After re-arrangement:

\[
\frac{1}{(1)^{s+1}} = \frac{1}{\left(\frac{(1)^{s+1}}{S}\right)}
\]

So,

The expected return on all possible holding periods are equal for all holding periods. The expected return on all possible holding periods is equal to the forward rate. The forward rate is equal to the expected return on a one-period bond.

A "Bad" Expectations Theory

The spot rate curve is inverted.

Consider, if the spot rate is expected to fall and only if

\[ (1)^{s+1} < \cdots < (1+f)^{s} < \cdots \]

Then the market expects the future spot rate to rise. Implies that a normal spot rate curve is due to the fact that the spot rate curve is inverted.

Unbiased Expectations Theory and Spot Rate Curve

That would mean investors are indifferent to risk are expected to earn the same return on the mortgage bonds. Regardless of their different maturities have expected interest rates to be 80% of the time. Since the term structure has been upward sloping about the possible exception of the period prior to 1962. The theory has been refected by some empirical studies.

More Implications

Forward rate equals the average future spot rate.
Define duration as

\[ \frac{\partial \theta}{(\partial \theta) \partial \theta} \]

associated with the cash flow. Let the price be the price

\[ \left\{ \sum (1 + S + 1)^{-1} \right\} \]

To handle more general types of spec rate curve changes.

Duration Revised

---

Local Expectations Theory

\[ \left\{ \begin{array}{c}
\left[ \frac{(1)S + 1}{1} \right] A = \frac{v((1)S + 1)}{(1)S + 1}
\end{array} \right. \]

The theory says the returns are equal.

\[ \left\{ \begin{array}{c}
(1)S + 1
\end{array} \right. \]

The expected return of return of any bond over a single

period equals the present value of a one-period bond with a return of

\[ \left\{ \begin{array}{c}
\frac{v((1)S + 1)}{(1)S + 1}
\end{array} \right. \]

The expected return is

\[ \left\{ \begin{array}{c}
\frac{v((1)S + 1)}{(1)S + 1}
\end{array} \right. \]

The returns are equal.

Now consider two one-period strategies.

A "bad" Expectations Theory (continued)
When \( X_i \) are uncorrelated

\[
\sum_{i=1}^{n} \text{Var}(X_i) = \sum_{i=1}^{n} \text{Var}(X_i)
\]

If becomes

\[
\sum_{i=1}^{n} \text{Cov}(X_i, X_i) = \sum_{i=1}^{n} \text{Var}(X_i)
\]

Variance of a weighted sum of random variables equals

Variance of Sum

Covariances

Random variables \( X \) and \( Y \) are uncorrelated if

\[
\text{Cov}(X, Y) = 0
\]

The covariance between random variables \( X \) and \( Y \) is defined as

\[
\text{Cov}(X, Y) = \text{E}[X - \mu_X][Y - \mu_Y]
\]

Moments

Fundamental Statistical Concepts

There are three kinds of lies:

1. Joseph Stalin (1879–1953) — but a million deaths and a statistic.
2. Benjamin Disraeli (1804–1881) — lies, damned lies, and statistics.
A random variable $X$ has the normal distribution with

\[ \mathbb{P}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad z \in \mathbb{R} \]

Distribution

Theorem: If $X_1, X_2, ..., X_n$ are independent and identically distributed (i.i.d.) random variables, then

\[ \sum_{i=1}^{n} X_i \sim N(n \mu, n \sigma^2) \]

Proof

The sum of independent normal random variables is also normal. The mean of the sum is $n \mu$ and the variance is $n \sigma^2$.

\[ \mathbb{E}(X) = \mu \quad \text{and} \quad \text{Var}(X) = \sigma^2 \]

\[ \mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y) \quad \text{and} \quad \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \]

Distribution of Sum

Moment Generating Function

\[ M_X(t) = \mathbb{E}(e^{tX}) = \exp\left( \mu t + \frac{1}{2} \sigma^2 t^2 \right) \]

Moment Generating Function

\[ M_{X_1 + X_2}(t) = M_{X_1}(t) M_{X_2}(t) \]

Theorem (Law of Iterated Expectations): If $X$ is a random variable and $A$ is an event, then

\[ \mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid A)) \]

Theorem (Bayes' Theorem): If $X$ is a discrete random variable conditioned on $A$, then

\[ \mathbb{P}(X \mid A) = \frac{1}{\mathbb{P}(A)} \mathbb{E}(X \mid A) \]

Theorem (Law of Total Expectation): If $X$ is a random variable and $A$ is an event, then

\[ \mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid A)) \]

