ORDINARY $p$-ADIC EISENSTEIN SERIES AND $p$-ADIC $L$-FUNCTIONS FOR UNITARY GROUPS

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Abstract. The purpose of this work is to carry out the first step in our four-step program towards the main conjecture for $GL_2 \times K^\times$ by the method of Eisenstein congruence on $GU(3,1)$, where $K$ is an imaginary quadratic field. We construct a $p$-adic family of ordinary Eisenstein series on the group of unitary similitudes $GU(3,1)$ with the optimal constant term which is basically the product of the Kubota-Leopoldt $p$-adic $L$-function and a $p$-adic $L$-function for $GL_2 \times K^\times$. This construction also provides a different point of view of $p$-adic $L$-functions of $GL_2 \times K^\times$.

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Introduction

Iwasawa main conjecture for totally real fields is proved by Wiles in one of his celebrated papers [Wil90]. His proof, modeled upon Ribet’s proof of the converse of Herbrand’s theorem, relies on the study of Eisenstein congruence for Hilbert cusp forms. One of the key ingredients in Wiles’ proof is to realize the Deligne-Ribet $p$-adic $L$-function as constant terms of a particular Hida family of Eisenstein series over totally real fields, and then construct congruence between this particular family of Eisenstein series and Hida families of Hilbert cusp forms modulo the Deligne-Ribet $p$-adic $L$-function.

The first generalization of Wiles’ work to Iwasawa main conjecture for non-abelian motives is due to E. Urban. In [Urb01] and [Urb06], he established one-sided divisibility result on Iwasawa main conjecture for adjoint representations. In particular, in [Urb06] he constructed a Hida family of Klingen-Eisenstein series on $GSp(4)$ such that the constant terms at all cusps are divisible by the $p$-adic $L$-function for adjoint representations, and he deduced Eisenstein congruence on $GSp(4)$ by proving the non-vanishing modulo $p$ of this Eisenstein series.

In this article, we consider certain Iwasawa main conjecture for $GL_2 \times K^\times$, where $K$ is an imaginary quadratic field. Following Ribet, Wiles and Urban, we propose a four-step program towards a one-sided divisibility result for this main conjecture. It turns out that we will need to consider Eisenstein congruence on certain unitary groups. Our main result is the construction of a particular Hida family of Eisenstein series on the unitary group $U(3,1)$ such that the constant terms (the image of Siegel boundary operator) at all cusps are divisible by a product of Kubota-Leopoldt $p$-adic $L$-function and a $p$-adic $L$-function for $GL_2 \times K^\times$.

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Main conjecture for $p$-adic Galois representations. In [Gre94], R. Greenberg reformulates the classical Iwasawa main conjecture in the context of $p$-adic Galois representations, and he proposes a more general main conjecture when the Galois representation satisfies the Panchishkin condition (for the definition, see [Gre94, §3 and §4]).

We now describe the Galois representation under consideration. Let $p$ be an odd rational number. Assume $p$ is split in $K$. Fix an embedding $i_p : \overline{\mathbb{Q}} \to \mathbb{C}$ and an isomorphism $\iota : \mathbb{C} \cong \mathbb{C}_p$, where $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$. Let $\iota_p = \iota_\infty \circ \overline{\mathbb{Q}} \to \mathbb{C}_p$ be their composition. Let $p$ and $\mathfrak{p}$ be primes in $K$ above $p$, where $p$ is the prime ideal induced by $\iota_p$. Let $G_K := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\hat{\chi} : G_K \to \mathbb{C}^\times$ be a $p$-adic character. We shall regard $\hat{\chi}$ as a character of $\hat{A}_K^\times$ by the geometrically normalized reciprocity law. We assume $\hat{\chi}$ is locally algebraic, namely there exist two integers $a$ and $b$ such that $\hat{\chi}(z_p) = \frac{z_p^a}{z_\mathfrak{p}^{b-a}}$ for $z_p = (z_p, z_\mathfrak{p})$ in the $p$-part of $A_K$ with $z_p$ and $z_\mathfrak{p}$ sufficiently close to 1 $p$-adically. In addition, the map $\hat{\chi} : \hat{A}_K^\times / K^\times \to \mathbb{C}^\times$ defined by

$$\hat{\chi}(z) = \iota_\infty \iota_p^{-1} (\hat{\chi}(z_f)) z_p^{-b-a} \frac{1}{\mathfrak{p}^b} z_\mathfrak{p}^{-a}$$

is a well-defined Hecke character. The Hecke character $\chi$ is called the complex avatar of $\hat{\chi}$ whereas $\hat{\chi}$ is called the $p$-adic avatar of $\chi$. We will say $\chi$ (or $\hat{\chi}$) has infinity type $(b, a)$. Let $E$ be an elliptic curve over $\mathbb{Q}$ and let $\rho_E : G_\mathbb{Q} \to \text{GL}(H^1_\text{et}(E, \mathbb{Q}_p))$ be the $p$-adic Galois representation associated with $E$. Let $\rho_E \otimes \hat{\chi} := \rho_E|_{G_K} \otimes \hat{\chi}$ be the $p$-adic Galois representation of $G_K$. We will consider the main conjecture for $\rho_E \otimes \hat{\chi}$ when $\hat{\chi}$ varies in a $p$-adic family of a fixed infinity type.

The formulation of this main conjecture depends on the infinity type of $\chi$. If the infinity type of $\hat{\chi}$ is $(1, 1)$ and $E$ has good ordinary reduction at $p$, the main conjecture for $\rho_E \otimes \hat{\chi}$ is the classical Iwasawa main conjecture for $E$ over $K$, which is the assertion for the equality between the $p$-adic $L$-function of $L_K(0, E \otimes \chi)$ divided by the period of $E$ and the characteristic power series associated to the Selmer group of $E \otimes \chi$. In this case, the conjecture has been settled down by Bertolini and Darmon in [BD05] if $\chi$ is further assumed to be anticyclotomic, and the general three-variable main conjecture for $\text{GL}_2 \times \mathbb{K}^\times$ is studied by Skinner and Urban in [SU08].

We are interested in the main conjecture for $\rho_E \otimes \hat{\chi}$ when the infinity type of $\hat{\chi}$ is $(k, 0)$, $k > 1$. This main conjecture is quite different from the one considered in [BD05] and [SU08]. On the analytic side, the $p$-adic $L$-function is related to the complex $L$-value $L_K(0, E \otimes \chi)$ divided by the CM-period attached to $\chi$ and $K$ instead of the period of $E$. On the algebraic side, the Selmer group also has different local conditions at $p$. Let us make precise the $L$-value and the Selmer group under consideration.

$L$-functions and Selmer groups.

