On the profile of the corona of two graphs

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Abstract

The concept of profile, together with bandwidth, originates from handling sparse matrices in solving linear systems of equations. Given a graph G , the profile minimization problem is to find a one-to-one mapping $f : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$ such that $\sum_{v \in V(G)} \max_{x \in N[v]} (f(v) - f(x))$ is as small as possible, where $N[v] = \{v\} \cup \{x : x \text{ is adjacent to } v\}$. This paper studies the profile of the corona $G \wedge H$ of two graphs G and H . In particular, bounds for the profile of the corona of two graphs are established. Also, exact values of the profiles of coronas $G \wedge H$ are obtained when G has certain properties, including when G is a caterpillar, a complete graph or a cycle.

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1. Introduction

The concept of profile, together with bandwidth, originates from handling sparse matrices in solving linear systems of equations, see [2,8,11,12,17] and their references. This problem can be re-formulated as the following problem in graphs.

A *numbering* (or *labeling* or *layout*) of a graph G is a one-to-one mapping f from $V(G)$ onto $\{1, 2, \dots, |V(G)|\}$. For a numbering f , the *profile-width* of a vertex v is defined as

$$w_f(v) = \max_{x \in N[v]} (f(v) - f(x)),$$

where $N[v] = \{x \in V(G) : x = v \text{ or } xv \in E(G)\}$ is the *closed neighborhood* of v . The *profile* of a numbering f of G is

$$P_f(G) = \sum_{v \in V(G)} w_f(v)$$

and the *profile* (or *skyline*) of G is

$$P(G) = \min\{P_f(G) : f \text{ is a numbering of } G\}.$$

A numbering f is called a *profile numbering* of G if $P_f(G) = P(G)$.

The profile minimization problem has many equivalent definitions, such as interval graph completion [1,20] and graph searching [5]. In this paper, we are more concerned about the interval graph completion. A graph $G = (V, E)$ is an *interval graph* if we can associate each vertex v with a (closed) interval I_v in the real line such that two different vertices x and y are

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adjacent if and only if $I_x \cap I_y \neq \emptyset$. The interval graph completion problem is for a graph G finding a supergraph of G (which is a graph G' with $V(G') = V(G)$ and $E(G) \subseteq E(G')$) of minimum size that is an interval graph.

Proposition 1 [1,20]. *For any graph G the profile of G is equal to the smallest number of edges in an interval supergraph of G .*

The interval graph completion problem is known to be NP-complete even for edge graphs (see [7]). On the other hand, there are linear-time algorithms for cographs [5,14] and an $O(n^{1.722})$ -time algorithm for trees of n vertices [14,15]. Because of its important applications, quite a few approximation algorithms have also been developed, see [13,23,28–30]. We say that the profile minimization problem is *solved* for a class of graphs if either a polynomial-time numbering algorithm which achieves the profile for each graph in the class is provided or a formula of the exact value is given for each graph in the class. The profile minimization problem is solved only for a few classes of graphs (most are graphs obtained from two graphs applying some graph operations), see [9,16–18,20,21,24]. Lai and Williams [19] provide a survey of recent works. The current paper is to study the profile of the corona of two graphs, which was introduced in [6].

Definition. Given graphs G and H with n and m vertices, respectively, the *corona* of G with respect to H is the graph $G \wedge H$ with vertex set $V(G \wedge H) = V(G) \cup \{n \text{ distinct copies of } V(H) \text{ denoted } V(H_1), V(H_2), \dots, V(H_n)\}$ and edge set $E(G \wedge H) = E(G) \cup \{n \text{ distinct copies of } E(H) \text{ denoted } E(H_1), E(H_2), \dots, E(H_n)\} \cup \{(u_i, v): u_i \in V(G), v \in V(H_i)\}$.

Fig. 1 shows $P_3 \wedge P_4$.

Graph parameters such as bandwidth, edgesum, and domination number have been studied on coronas of graphs, see [3,4,10,27,31]. Also, the incidence

Fig. 1. $P_3 \wedge P_4$.

coloring of corona graphs has been studied by several researchers, see [22,25,26]. In this paper we establish bounds for the profile of the corona of two graphs. Exact values of the profiles of coronas $G \wedge H$ are determined when G has certain properties, including when G is a caterpillar, a complete graph or a cycle.

