n)$ is an odd function, then the even part of its Hartley transform is zero, i.e., $\text{even}[H_h(k)] = 0$ and $H_h(k) = \text{odd}[H_h(k)]$, the convolution theorem of (12) simplifies into the multiplication product of the two separate Hartley transforms:

$$H_f(k) = H_f(-k) \cdot \text{odd}[H_h(k)] = H_f(-k) \cdot H_h(k) \quad (13)$$

where

$$H_h(k) = \begin{cases} 1, & k = 1, 2, \dots, \frac{N}{2} - 1 \\ 0, & k = 0, \frac{N}{2} \\ -1, & k = \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, N-1 \end{cases} \quad (14)$$

and $H_f(-k)$ is the time reversal of $H_f(k)$, i.e.,

$$H_h(-k) = \begin{cases} H_h(k), & k = 0 \\ H_h(N-k), & k = 1, 2, \dots, N-1. \end{cases}$$

Then it is very efficient to compute the discrete Hilbert transform $\hat{f}_n$ of the sequence f_n through FHT shown in Fig. 2. Right now, the computational complexity is greatly reduced from 2 complex FFT's into 2 real FHT's.

V. CONCLUSIONS

In this paper, we have presented a new fast algorithm for computing the discrete Hilbert transform through the FHT. Compared with the conventional FFT approach, it is performed entirely in the real domain, also many efficient FHT algorithms exist and can be applied here, and the computation complexity is greatly reduced from 2 complex FFT's into 2 real FHT's.

APPENDIX

Proof of the Hartley transform convolution theorem if $x(n)$ is the convolution of $x_1(n)$ with $x_2(n)$, i.e.,

$$X(n) = x_1(n) * x_2(n) = \sum_{m=0}^{N-1} x_2(m) \cdot x_1(n-m). \quad (A1)$$

Taking discrete Hartley transformation on both sides:

$$H_x(k) = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x_2(m) \cdot x_1(n-m) \cdot \text{cas}\left(\frac{2nk}{N}\pi\right) \quad (A2)$$

$$H_x(k) = \sum_{m=0}^{N-1} x_2(m) \left[\sum_{n=0}^{N-1} x_1(n-m) \text{cas}\left(\frac{2nk}{N}\pi\right) \right]. \quad (A3)$$

By the Hartley transform shift theorem [4], [5]

$$x_1(n-m) \xrightarrow{\text{DHT}} H_{x_1}(k) \cos\left(\frac{2mk}{N}\pi\right) - H_{x_1}(-k) \sin\left(\frac{2mk}{N}\pi\right). \quad (A4)$$

So (A3) becomes

$$\begin{aligned} \therefore H_x(k) = \sum_{m=0}^{N-1} x_2(m) & \left[H_{x_1}(k) \cos\left(\frac{2mk}{N}\pi\right) \right. \\ & \left. - H_{x_1}(-k) \sin\left(\frac{2mk}{N}\pi\right) \right] \end{aligned} \quad (A5)$$

$$\begin{aligned} \therefore H_x(k) = H_{x_1}(k) & \left[\sum_{m=0}^{N-1} x_2(m) \cos\left(\frac{2mk}{N}\pi\right) \right] \\ & - H_{x_1}(-k) \left[\sum_{m=0}^{N-1} x_2(m) \sin\left(\frac{2mk}{N}\pi\right) \right] \end{aligned} \quad (A6)$$

$$\therefore H_x(k) = H_{x_1}(k) \cdot \text{even}[H_{x_2}(k)] - H_{x_1}(-k) \cdot \text{odd}[H_{x_2}(k)]. \quad (A7)$$

Then the proof is complete.

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Implementation of 2-D Filters for Real-Time Processing

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Abstract—A systematic way for implementation of high speed 2-D linear filters for real-time image processing is developed. Combining a consequent parallelization and a new RAM-based implementation tech-

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