$L$-functions. Thanks to the works of Wiles, Taylor and many other people, one can associate to $E$ a weight two holomorphic cuspidal eigenform $f = \sum_{n=1}^{\infty} a_n(E)q^n$. We choose a prime-to-$p$ integral ideal $\epsilon$ of $\mathbb{Q}$ such $\epsilon$ is divisible by the conductor of $E$. We let $S$ be the set of prime factors of $\epsilon$ and put

$$L_{K}^{S_\epsilon}(s, \rho_E \otimes \chi) = \sum_{(n, \mathfrak{p}) = 1} a_n(E)\chi(n)\|n\|^{-s}, \quad (n) = n \cap \mathbb{Z},$$

where $n$ runs over integral ideals of $K$.

Throughout we assume $E$ has ordinary good reduction at $p$. In order to have nice arithmetic and $p$-adic properties of the above $L$-value when $\chi$ varies in a $p$-adic family, we need to take normalization as follows. We begin with some notations. Let $\alpha_1$ and $\alpha_2$ be two roots of the equation $x^2 - a_p(E)x + p$, and we define $\delta_1$ and $\delta_2$ to be the unramified characters on $\mathbb{Q}_p^\times$ such that $\delta_i(p) = \alpha_i, \quad i = 1, 2$. For a character $\mu : \mathbb{Q}_p^\times \to \mathbb{C}$, we let $L(s, \mu)$ be the local $L$-factor associated to $\mu$. Following [Coo91, Lemma 7], we define the modified Euler factor $E_{p}(0, \mu)$ by

$$E_p(0, \mu) = \frac{L(0, \mu)}{\epsilon(0, \mu)L(1, \mu^{-1})},$$

where $\epsilon(s, \mu)$ is Tate’s $\epsilon$-factor of $\mu$. Now we make the following definition of the normalized $L$-value.
Definition 1.

\[ L_{\mathcal{K}}^{\text{alg},S}(0, \rho_E \otimes \chi) := \frac{\Gamma(k) \Gamma(k-1)}{(2\pi i)^{2k-1}} \cdot E_p(0, \chi p \delta_1) E_p(0, \chi p \delta_2) \times \frac{(2\pi i)^{2k}}{\Omega_{\mathcal{K}}^{k}} \cdot L_{\mathcal{K}}^{\text{S}^0}(0, \rho_E \otimes \chi), \]

where \( \Omega_{\mathcal{K}} \) is the CM period associated to \( \mathcal{K} \).

We remark that \( L_{\mathcal{K}}^{\text{alg},S}(0, \rho_E \otimes \chi) \) actually lies in \( \mathbb{T} \) by works of Shimura and that such normalization is suggested by J. Coates’ recipe ([Coa91, Conj. A p.168]).

To define \( p \)-adic \( L \)-functions, we need to introduce more notations. Let \( \mathcal{K}_\infty \) be the maximal \( \mathbb{Z}_p^2 \)-extension and \( \Gamma = \text{Gal}(\mathcal{K}_\infty/\mathcal{K}) \cong \mathbb{Z}_p^2 \). Let \( \psi \) be a branch character. Namely \( \psi \) is a character of \( G_\mathcal{K} \) of finite order such that \( \mathcal{K}_\psi = \mathcal{K}^{\text{ker} \psi} \) is linearly disjoint from \( \mathcal{K}_\infty \). Let \( \Delta = \text{Gal}(\mathcal{K}_\psi/\mathcal{K}) \). Let \( \mathcal{K}_\psi' = \mathcal{K}_\psi^\infty \mathcal{K}_\infty \) and \( \mathcal{G} = \text{Gal}(\mathcal{K}_\infty/\mathcal{K}) \). Then \( \mathcal{G} = \Delta \times \Gamma \). Put \( \mathcal{O} = \mathbb{Z}_p[\text{Im} \psi] \), the ring of values of \( \psi \). Let \( \Lambda \) be \( \mathcal{O}[[\Gamma]] \) a two-variable Iwasawa algebra. Let \( \Psi \) be the \( \Lambda \)-valued character of \( G_\mathcal{K} \) defined by

\[ \Psi : G_\mathcal{K} \longrightarrow \Lambda^\times \]

\[ g \longrightarrow \psi(g)|_{\mathcal{K}_\infty}. \]

Let \( \mathcal{W}_0 \) be the set of locally algebraic \( p \)-adic characters of \( \Gamma \) of infinity type \((k,0), k \geq 2 \). For \( x \in \mathcal{W}_0 \) and \( \lambda \in \Lambda \), put \( \lambda(x) = x \circ \lambda \in \mathcal{C}_p \). We let \( \Psi_\lambda \) denote the complex avatar of \( x \circ \Psi \). Then \( \mathcal{W}_0 \) can be regarded as a Zariski-dense subset in \( \text{Spec} \Lambda(\mathcal{G}_p) \). Moreover \( \mathcal{W}_0 \) is an ample subset of the set of critical specializations for \( \rho_E \otimes \Psi \) in the sense of Greenberg ([Gre94, §4]).

We further assume \( \iota \) is divisible by the prime-to-\( p \) conductor of \( \psi \). Let \( L_{p}^{S}(\rho_E \otimes \Psi) \) denote the unique element in \( \Lambda \) such that

\[ L_{p}^{S}(\rho_E \otimes \Psi)(x) = t_p^{-1}(L_{\mathcal{K}}^{\text{alg},S}(0, \rho_E \otimes \Psi_\lambda)), \]

\( x \in \mathcal{W}_0 \).

The existence of \( L_{p}^{S}(\rho_E \otimes \Psi) \) is known. In fact it is a special case of \( p \)-adic Rankin products constructed by Hida (cf. [Hid93]). We call \( L_{p}^{S}(\rho_E \otimes \Psi) \) the (non-primitive) \( p \)-adic \( L \)-function for \( \text{GL}_2 \times \mathcal{K}_\infty^\times \) associated to \( \mathcal{E} \) and \( \psi \).

Selmer groups. Let \( \mathcal{K}_S \) be the maximal \( S \cup \{p\}\)-ramified extension of \( \mathcal{K} \). Let \( \Lambda^* = \text{Hom}_{\text{cont}}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p) \) be the Pontryagin dual of \( \Lambda \). Let \( T = H^1_{I}(E, \mathbb{Z}_p) \). We consider a discrete a \( \Lambda \)-module \( T \otimes \Lambda^* \) of corank two equipped with Galois action by \( \rho_E \otimes \Psi \). According to the Panchishkin condition of \( \rho_E \otimes \Psi_\lambda \) for \( x \in \mathcal{W}_0 \), we define the non-primitive \( \Lambda \)-adic Selmer group to \( \rho_E \otimes \Psi \) by

\[ \text{Sel}_{\mathcal{K}}^{S}(\rho_E \otimes \Psi) := \ker \{ H^1(\mathcal{K}_S, T \otimes \Lambda^*) \rightarrow H^1(T_{\mathcal{G}}, T_{\mathcal{G}} \otimes \Lambda^*) \}. \]

If \( S \) is empty, we shall drop \( S \) in the above definition and simply write \( \text{Sel}_{\mathcal{K}}(\rho_E \otimes \Psi) \). It is well-known that \( \text{Sel}_{\mathcal{K}}^{S}(\rho_E \otimes \Psi) \) is a discrete and finitely generated \( \Lambda \)-module. Let \( \text{ht}_1(\Lambda) \) denote the set of height one prime ideals of \( \Lambda \). For a finitely generated \( \Lambda \)-module \( S \), let \( S^* \) be the Pontryagin dual of \( S \). For \( P \in \text{ht}_1(\Lambda) \), put

\[ \ell_P(S) = \text{length}_{\Lambda_P}(S^* \otimes \Lambda_P). \]

The main conjecture: connection between \( L \)-functions and Selmer groups. The following main conjecture for \( \rho_E \otimes \Psi \) is formulated in ([Gre94, Conj. 4.1]).