2. Bounds for general graphs

In this section we establish bounds for the profile of the corona of two general graphs. For convenience, we introduce the following terminology.

An *interval-labeling* of a graph G (which is not necessary an interval graph) is a mapping f on $V(G)$ that maps each vertex v to a (closed) interval $f(v)$ such that $xy \in E(G)$ implies $f(x) \cap f(y) \neq \emptyset$. Let $I_f(G)$ denote the number of unordered pairs $\{f(x), f(y)\}$ with $x \neq y$ for which $f(x) \cap f(y) \neq \emptyset$; and $I(G)$ denoted the minimum value of $I_f(G)$, where f is taken over all interval-labellings of G . An interval-labeling f is called *I-optimal* if $I_f(G) = I(G)$. According to Proposition 1, $P(G) = I(G)$ for any graph G . We will use $P(G)$ and $I(G)$ interchangeably.

In the above definition, we may assume that f is *canonical* (i.e., the $2|V(G)|$ endpoints of the intervals $f(v)$ are all distinct) as this does not affect the value of $I(G)$. In particular, each interval $f(v)$ is non-trivial (i.e., has a positive length) and the intersection of any two intersecting intervals $f(x)$ and $f(y)$ is also non-trivial. We assume that all interval-labellings are canonical throughout this paper.

Suppose f is an interval-labeling of a graph G . For any vertex v , denoted by $M_f(v)$ the minimum number of intervals $f(x)$ intersecting a non-trivial interval J_v with $J_v \cap f(v) \neq \emptyset$, where x is taken over all vertices in G and J_v is taken over all possible intervals with $J_v \cap f(v) \neq \emptyset$. Notice that $M_f(v)$ is a finite number even if there are infinitely many possible J_v as there are only finitely many $f(x)$. Denote by $M_f(G)$ the sum $\sum_{v \in V} M_f(v)$, and $M(G) = \min\{M_f(G): f \text{ is an interval-labeling of } G\}$. An interval-labeling f is called *M-optimal* if $M_f(G) = M(G)$.

In the above definition, we may assume that J_v is a non-trivial subinterval of $f(v)$ and all J_v 's are pairwise disjoint, as this does not affect the value of $M(G)$.

We are now ready to establish a lower bound for the profile $P(G \wedge H)$ in terms of $P(G)$, $P(H)$

and $M(G)$. We assume that $V(G) = \{u_1, u_2, \dots, u_n\}$ and the copy of H corresponding to u_i is H_i with vertex set $V(H_i) = \{v_{i1}, v_{i2}, \dots, v_{im}\}$ for $1 \leq i \leq n$.

Lemma 2. *If G and H are two graphs of orders n and m , respectively, then*

$$\begin{aligned} P(G \wedge H) \\ = \min \{ I_g(G) + nP(H) + M_g(G)m : \\ g \text{ is an interval-labeling of } G \}. \end{aligned} \quad (1)$$

Proof. Suppose f^* is an I -optimal interval-labeling of $G \wedge H$. Let g' be the restriction of f^* on $V(G)$ and h'_i the restriction of f^* on $V(H_i)$ for $1 \leq i \leq n$. Then

$$\begin{aligned} P(G \wedge H) &= I_{f^*}(G \wedge H) \\ &\geq I_{g'}(G) + \sum_{i=1}^n I_{h'_i}(H_i) + \sum_{j=1}^m \sum_{i=1}^n b_{ij}, \end{aligned}$$

where b_{ij} is the number of intervals $g'(u_k)$ s that intersect the interval $h'_i(v_{ij})$. Notice that the inequality may possibly be strict if $f^*(v_{ij})$ intersects $f^*(v_{i'j'})$ for some $i \neq i'$. As $h'_i(v_{ij}) \cap g'(u_i) \neq \emptyset$, we may choose $h'_i(v_{ij})$ as J_{u_i} in the definition of $M_{g'}$ to get $b_{ij} \geq M_{g'}(u_i)$. Then

$$\begin{aligned} P(G \wedge H) &\geq I_{g'}(G) + \sum_{i=1}^n P(H_i) + \sum_{j=1}^m \sum_{i=1}^n M_{g'}(u_i) \\ &\geq \text{right-hand side of Eq. (1)}. \end{aligned}$$