Theorem (Bayes' Rule): If $X$ is a random variable and $A$ is an event, then

\[ \mathbb{P}(A \mid X) = \frac{1}{\mathbb{P}(X)} \mathbb{E}(\mathbb{P}(A \mid X)) \]

Conditional Expectation

\[ \mathbb{E}(X \mid A) = \frac{\mathbb{E}(X 1_A)}{\mathbb{P}(A)} \]

Conditional Expectation

\[ \mathbb{E}(Z \mid X = x) = \frac{\int z f_{Z \mid X}(z \mid x) \, dz}{f_X(x)} \]

Conditional Expectation

\[ f_{Z \mid X}(z \mid x) = \frac{f_{Z, X}(z, x)}{f_X(x)} \]

Conditional Expectation

\[ f_{Z \mid X}(z \mid x) = f_{Z \mid X}(z \mid x) \]

Conditional Expectation
Generation of Bivariate Normal Distributions

If $X_1$ and $X_2$ are independent standard normal variables, then:

- $\frac{\alpha}{\sqrt{1-\rho^2}}$ and $\frac{\beta}{\sqrt{1-\rho^2}}$ are independent normal variables where $\alpha$ and $\beta$ are independent standard normal variables. Then $(\alpha, \beta)$ are independent.
- $(\alpha \xi, \beta \xi)$ are independent.
- Randomly draw two samples $\eta_1$ and $\eta_2$ from $X$ until $\eta_1 > \gamma, \eta_2 > \gamma$.
- If $A \leq X \leq B$ for $0 < x < 1$, then $X \leq B$ be uniformly distributed over $(0,1]$ so that $X$ can be generated.

A Dirty Trick and a Right Attitude (concluded)

Generation of Univariate Normal Distributions
How to price options?

- An embedded option has to be embedded along with the underlying asset for the strike price.
- A put gives the holder the right to sell a number of the underlying asset by paying a strike price.
- A call gives the holder the right to buy a number of the underlying asset.

Calls and Puts

Option Basics

\[ \sum_{\nu=0}^{n+1} \frac{\nu!}{\nu!(n+1-\nu)!} = \frac{n+1}{\nu+1} \]

The mean of the distribution of a random variable \( X \) is said to have a logarithmic distribution.

The Lognormal Distribution
Payoff

- The payoff of a put at expiration is $S - X$.
- The payoff of a call at expiration is $X - S$.
- $S$ (the stock price) is the strike price of the option.
- $C$ (call value) is the value of the option at expiration.
- $P$ (put value) is the value of the option at expiration.

American and European

- An American option can be exercised at any time up to the expiration date.
- An American option is worth at least as much as an equivalent European option because of the early exercise feature.
- European options can only be exercised at expiration.
- When a put is exercised, the holder receives the difference between the strike price and the stock price.
- When a call is exercised, the holder pays the strike price and receives the stock.
- $d$: dividend.
The opposite is true for puts:
- Cash dividends are determined for calls.
  - The amount of the cash dividend is $\frac{\text{dividend}}{\text{stock price}}$.
  - The stock price falls by an amount roughly equal to the dividend.
- Exchange-traded stock options are not cash dividends.
  - The option contract is not adjusted for cash dividends.
  - The exchange-traded option contract (or simply contract).

Call Options

\[
\text{Payoff} = \begin{cases} 
K - S & \text{if } X < S \\
0 & \text{if } X = S \\
S - K & \text{if } X > S 
\end{cases}
\]

Put Options

\[
\text{Payoff} = \begin{cases} 
S - K & \text{if } X < S \\
0 & \text{if } X = S \\
K - S & \text{if } X > S 
\end{cases}
\]
Covered Position: Hedged

A covered position has a profit or loss opposite in sign to that of a long

- a short position
- buying a call option
- writing a call option
- buying a put option
- writing a put option
- buying a stock
- writing a stock

So they are bullish.

- Both strategies break even only if the stock price rises.
- Covered only: A long position in stock with a short call.
- Protection puts: A long position in stock with a long put.

Example

Options are assumed to be unprotected.

- Dividends.
- Exchange-traded stock options are adjusted for stock dividends.
- Provisional exchange.
- Covered by one contract, becomes m times the
  covered by one contract becomes m times the
  number of shares
  number of shares

After an n-for-1 stock split, the strike price is only

Stock splits and stock dividends

Not all assets can be shorted.

- Clearly, the investor profits if the stock price falls.
  buying 1,000 XYZ shares.
  The investor can close out the short position by
  selling 1,000 XYZ shares, the broker borrows
  If you short 1,000 XYZ shares, the broker borrows
  back later.

  Short selling (or simply shorting) involves selling an
  asset that is not owned with the intention of buying it back later.