Conjecture 1 (The Main Conjecture).

(1) \( \text{Sel}_{\mathcal{K}}(\rho_E \otimes \Psi) \) is cotorision over \( \Lambda \).

(2) For any \( P \in \text{ht}_1(\Lambda) \),

\[ \text{ord}_P(L_p(\rho_E \otimes \Psi)) = \ell_P(\text{Sel}_{\mathcal{K}}(\rho_E \otimes \Psi)). \]

Remark 1. The formulation of the main conjectures depends not only on the Galois representation itself but also the choice of an ample subset of critical specializations (or a choice of local conditions at \( p \)). The main conjectures studied in [BD05] and [SU08] are for the same Galois representation \( \rho_E \) with the critical specialization at locally algebraic characters of \( \Gamma \) with infinity type \((1,1) \). The \( p \)-adic \( L \)-function and Selmer group considered in this article are quite different from theirs.
We shall consider the dual version of Conjecture 1 which has the advantage of including non-primitive $p$-adic $L$-functions and Selmer groups. Let $\epsilon : G_K \to \mathbb{Z}_p^\times$ be the cyclotomic character and let $c$ be the complex conjugation. We define the $\Lambda$-valued Galois character $\Psi^\perp$ by
\[
\psi^\perp(g) = \psi^{-1}\epsilon^2(egc).
\]

**Conjecture 2.** For every $P \in \text{ht}_1(\Lambda)$,
\[
\text{ord}_P(L^S_p(\rho_E \otimes \Psi)) = \ell_P(Sel^S_E(\rho_E \otimes \Psi^\perp)).
\]

We propose the following weaker conjecture.

**Conjecture 3 (Lower bound of Selmer groups).** For every $P \in \text{ht}_1(\Lambda)$,
\[
\text{ord}_P(L^S_p(\rho_E \otimes \Psi)) \leq \ell_P(Sel^S_E(\rho_E \otimes \Psi^\perp)).
\]

We wish to show Conjecture 3 by the method of Eisenstein congruences.

**Eisenstein congruences.** The method of congruences among modular forms provides a general strategy to construct a nontrivial Galois extension. Roughly speaking, a Galois representation over a PID which is generically irreducible and residually reducible should give rise to a nontrivial Galois extension. This basic idea was due to K. Ribet in [Rib76].

In our project, we consider the congruences between Eisenstein series and cusp forms on the unitary groups $U(3, 1)$. The application of Eisenstein congruences to various main conjectures has been a success in [MW84], [Wil90], [Urb01] and [SU08]. Following the exposition in [Ski06], we describe our project as follows.

1. Construct an ordinary $\Lambda$-adic Eisenstein series $E_{\text{ord}}^{\text{ord}}$ on the unitary group $U(3, 1)$ with the constant terms at all cusps are divisible by $L^S_p(-2, \Psi_+) \cdot L^S_p(\rho_E \otimes \Psi)$, where $\Psi_+ = \Psi \circ V$ is the composition of $\Psi$ and the Verschiebung map $V : G_Q^{ab} \to G_K^{ab}$, and $L^S_p(-2, \Psi_+)$ is an imprimitive $p$-adic $L$-function with the following specialization property:
\[
L^S_p(-2, \Psi_+)(x) = \ell_P(L^S_{\text{alg}}(-2, \Psi_+(x)), \ x \in \mathfrak{m}_0.
\]

2. Show that $E_{\text{ord}}^{\text{ord}} \not\equiv 0$ mod $\mathfrak{m}_\Lambda$, where $\mathfrak{m}_\Lambda$ is the maximal ideal of $\Lambda$.

3. For $P \in \text{ht}_1(\Lambda)$, we let $r_c = \text{ord}_P(L^S_p(-2, \Psi_+))$ and $r_s = \text{ord}_P(L^S_p(\rho_E \otimes \Psi))$. Using Hida theory of ordinary $p$-adic modular forms, we can show that there is a $p$-adic ordinary cusp form $\mathcal{F}$ such that
\[
E_{\text{ord}}^{\text{ord}} \equiv \mathcal{F} \mod P^{r_c + r_s}.
\]

4. We use the Galois representations associated to cuspidal automorphic representations of $U(3, 1)$ together with the above congruences to construct elements in the Selmer group $Sel^S_E(\rho_E \otimes \Psi^\perp)$. The key here is to perform the lattice construction ([Urb01] and [SU08]). Assuming the existence of the Galois representation attached to $\mathcal{F}$, in favorable cases for instance $r_c = 0$, we can construct a $\Lambda$-lattice $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ in $K^4 = (\text{Frac} \Lambda)^4$ with $\dim \mathcal{L}_1 \otimes K = \dim \mathcal{L}_2 \otimes \mathcal{L}_3 \otimes K = 1$ and a representation $\rho_F$.
\[
\rho_F : G_K \to \text{GL}(\mathcal{L}), \quad \rho_F(\sigma) = \begin{pmatrix}
\psi^{-1}(c\sigma c) & * & * & *
\end{pmatrix} \mod P^{r_c},
\]

which is unramified away from $S \cup \{p\}$. Moreover under the assumption that $r_c = 0$ we can show
\[
\rho_F(\sigma) = \begin{pmatrix}
\rho_E\epsilon^{-1}(\sigma) & c_\sigma \\
0 & \psi\epsilon^{-3}(\sigma)
\end{pmatrix} \mod P^{r_c},
\]
is indeed a non-split representation of $G_K$, but $\rho_F|_{D_\sigma}$ is split. Then it can be shown that the cocycle $c_\sigma$ provides a $\Lambda_F$-submodule of length at least $r_s$ in $Sel^S_E(\rho_E \otimes \Psi^\perp)_P$. 