On the other hand, suppose g^* is an interval-labeling of G that attains the minimum in the right-hand side of Eq. (1). For each u_i in $V(G)$ there is an interval J_{u_i} with $J_{u_i} \cap g^*(u_i) \neq \emptyset$ that intersects $M_{g^*}(u_i)$ intervals $g^*(u_k)$ s. Without loss of generality, we may assume that each J_{u_i} is a non-trivial subinterval of $g^*(u_i)$ and all J_{u_i} s are pairwise disjoint. For each i we may choose an I -optimal interval-labeling h_i^* of H_i such that each $h_i^*(v_{ij})$ is a subinterval of J_{u_i} . Define the mapping f' on $V(G \wedge H)$ by

$$f'(x) = \begin{cases} g^*(x), & \text{if } x \in V(G); \\ h_i^*(x), & \text{if } x \in V(H_i). \end{cases}$$

It is clear that f' is an interval-labeling of $G \wedge H$ with

$$\begin{aligned} P(G \wedge H) &\leq I_{f'}(G \wedge H) \\ &= I_{g^*}(G) + \sum_{i=1}^n I_{h_i^*}(H_i) + \sum_{j=1}^m \sum_{i=1}^n b_{ij}, \end{aligned}$$

where b_{ij} is the number of intervals $g^*(u_k)$ s that intersects the interval $h_i^*(v_{ij})$. Then $b_{ij} \leq M_{g^*}(u_i)$ and so

$$\begin{aligned} P(G \wedge H) &\leq I_{g^*}(G) + \sum_{i=1}^n P(H_i) + \sum_{j=1}^m \sum_{i=1}^n M_{g^*}(u_i) \\ &= I_{g^*}(G) + nP(H) + M_{g^*}(G)m. \end{aligned}$$

Both inequalities prove the theorem. $\square$

Consequently, we have the following lower bound for the profile of the corona of two graphs.

Theorem 3. *If G and H are two graphs of orders n and m , respectively, then*

$$P(G \wedge H) \geq P(G) + nP(H) + M(G)m.$$

Proof. The theorem follows from Lemma 2 and the fact that $I_g(G) \geq P(G)$ and $M_g(G) \geq M(G)$ for any interval-labeling g of G . $\square$

The minimization in Eq. (1) depends not only on G but also on the value m . It is also the case that the minimization is attained for the “mixed” value of both I_g and M_g , rather than individual I_g and M_g . However, if there is a g^* that attains the minimum individually for I_g and M_g at the same time, then the lower bound in Theorem 3 is in fact the exact value.

Theorem 4. *If G has an interval-labeling g^* that is both I -optimal and M -optimal, then*

$$P(G \wedge H) = P(G) + nP(H) + M(G)m.$$

Proof. The theorem follows from Lemma 2 and the fact that g^* attains the minimum of the right-hand side of Eq. (1). $\square$

Next, we establish an upper bound for the profile of the corona of two graphs in terms of their profiles.

Theorem 5. *If G and H are two graphs of orders n and m , respectively, then*

$$P(G \wedge H) \leq (m+1)P(G) + nP(H) + nm.$$

Proof. Let g be a profile numbering of G and h be a profile numbering of H . Define a numbering f of $G \wedge H$ as follows:

$$f(x) = \begin{cases} (m+1)g(x), & \text{if } x \in V(G); \\ (m+1)(g(y)-1) + h(x), & \text{if } x \in V(H) \text{ and } (x, y) \in E(G \wedge H). \end{cases}$$

Then $P(G \wedge H) \leq P_f(G \wedge H) \leq (m+1)P(G) + nP(H) + nm$. $\square$

The upper bound in Theorem 5 is tight as illustrated by $E_n \wedge H$, where E_n is the complement of the complete graph K_n . Note that $E_n \wedge H$ is n separate copies of $K_1 \wedge H$ whose profile $P(K_1 \wedge H) = P(H) + m$, and then $P(E_n \wedge H) = n(P(H) + m) = (m+1)P(E_n) + nP(H) + nm$.