- Selling short.
and short two X \bar{Y} \bar{X} calls.
- The spread is long one X \bar{Y} \bar{X} call, long one X \bar{Y} call.

The same expiration date creates a butterfly spread.

Three calls or three puts with different strike prices and
If profits from defining stock prices.
A bear spread amounts to selling a bull spread.
A put spread consists of a long X \bar{Y} \bar{X} call and a short
Writing an \bar{X} \bar{Y} \bar{X} put and buying an X \bar{Y} \bar{X} put with
Covered Position: Spread (continued)
Stock price

The price of a state contingent claim is called a state

Particular state results.

A state contingent claim, which pays $1 only when a

A butterfly spread with a small $X_H - X_L$ approximation

A butterfly spread only if the asset price falls between $X_L$ and

A butterfly spread pays on a positive amount at

Covered Position: Spread (continued)
as good under all circumstances and better under some
should be more valuable than $A$'s portfolio at least

- The portfolio dominance principle says portfolio A
  (or long).
- In all efficient markets, such opportunities do not exist.

Anti-inequity

Anti-triangle in Option Pricing

— David Hume (1711–1776)

When applied to particular cases,
All general laws are attended with circumstances
Two More Examples

Reversing the trades

If the price $p < P$, a riskless profit can be realized by

- selling the stock at a price $P$
- buying the option
- buying the security
- selling the option

If the price $p > P$, short the stock and use the proceeds to buy the security. The loss in cash flows from this procedure is

$$ p \Delta t - P \Delta t = \sum_{t=1}^{n} C_t \Delta t $$

The PV formula justified

- In both cases, a riskless position is created.
- Buy if its PV is positive.
- Sell the portfolio if its PV is positive.
-ipe must have a zero PV.

A Collar
Consequences of Put-Call Parity

\[ P(x) \text{ of the strike price} \]

The no-arbitrage principle implies that at the initial moment to set up the portfolio must be no loss as well.

The no-arbitrage principle implies that the initial condition is zero in either case.

After the loan, now, \( X \) is repaid, the net future cash flow is zero. The loan \( S^2 - X \) and the put \( X \) at maturity will expire worthless.

On the other hand, if \( S^2 > X \), the call \( X \) will be worth \( P(x) \).

\[ P(x) = P_d(x) \]

where \( P_d(x) \) is the time to expiration.

\[ P_d(x) \] stands for the \( P_d(x) \) of \( x \) dollars at expiration.

Let the current time be time zero. The option prices are nonnegative, and there are no interest rates are nonnegative, and there are no transaction costs or margin requirements. Portfolio

assume among other things, that there are no basis for stock prices.

These relations hold regardless of the probabilistic

Relative Option Prices
Stock price dynamics and the Stock Price Dividend:

- When the stock price is less than the strike price $S$, exercise the call option.
- By an exercise in text, $C > \max(0, S - X)$.

Non-dividend-paying stocks continue to exhibit the same behavior as in European options. However, American options have the additional feature of early exercise.

Remarks:

- If the call is exercised, the value is the smaller $S - X$.
- An American option also cannot be worth less than its intrinsic value.

Explain the right-hand equation on p. 155 why $P > S - X$.

Proof: It may be called early due to:

- The above theorem does not mean American calls.
- The above theorem does not mean American calls.
- The above theorem does not mean American calls.
- The above theorem does not mean American calls.
- The above theorem does not mean American calls.

Intrinsic Value (concluded)

Theorem 3 (American 1972) An American call on a non-dividend-paying stock is never worth less than its intrinsic value.

- The intrinsic value of an American call is always greater than or equal to zero.
- An American option can also be exercised early.

Since $C \geq 0$, it follows that $C \geq \max(0, S - X)$, the intrinsic value.

Theorem 2 For European and $P \geq \max(0, p(Y) - S)$

A European put on a non-dividend-paying stock may be
The same result holds for European puts.

Let $X^*$ be the strike price. If all options expire on the same date, the portfolio is\footnote{Early Exercise of American Calls: Dividend Case.}:

$$\left\{ \begin{array}{l}
(\mathbb{E}^* - X^*)/\mathbb{E} X^* \\
\mathbb{E} X^* - X
\end{array} \right. $$

Here

$$P X^* > X^* \geq p X^*$$

Theorem 6 Consider a portfolio of non-dividend-paying options on one stock. Let $C_i$ denote the price of a European call on asset $i$ with strike price $X_i$. Then the call

European stocks do not pay dividends. It might be optimal to exercise an American put even if the expiration of that option is before an ex-dividend date. Theorem 4 An American call will only be exercised at a low date. Supposing an American call should be exercised only at a