The main result. Our main result in this paper is to fulfill Step (1) of this program when $E$ has good ordinary reduction at $p$, namely $p \not| \phi_p(E)$. Let $\pi = \pi_f$ be the automorphic representation of $G_L$ generated by $f$. We further assume that the conductor $N_E$ of $E$ has a decomposition $N_E = N_+ N_-$ such that $(N_+, N_-) = 1$ and $N_-$ is a product of an odd number of distinct primes. This is equivalent to saying that $p = JL(\pi^B)$ is in the image of Jacquet-Langlands correspondence of the definite quaternion algebra $B$ ramified exactly at infinity and $N_-$. Let $f$ be a $p$-primitive new form in $\pi^B$ and choose $\xi$ an auxiliary Dirichlet character of $G_K$ such that $\xi_+ = 1$. Thus $F := f \otimes \xi$ can be regarded as a modular form of $GU(2)$ (See §4.1). Let $D_K$ be the discriminant of $K$. We further assume $c$ is divisible by the conductors of $\psi$ and $\xi$, $D_K$ and $N_E$.

Our main result is as follows.

**Theorem 1.** Under the above assumptions there exists a measure $d\xi^\ord \chi^\ord f^\ord \xi^\ord c \xi^\ord c$ with values in the space of $p$-adic modular forms on $GU(3, 1)$ such that for any $\chi$ a $p$-adic algebraic character of $\mathfrak{S}$ with infinity type $(k, 0)$, $k \geq 4$, then we have

$$\int_{\mathbf{Q}_p} \chi^\ord d\xi^\ord c = E^\ord(\chi | f, \xi, c),$$

where $E^\ord(\chi | f, \xi, c)$ is an ordinary $p$-adic Eisenstein series. Let $d\xi^\ord c(x)$ be the $p$-adic measure induced by the constant term of $E^\ord(\chi | f, \xi, c)$ at a cusp $x$, which is a measure with value in the space of $p$-adic modular forms of $GU(2)$. Then there exists a well-chosen unramified cusp $x_0$ and a $p$-adic period $\Omega_p \in \mathbb{Z}_p^{\times}$ depending on $x_0$ such that

$$\frac{1}{\Omega_p} \cdot \int_{\mathbf{Q}_p} \chi^\ord d\xi^\ord c(x_0) = |D_K|^{\frac{1}{2}} (p - 1) \cdot (\hat{\chi} \xi^{-1}) p(- \det \theta \cdot p) \times L^{alg,S}(-2, \chi_+) \cdot L^{alg,S}(0, \rho_E \otimes \chi) \cdot F.$$

Moreover $E(\chi | f, \xi, c)$, the ideal generated by the constant term at all cusps, is

$$E(\chi | f, \xi, c) = L^{alg,S}(-2, \chi_+) \cdot L^{alg,S}(0, \rho_E \otimes \chi).$$

Let us put a few words on the current progress about Step (2). To carry out Step (2), one usually needs

- an explicit computation of Fourier-Jacobi coefficients of an Eisenstein series, and
- a non-vanishing result of $L$-values mod $p$ that are related to the above one.

Bei Zhang in her thesis [Zha07] has made a preliminary computation of Fourier-Jacobi coefficients of an Eisenstein series on our unitary groups. Her computation does relate the Fourier-Jacobi coefficients of Eisenstein series to certain $L$-values for $GL_2 \times K^\times$, but the desired non-vanishing mod $p$ result of the corresponding $L$-values is not available yet. Nonetheless a wishing thinking (if we believe the main conjecture) is that since our Eisenstein series has the optimal constant term, Step (2) should hold as well. We hope to work out this problem in the future.

**Sketch of the construction.** The easiest way to construct an Eisenstein measure is to interpolate its Fourier coefficients as we have seen in the examples of classical $p$-adic Eisenstein series for $GL_2$. However it is difficult to do so for our non-quasi-split unitary group $GU(3, 1)$, since modular forms for this group only have Fourier-Jacobi expansion, which is difficult to interpolate directly. To bypass this difficulty, our approach is as follows.

- Construct a nice $p$-adic Siegel-Eisenstein series on the quasi-split unitary group $GU(3, 3)$.
- Apply the pull-back formula to obtain a $p$-adic Eisenstein series on $GU(3, 1)$.
- Construct an ordinary $p$-adic Eisenstein series on $GU(3, 1)$ by taking the ordinary projection.

To study congruences, we further require that this ordinary Eisenstein series have optimal constant terms.

Our construction of Siegel-Eisenstein series on $GU(3, 3)$ was inspired by [Kat78], [HT91] and [HLS06]. In particular, Harris, Li and Skinner in [HLS06] generalize Katz’s construction in [Kat78] to obtain a $p$-adic Siegel-Eisenstein series of several variables with which they can construct $p$-adic $L$-functions for general unitary groups. However the ordinary projection of the pull-back of their Eisenstein series is zero. A heuristic reason is that the Fourier coefficients of the Eisenstein in [HLS06] are only supported in the matrices which are non-degenerate modulo $p$. Therefore, we need to modify their section to fit our purpose. The new ingredients in our paper are the choice of this modified section at $p$ (3.15) and also the calculation of the ordinary projection of its local pull back section in §6.4. Here is the summary of this paper:
In §2, we review the theory of $p$-adic modular forms on unitary groups. This theory is due to Hida ([Hid02] and [Hid04]).

In §3, following the same spirit in [HLS06], we construct the Siegel-Eisenstein series on $GU(3,3)$ and compute its Fourier coefficients explicitly.

In §4, we review the pull-back formula in [Shi97], and in §5, we study the constant term of the pull back of our Siegel-Eisenstein series. It turns out that the constant term is simply the pull back section itself. We also calculate the local pull back section at places other that $p$ in this section.

In §6, we calculate the ordinary projection of the pull back section at the place $p$ by employing Jacquet’s functor. We find that indeed it has correct modified Euler factors at $p$ as suggested by J. Coates. The main result of this section is Prop. 6.7.

In §7, we give the explicit formula for the constant term Theorem 7.2.

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1. Notation and definitions

1.1. Throughout $\mathcal{K}$ is an imaginary quadratic field with the ring of integers $\mathcal{O}_\mathcal{K}$. Let $D_\mathcal{K}$ (resp. $D_\mathcal{K}^-$) be the discriminant (resp. different) of $\mathcal{K}$.

Fix an odd rational prime $p$ split in $\mathcal{K}$. Fix an embedding once and for all $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and an isomorphism $\iota : \mathbb{C} \cong \mathbb{C}_p$, where $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$. Let $\iota_p = \iota\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ be their composition. Write $p\mathcal{O}_\mathcal{K} = p\mathcal{O}_\mathcal{K}$, where $p$ is the prime ideal induced by $\iota_p$.