3. Exact values

This section establishes exact values of the profiles of coronas $G \wedge H$ by means of Theorem 4. These include when G is a caterpillar, a complete graph or a cycle, for which the values of $P(G)$ and $M(G)$ are also determined.

We first consider the case when G is a caterpillar. A caterpillar is a tree from whom the removing of all leaves results a path (possibly empty). More precisely, suppose $n = r + \sum_{1 \leq i < r} s_i$, where $r \geq 2$ and each $s_i \geq 0$. A *caterpillar* with the parameters $(n; r; s_2, s_3, \dots, s_{r-1})$ is the tree T with

vertex set $V(T) = \{x_1, x_2, \dots, x_r\}$

$$\cup \bigcup_{1 \leq i < r} \{y_{i1}, y_{i2}, \dots, y_{is_i}\}$$

and

edge set $E(T) = \{x_1x_2, x_2x_3, \dots, x_{r-1}x_r\}$

$$\cup \bigcup_{1 \leq i < r} \{x_iy_{i1}, x_iy_{i2}, \dots, x_iy_{is_i}\}.$$

Theorem 6. *If T is a caterpillar with parameters $(n; r; s_2, s_3, \dots, s_{r-1})$ and H is a graph of m vertices, then $P(T) = n - 1$ and $M(T) = 2n - r$ and*

$$\begin{aligned} P(T \wedge H) &= P(T) + nP(H) + M(T)m \\ &= n - 1 + nP(H) + (2n - r)m. \end{aligned}$$

Proof. First, it is clear that $P(T) \geq |E(T)| = n - 1$.

Next, suppose g is an M -optimal interval-labeling of T . Suppose u is the vertex for which $g(u)$ has the smallest left endpoint, and v is the vertex other than u for which $g(v)$ has the largest right endpoint. Let Q be the u - v path in T , and assume it has q vertices. Then, $\bigcup_{w \in V(Q)} g(w)$ is an interval in the real line that includes $g(z)$ for any vertex $z \in V(T)$. Consequently, $M_g(w) \geq 1$ for each $w \in V(Q)$ and $M_g(z) \geq 2$ for each $z \in V(T) - V(Q)$. As $q \leq r$, we have $M(T) \geq q + 2(n - q) = 2n - q \geq 2n - r$.

On the other hand, define an interval-labeling g^* by

$$g^*(z) = \begin{cases} [4i, 4i + 5], & \text{if } z = x_i, 1 \leq i \leq r; \\ [4i + 2 + \frac{2j}{2(s_i+1)}, 4i + 2 + \frac{2j+1}{2(s_i+1)}], & \text{if } z = y_{ij}, 1 < i < r, 1 \leq j \leq s_i. \end{cases}$$

It is clear that g^* is an interval-labeling of T with $I_{g^*}(T) = n - 1$ and $M_{g^*}(T) = 2n - r$. Consequently, g^* is both I -optimal and M -optimal, and $P(T) = n - 1$ and $M(T) = 2n - r$. The theorem then follows from Theorem 4. $\square$

We next consider the case when G is the complete graph K_n .

Theorem 7. *If K_n is a complete graph of n vertices and H is a graph of m vertices, then $P(K_n) = n(n-1)/2$ and $M(K_n) = \lfloor (n+1)^2/4 \rfloor$ and*

$$\begin{aligned} P(K_n \wedge H) &= P(K_n) + nP(H) + M(K_n)m \\ &= n(n-1)/2 + nP(H) \\ &\quad + \lfloor (n+1)^2/4 \rfloor m. \end{aligned}$$

Proof. First, it is clear that $P(K_n) \geq |E(K_n)| = n(n-1)/2$.