1.2. For a finite set $\square$ of rational primes, we define $\mathbb{Z}(\square)$ by

$$\mathbb{Z}(\square) = \left\{ \frac{a}{q} \in \mathbb{Q} \mid b\mathbb{Z} + q\mathbb{Z} = \mathbb{Z} \text{ for all } q \in \square \right\}.$$  

By definition, $\mathbb{Z}(\square) = \mathbb{Q}$ if $\square$ is empty. Write $\mathbb{Z}(p)$ for $\mathbb{Z}(\square)$ if $\square = \{ p \}$. Let $\mathbb{Z}(\square) = \{ a \in \mathbb{Z}(\square) \mid a > 0 \}$ and $\mathfrak{O} := \mathcal{O}_\mathcal{K} \otimes_{\mathbb{Z}} \mathbb{Z}(p)$.

1.3. Denote by $\text{SET}$ the category of sets and by $\text{SCH}_R$ the category of $R$-schemes for a ring $R$. The complex conjugation $c$ induces a natural involution on $R \otimes_{\mathbb{Z}} \mathcal{K}$ by $r \otimes x \mapsto r \otimes c(x)$. Define the $n \times n$ the Hermitian matrices $\mathcal{H}_n(R)$ over $R \otimes_{\mathbb{Z}} \mathcal{K}$ by

$$\mathcal{H}_n(R) = \{ g \in M_n(R \otimes_{\mathbb{Z}} \mathcal{K}) \mid g = g^* \},$$

where $g^* = c(g^t)$, $^t$ is the transpose of $g$.

1.4. We write

$$\mathfrak{O}_\mathcal{K} \otimes_{\mathbb{Z}} \mathfrak{O}_\mathcal{K} = \mathfrak{O}_\mathcal{K} e^+ \oplus \mathfrak{O}_\mathcal{K} e^-,$$

where $e^+$ (resp. $e^-$) is the idempotent corresponding to the identity map $\Sigma : \mathcal{K} \to \mathcal{K}$ (resp. the complex conjugation $\Sigma^c = \Sigma \circ c : \mathcal{K} \to \mathcal{K}$). If $M$ is an $\mathfrak{O}_\mathcal{K}$-module, we define

$$M_{\Sigma} := e^+(M \otimes_{\mathbb{Z}} \mathbb{Z}_p) \text{ and } M_{\Sigma^c} = e^-(M \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

Let $\mathbb{C}(\Sigma)$ (resp. $\mathbb{Z}_p(\Sigma)$) be the $\mathfrak{O}_\mathcal{K}$-module $\mathbb{C}$ (resp. $\mathbb{Z}_p$) on which $\mathfrak{O}_\mathcal{K}$ acts via $\iota_\infty$ (resp. $\iota_p$). Similarly let $\mathbb{C}(\Sigma^c)$ (resp. $\mathbb{Z}_p(\Sigma^c)$) be the $\mathfrak{O}_\mathcal{K}$-module $\mathbb{C}$ (resp. $\mathbb{Z}_p$) on which $\mathfrak{O}_\mathcal{K}$ acts through $c$. 


1.5. Unitary groups. Let \( r \geq s \) be two non-negative integers. Let \((W, \theta)\) be a skew-Hermitian space of dimension \( r - s \) such that \( \delta^{-1} \theta \) is positive definite, where \( \delta = -\sqrt{D_K} \) and \( D_K \) is the discriminant of \( \mathcal{K} \). We fix a \( \mathcal{K}\)-basis \( \{ w^i \}_{i=1}^{r-s} \) and we regard \( \theta \) as a \((r-s) \times (r-s)\) matrix according to this basis. We further assume \( t_p(\det \theta) = t_p(\det(\theta(w^i, w^j))) \) is a \( p \)-adic unit. Let \( V = \oplus_{i=1}^s \mathcal{K} y^i \oplus W \oplus_{i=1}^s \mathcal{K} x^i \) and \( \theta_{r,s} \) be the skew-Hermitian form on \( V \) such that according to the basis \( \{ y^i, w^i, x^i \} \),

\[
\theta_{r,s} = \begin{bmatrix}
\theta & -1_s \\
1_s & \theta
\end{bmatrix}.
\]

Let \( \langle \cdot, \cdot \rangle_{r,s} : V \times V \to \mathbb{Q} \) be the alternating pairing defined by \( \langle v, v' \rangle_{r,s} = \text{Tr}_{\mathcal{K}}(\theta_{r,s}(v, v')) \).

Let \( G = GU(r, s) \) be the group of unitary pairing defined by \( \langle V, \theta_{r,s} \rangle \), i.e. for a \( \mathbb{Q}\)-algebra \( R \),

\[
G(R) = \{ g \in \text{End}_{\mathcal{K}}(V \otimes R) \mid \langle \theta_{r,s}(xg, yg) = \nu(g) \theta_{r,s}(x, y), x, y \in V \otimes R \}.
\]

Let \( GU(0, r - s) \) be the group of unitary similitudes attached to \((W, -\theta)\). The unitary group \( U(V) \) is defined by

\[
U(V)(R) := \{ g \in G(R) \mid \nu(g) = 1 \}.
\]

1.6. Lattices and polarization. In what follows we make the specific choice of the lattice \( M \) in \( V \) and define the standard polarization \( \text{Pol}_p^0 \) of \( M_p \). We let \( X^r = \mathcal{O}_K x^1 \oplus \cdots \oplus \mathcal{O}_K x^r = \mathcal{O}_K^r \) and \( Y = \mathcal{O}_K y^1 \oplus \cdots \oplus \mathcal{O}_K y^s = \mathcal{O}_K^s \) be the standard \( \mathcal{O}_K \)-lattices in \( X^r \) and \( Y^s \) respectively. We choose an \( \mathcal{O}_K \)-lattice \( L \) in \( W \) such that \( L \) is \( \mathbb{Z}\)-maximal with respect to the Hermitian form \( \delta^{-1} \theta \). Let \( L_p = L \otimes \mathbb{Z}_p = \sum_{i=1}^{r-s} (\mathcal{O}_\mathcal{K} \otimes \mathbb{Z}_p) w^i = (\mathcal{O}_\mathcal{K} \otimes \mathbb{Z}_p)^{r-s} \). Then we define the \( \mathcal{O}_K \)-lattice \( M \) in \( V \) by

\[
M := Y \oplus L \oplus X^r.
\]

Let \( M_p = M \otimes \mathbb{Z}_p \). A pair of sublattices \( \text{Pol}_p^0 \) of \( M_p \) is called an ordered polarization of \( M_p \) if \( N^{-1} \) and \( N^0 \) are maximal isotropic direct summands in \( M_p \) and they are dual to each other with respect to \( \langle \cdot, \cdot \rangle_{r,s} \). Moreover, we require that

\[
\text{rank } N^{-1}_{\Sigma^1} = \text{rank } N^0_{\Sigma^1} = r, \quad \text{rank } N^{-1}_{\Sigma^2} = \text{rank } N^0_{\Sigma^2} = s.
\]

We endow \( M_p \) with the standard polarization as follows. Put

\[
M^{-1} = Y^r \oplus L^r \oplus Y^s, \quad M^0 = X^r \oplus L^r \oplus X^s.
\]

We call \( \text{Pol}_p^0 \) the standard polarization of \( M_p \). We make the following identification according to the basis \( \{ y^i, w^i, x^i \} \).

\[
\begin{align*}
M^0_{\Sigma^1} &= X^r_{\Sigma^1} = Z_p(\Sigma^s)^s \\
M^{-1}_{\Sigma^1} &= Y^s \oplus L^s = Z_p(\Sigma^s)^r \\
M^0_{\Sigma^2} &= X^r_{\Sigma^2} \oplus L^r \oplus X^s
\end{align*}
\]

1.7. Let \( n \) be a positive integer. Denote by \( 1_n \) the identity matrix in \( \text{GL}_n \). Denote by \( B_n \) the upper unipotent subgroup of \( \text{GL}_n \), by \( T_n \) the diagonal matrices and by \( N_n \) the unipotent radical of \( B_n \).