Next, suppose g is an M -optimal interval-labeling of K_n . We recursively define $u_1, u_2, \dots, u_{\lfloor n/2 \rfloor}$ and $v_1, v_2, \dots, v_{\lfloor n/2 \rfloor}$ as follows. Having defined u_j and v_j for all $j < i$, let u_i be the vertex (other than the above vertices) for which $g(u_i)$ has the smallest left endpoint, and v_i is the vertex (other than the above vertices and u_i) for which $g(v_i)$ has the largest right endpoint. Repeat this process until all vertices of K_n are scanned. It is clear that for each $1 \leq i \leq \lfloor n/2 \rfloor$ we have that $g(u_i) \cup g(v_i)$ is an interval in the real line that includes $g(u_j)$ and $g(v_j)$ for all $j \geq i$. Consequently, $M_g(u_i) \geq i$ for $1 \leq i \leq \lfloor n/2 \rfloor$ and

$M_g(v_i) \geq i$ for $1 \leq i \leq \lfloor n/2 \rfloor$. Then $M_g(K_n) \geq 1 + 2 + \dots + \lfloor n/2 \rfloor + 1 + 2 + \dots + \lfloor n/2 \rfloor = \lfloor (n+1)^2/4 \rfloor$.

On the other hand, suppose $V(K_n) = \{x_1, x_2, \dots, x_n\}$. Define g^* by $g^*(x_i) = [i, i+n]$ for $1 \leq i \leq n$. It is clear that g^* is an interval-labeling of K_n with $I_{g^*}(K_n) = n(n-1)/2$ and $M_{g^*}(K_n) = \lfloor (n+1)^2/4 \rfloor$. Consequently, g^* is both I -optimal and M -optimal, and $P(K_n) = n(n-1)/2$ and $M(K_n) = \lfloor (n+1)^2/4 \rfloor$. The theorem then follows from Theorem 4. $\square$

Finally, we consider the case when G is the cycle C_n .

Theorem 8. *If C_n is a cycle of n vertices and H is a graph of m vertices, then $P(C_n) = 2n - 3$ and $M(C_n) = 2n - 2$ and*

$$\begin{aligned} P(C_n \wedge H) &= P(C_n) + nP(H) + M(C_n)m \\ &= 2n - 3 + nP(H) + (2n - 2)m. \end{aligned}$$

Proof. We first prove that $P(C_n) \geq 2n - 3$ by induction on n . The case of $n = 3$ is clear as $P(C_3) \geq |E(C_3)| = 3$. Suppose $P(C_n) \geq 2n - 3$ for $n \geq 3$. Choose an I -optimal interval-labeling g of C_{n+1} . Let x be the vertex whose $g(x)$ has a smallest right endpoint. Then its two neighbors y and z both contain this right endpoint. Consequently $g(y) \cap g(z) \neq \emptyset$. Hence the restriction g' of g on $V(C_{n+1}) - \{x\}$ is an interval-labeling of C_n . Then $P(C_{n+1}) = I_g(C_{n+1}) \geq 2 + I_{g'}(C_n) \geq 2 + P(C_n) \geq 2 + 2n - 3 = 2(n+1) - 3$. Therefore, $P(C_n) \geq 2n - 3$ for all $n \geq 3$.

Next, suppose g is an M -optimal interval-labeling of C_n . Let u be the vertex for which $g(u)$ has the smallest left endpoint, and v be the vertex other than u for which $g(v)$ has the largest right endpoint. Then C_n is the union of two internally vertex-disjoint $u-v$ paths Q_1 and Q_2 . It is the case that $g(x) \subseteq \bigcup_{y \in V(Q_{3-i})} g(y)$ for each vertex $x \in V(Q_i)$, $i = 1, 2$. Consequently, $M_g(x) \geq 2$ for all vertices in C_n except $M_g(u) \geq 1$ and $M_g(v) \geq 1$. Then $M_g(C_n) \geq 2(n-2) + 1 + 1 = 2n - 2$.

On the other hand, suppose $V(C_n) = \{x_1, x_2, \dots, x_n\}$. Define g^* by $g^*(x_i) = [2i, 2i+3]$ for $1 \leq i \leq n-1$ and $g^*(x_n) = [3, 2n+2]$. It is clear that g^* is an interval-labeling of C_n with $I_{g^*}(C_n) = 2n - 3$ and $M_{g^*}(C_n) = 2n - 2$. Consequently, g^* is both I -optimal and M -optimal, and $P(C_n) = 2n - 3$ and

$M(C_n) = 2n - 2$. The theorem then follows from Theorem 4. $\square$

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