Since \( p \) splits in \( K \), \( G(\mathbb{Q}_p) \supseteq \text{GL}_{r+s}(\mathbb{Q}_p) \times \mathbb{Q}_p^* \) via the map \( g \mapsto (g|_{\Sigma^1}, \nu(g)) \). For \( v \in \mathfrak{h} \), we put

\[
K^0_v = \{ g \in G(\mathbb{Q}_v) \mid M_v g = M_v \} \quad \text{and} \quad K^0 = \prod_{v \in \mathfrak{h}} K^0_v.
\]

For \( g_p \in K_p^0 \cong \text{GL}(M_\Sigma) \), we write \( g_p = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) according to the decomposition \( M_\Sigma = M^{-1}_{\Sigma} \oplus M^0_{\Sigma} \). For \( K \) an open-compact subgroup in \( K^0 \) with \( K_p = K_p^0 \), we put

\[
\begin{align*}
K^0 &= \{ g \in K^0 \mid g_p \equiv 1 \quad \text{mod } p^0 \} \\
K^1 &= \{ g \in K^0 \mid g_p \equiv N_{r+s}(\mathbb{Z}_p) \text{ mod } p^0 \}.
\end{align*}
\]
2. Modular forms on unitary groups

2.1. In this section, we give a brief account of the theory of p-adic modular forms on unitary groups. This theory is due to Katz for \text{GL}(2) ([Kat78]) and to Hida for general reductive groups ([Hid04]).

**Definition 2.1.** [S-quadruples] Let \( \square \) be a finite set of rational primes. Let \( U \subset K^0 \) be an open-compact subgroup in \( G(\mathcal{A}_f^{(\square)}) \). Let \( S \) be a connected, locally noetherian \( \mathcal{O} \)-scheme and \( \pi \) be a geometric point. A **S-quadruple** of level \( U^{(\square)} \) is a quadruple \( \mathcal{A} = (A, \overline{A}, \iota, \eta^{(\square)})_S \) consisting of the following data:

- \( A \) is an abelian scheme of dimension \((r + s)d\) over \( S \).
- \( \lambda \) is a prime-to-\( \square \) polarization of \( A \) over \( S \) and \( \overline{\lambda} \) is the \( \mathbb{Z}_{(\square),+} \)-orbit of \( \lambda \). Namely,
  \[
  \overline{\lambda} = \mathbb{Z}_{(\square),+} \lambda := \{ \lambda' \in \text{Hom}(A, A^\prime) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\square)} \mid \lambda' = \lambda \circ a, a \in \mathbb{Z}_{(\square),+} \}.
  \]
- \( \iota : \mathcal{O}_K \hookrightarrow \text{End}_S A \otimes_{\mathbb{Z}} \mathbb{Z}_{(\square)} \).
- \( \eta^{(\square)} = U\eta^{(\square)} \) is a \( \pi_1(S, \pi) \)-invariant \( U \)-orbit of isomorphisms of \( \mathcal{O}_K \)-modules \( \eta^{(\square)} : M \otimes \hat{\mathbb{Z}}^{(\square)} \to T^{(\square)}(A_\pi) \).

Furthermore, the quadruple \( (A, \overline{A}, \iota, U\eta^{(\square)})_S \) satisfies the following conditions (K1)-(K3):

(K1) Let \( t \) denote the Rosati involution induced by \( \lambda \) on \( \text{End}_S A \otimes \mathbb{Z}_{(\square)} \). Then \( t(b)t = \iota(c(b)), \forall b \in \mathcal{O}_K \).

(K2) Let \( e^\lambda \) be the Weil pairing induced by \( \lambda \). We fix an isomorphism \( \xi : \mathcal{A}_f \cong \mathcal{A}_f(1) \) once and for all, with which we regard \( e^\lambda \) as a skew-Hermitian form \( e^\lambda : T^{(\square)}(A_\pi) \times T^{(\square)}(A_\pi) \to D_{\mathcal{O}_K} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{(\square)} \). Let \( e^\eta \) denote the skew-Hermitian form on \( T^{(\square)}(A) \) induced by \( e^\eta(x, x') = \theta_{\pi, \eta}(x, \eta(x)) \). We require that
  \[
  e^\lambda = u \cdot e^\eta \quad \text{for some} \quad u \in K_f^{(\square)}.
  \]

(K3) The determinant condition:

\[
\text{det}(X - \iota(b)| \text{Lie} A) = (X - (\sigma c)(b))^\ast(X - \sigma(b))^\ast \in \mathcal{O}_S[X], \forall b \in \mathcal{O}_K.
\]

Define the fibered category \( \mathcal{E}_U^{(\square)} \) over \( \text{SCH}_{\mathcal{O}_K^{(\square)}} \) as follows. Objects over \( S \) are \( S \)-quadruples. For \( \mathcal{A} = (A, \overline{A}, \iota, U\eta^{(\square)})_S \) and \( \mathcal{A}' = (A', \overline{A}', \iota', U(\eta')^{(\square)})_S \), we define

\[
\text{Hom}_{\mathcal{E}_U^{(\square)}}(\mathcal{A}, \mathcal{A}') = \left\{ \phi \in \text{Hom}_{\mathcal{O}_K}(A, A') \mid \phi^\ast \overline{\lambda} = \overline{\lambda}, U\phi(\eta')^{(\square)} = U\eta^{(\square)} \right\}.
\]

We say \( \mathcal{A} \cong \mathcal{A}' \) if there exists an isomorphism in \( \text{Hom}_{\mathcal{E}_U^{(\square)}}(\mathcal{A}, \mathcal{A}') \).

2.2. Shimura varieties.

2.2.1. Let \( \square = \emptyset \) be the empty set and \( U \) be an open-compact subgroup in \( G(\mathcal{A}_f) \). We define the functor \( \mathcal{E}_U : \text{SCH}_{/K} \to \text{SET} \) by

\[
\mathcal{E}_U(S) = \left\{ \mathcal{A} = (A, \overline{A}, \iota, U\eta)_S \in \mathcal{E}_U(T) \right\} / \cong.
\]

By the theory of Shimura-Deligne, \( \mathcal{E}_U \) is represented by \( S_G(U)_{/K} \) which is a quasi-projective scheme over \( K \). \( S_G(U)_{/K} \) is the Shimura variety associated to \( G \).

2.2.2. **Kottwitz model.** Let \( \square = \{ p \} \). Let \( K \) be an open-compact subgroup such that \( K_p = K_p^0 \). We define functor \( \mathcal{E}_{K(p)}^{(p)} : \text{SCH}_{/\mathcal{O}} \to \text{SET} \) by

\[
\mathcal{E}_{K(p)}^{(p)}(S) = \left\{ \mathcal{A} = (A, \overline{A}, \iota, \eta^{(p)})_S \in \mathcal{E}_{K(p)}^{(p)} \right\} / \cong.
\]

In [Kot92], Kottwitz shows \( \mathcal{E}_{K(p)}^{(p)} \) is representable by a quasi-projective scheme \( S_G(K)_{/\mathcal{O}} \) over \( \mathcal{O} \) if \( K \) is neat.

2.3. Igusa schemes.
2.3.1. For a quadruple $V = (V, \theta_{r,s}, M, \text{Pol}_p)$, where $(V, \theta_{r,s})$ is the skew-Hermitian space defined in §1.5, $M$ is the $\theta_K$-lattice in (1.1) and $\text{Pol}_p = \{N^{-1}, N^0\}$ is a polarization of $M_p$, we review the (open) Igusa schemes associated to $V$ following the exposition in [HLS06, 2.1].

**Definition 2.2** (S-quintuples). Let $n$ be a positive integer. We define the fibered category $\mathcal{C}_{K,n,\text{Pol}_p}^{(p)}$ whose objects over a base scheme $S$ are $S$-quintuples $(A, j)_S = (A, \overline{\lambda}, \iota, \overline{\rho}, j)_S$ of level $K^n$, where $A_S \in \mathcal{C}_{K,n}^{(p)}$ is a $S$-quadruple and

$$j : \mu_{pn} \otimes \mathbb{Z} N^0 \hookrightarrow A[p^n]$$

is a monomorphism as $\theta_K$-group schemes over $S$. We call $j$ a level-$p^n$ structure of $A$. Morphisms between $S$-quintuples are

$$\text{Hom}_{\mathcal{C}_{K,n,\text{Pol}_p}^{(p)}}((A, j), (A', j')) = \left\{ \phi \in \text{Hom}_{\mathcal{C}_{K,n}^{(p)}}(A, A') \mid \phi j = j' \right\}.$$ 

Define the functor $\mathcal{C}_{K,n,\text{Pol}_p}^{(p)} : SCH/\mathcal{O} \rightarrow SET$ by

$$\mathcal{C}_{K,n,\text{Pol}_p}^{(p)}(S) = \left\{ (A, j) = (A, \overline{\lambda}, \iota, \overline{\rho}, j)_S \in \mathcal{C}_{K,n}^{(p)} \right\} / \cong.$$ 

Let $\underline{A}$ be the universal quadruple of level $K^{(p)}$ over $S_G(K)$. Then equivalently, we have

$$I_G(K^n) = \operatorname{Inj}_{\theta_K}(\mu_{pn} \otimes \mathbb{Z} M^0, A).$$

It is known that $\mathcal{C}_{K,n,\text{Pol}_p}^{(p)}$ are relatively representable over $S_G(K)/\mathcal{O}$ (cf. [HLS06, Lemma(2.1.6.4)] and [SGA64, Prop. 3.12]), and thus it is represented by a scheme. We denote by $I_G(K^n)/\mathcal{O}$ the scheme that represents $\mathcal{C}_{K,n,\text{Pol}_p}^{(p)}$ for the standard polarization $\text{Pol}_p = \{M^{-1}, M^0\}$ defined in (1.2).

2.3.2. $p$-adic forms. Suppose $p$ is nilpotent in $R$ and $p^m R = 0$ for some $m \geq 1$. Let $(A, j)$ be a $R$-quintuple of level $K^n$. Identity $M^0 = M_{\Sigma}^{-1} \oplus M_{\Sigma^e}$ with the basis in (1.2). Then if $n \geq m$, the level $p^n$-structure $j$ over $R$ induces a trivialization of $\text{Lie} A$:

$$j^+ : M_{\Sigma}^0 \otimes R \rightarrow e^+ \text{Lie} A[p^n] = e^+ \text{Lie} A; \quad j^- : M_{\Sigma^e}^0 \otimes R \rightarrow e^- \text{Lie} A[p^n] = e^- \text{Lie} A.$$

Let $\omega_A = \text{Hom}(\text{Lie} A, R)$ be the $R$-module of invariant one forms of $A$. Then via the identification in (1.2), $j$ induces an isomorphism:

$$\omega(j)^+ = \omega(j^-) : \mathbb{Z}_p^\times \otimes R \rightarrow e^+ \omega_A; \quad \omega(j)^- = \omega(j^+) : \mathbb{Z}_p^\times \otimes R \rightarrow e^- \omega_A.\quad (2.2)$$

2.3.3. Change of the polarization. We call $j$ a level-$p^n$ structure of $A$. It is clear that the notion of level-$p^n$ structures depends on the choice of the polarization of $M$. Choose $\gamma \in K^n_\text{pol}$ such that $N^{-1} = M^{-1} \gamma$ and $N^0 = M^0 \gamma$. Then we see that $j \rightarrow \gamma j$ is an isomorphism from the level-$p^n$ structures with respect to $\text{Pol}_p$ to those of $\text{Pol}_p$. Therefore the map $(A, j) \rightarrow (A, \gamma j)$ induces an isomorphism between $\mathcal{C}_{K,n,\text{Pol}_p}^{(p)}$ and $\mathcal{C}_{K,n,\text{Pol}_p}^{(p)}$.

2.4. Complex uniformization.

2.4.1. Let $G = GU(V)$ and $U \subset K^0$ be an open compact subgroup in $G(A_f)$. We recall the description of the complex points of $S_G(U)(\mathbb{C})$ following [ST61].

We begin with the Hermitian symmetric domain attached to unitary groups with signature $(r, s)$. We treat two cases $r \geq s > 0$ and $r = s$ 0 individually. If $r \geq s > 0$, we put

$$X_{r,s} = \{ (x, y) \mid x \in M_s(\mathbb{C}), y \in M_{(s-r) \times s}(\mathbb{C}), i(x^* - x) > -iy^* \theta^{-1} y \}.$$ 

For $\alpha \in G(\mathbb{R})$, we write

$$\alpha = \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & d \end{bmatrix}$$

according to the standard basis of $V$. The action of $\alpha \in G(\mathbb{R})^+$ on $X_{r,s}$ is defined by

$$\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by + c \\ gx + ey + f \end{bmatrix} \cdot (hx + ly + d)^{-1}.\quad (2.3)$$
If \( rs = 0 \), \( X_{r,s} \) consists of a single point written \( x_0 \) with the trivial action of \( G \).

Then \( X^+ = X_{r,s} \) is the Hermitian symmetric domain associated to \( G \). Put

\[
M_G(X^+, U) := G(\mathbb{Q})^+ \backslash X^+ \times G(\mathbb{A}_f)/U,
\]

where \( G(\mathbb{Q})^+ = \{ g \in G(\mathbb{Q}) \mid \nu(g) > 0 \} \). Then \( M_G(X^+, U) \) is a complex manifold when \( U \) is neat. The group \( G = GU(r, s) \) satisfies the Hasse principle ([Hid04, 7.1.5, p.319]). Hence we have

\[
(2.4) \quad M_G(X^+, U) \cong S_G(U)(\mathbb{C}).
\]

### 2.4.2. Analytic construction of the universal abelian scheme over \( \mathbb{C} \)

Let \( \mathcal{A}(V)/\mathbb{C} \) be the universal quadruple of level \( U \) over \( S_G(U)/\mathbb{C} \). After introducing some notations, we shall recall a construction of \( \mathcal{A}(V)_{/\mathbb{C}} \). Define \( \mathbb{C}^{r,s} \) a vector space over \( \mathbb{C} \) of dimension \( r + s \) with \( K \)-action by

\[
(2.5) \quad \mathbb{C}^{r,s} = \mathbb{C}^*(\Sigma^r) \oplus \mathbb{C}^{-s}(\Sigma^r) \oplus \mathbb{C}^s(\Sigma).
\]

Here \( \mathbb{C}^{r,s} \) and \( \mathbb{C}^s \) are regarded as spaces of row vectors. According to the above decomposition (2.5), we define \( c_{r,s} : \mathbb{C}^{r,s} \rightarrow \mathbb{C}^{r,s} \) by

\[
(2.6) \quad B(\tau) = \begin{bmatrix} x^* & y^* & x \\ 0 & \theta & y \\ 1_s & 0 & 1_s \end{bmatrix} \in M_{r+s}(\mathbb{C}).
\]

If \( rs = 0 \), we put \( B(x_0) = \theta \). Via the isomorphism \( K \otimes \mathbb{R} \cong \mathbb{C} \), we regard \( V_K \) as a \( \mathbb{C} \)-space of row vectors according to the \( K \)-basis \( \{ y^i, w^i, x^i \} \). For each \( \tau \in X^+ \), we define the map \( p(\tau) : V_K \rightarrow \mathbb{C}^{r,s} \) by \( p(\tau)v = vB(\tau)c_{r,s} \). Then \( V \) acts on \( (\tau, z) \in X^+ \times \mathbb{C}^{r,s} \) by

\[
v \cdot (\tau, z) = (\tau, p(\tau)v + z).
\]

We define a left action of \( G \) on \( V \) by

\[
(2.7) \quad g \ast v := vg^{-1}v^g = vg^{-1}\nu(g)
\]

and put

\[
M_{[g]} = g \ast M = Mg^{-1} \text{ and } M_{[g]}(\tau) = p(\tau)(M_{[g]}).
\]

To each point \( (\tau, g) \in X^+ \times G(\mathbb{A}_f) \) we can attach a \( \mathbb{C} \)-quadruple \( A(V)g(\tau) = (A(V)_g(\tau), \overline{\cdot}, \text{can}, [\cdot], U\eta_g) \) of level \( K^n \) defined by the following data

- **Abelian variety**: \( A(V)g(\tau) = \mathbb{C}^{r,s}/M_{[g]}(\tau) \cong V_K/M_{[g]} \).
- **Polarization**: \( \overline{\cdot}, \text{can} \) is the \( \mathbb{Q}_1 \)-orbit of the polarization induced by the Riemann form defined by the pull back of \( \langle \ , \rangle_{\text{can}} \) via \( p(\tau) \).
- **Endomorphism**: \( \eta_g : \mathcal{O}_K \rightarrow \text{End}(A_g(\tau) \otimes_{\mathbb{Z}} \mathbb{Q}) \) is the \( \mathcal{O}_K \)-action induced by the action on \( V \) via \( p(\tau) \).
- **Prime-to-\( p \) level structure**: \( \eta_g^{(p)} : M \otimes \mathbb{Z}(p) \rightarrow M_{[g]} = H_1(A_g(\tau), \mathbb{Z}(p)) \) is defined by \( \eta_g^{(p)}(x) = g \ast x \) for \( x \in M \).
- **Level structure at \( p \)**: Taking a primitive \( p^n \)-th root \( \zeta = e^{2\pi i/p^n} \), we have an isomorphism \( \zeta : \mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{C}^{p^n} \). Similarly as above we have

\[
j_\zeta : \mu_{p^n} \otimes M^0 \cong \mathbb{Z}/p^n\mathbb{Z} \otimes M^0 \rightarrow A_g(\tau)[p^n] = Z/p^n\mathbb{Z} \otimes M_{[g]}, j(x^0) = g \ast x^0 \text{ for } x^0 \in M^0.
\]

Let \( \eta_g \) be the full level structure \( \eta_g^{(p)} \times \eta_g(j_\zeta) \). Thus the isomorphism in (2.4) can be described explicitly as follows.

\[
(2.8) \quad (\tau, g) \rightarrow \mathcal{A}(V)_g(\tau) := \left[ (A(V)_g(\tau), \overline{\cdot}, \text{can}, [\cdot], K^n\eta_g) \right].
\]
Note that for \( u \in K^0 \), we have
\[
\tau, gu = \left( (A_g(\tau), \sum_{\ell \in \mathcal{C}_+} [\ell], K u^\ell \eta^{(p)} \nu^{(p)}, u_p \nu_j \right).
\]

We put
\[
dz V(\Sigma) = \left\{ dz^{(i)}_V(\Sigma), \ldots, dz^{(r)}_V(\Sigma) \right\}
\]
and
\[
dz V(\Sigma^c) = \left\{ dz^{(r+s)}_V(\Sigma^c), \ldots, dz^{(t+s)}_V(\Sigma^c) \right\}.
\]
Then \( dz := (dz_V(\Sigma), dz_V(\Sigma^c)) \) form a basis of \( \Omega \). We define a \( r \)-form \( \omega_V/\mathcal{C}(\Sigma) \) and a \( s \)-form \( \omega_V/\mathcal{C}(\Sigma^c) \) by
\[
\omega_V/\mathcal{C}(\Sigma) = \sum_{i=1}^r dz^{(i)}_V(\Sigma); \omega_V/\mathcal{C}(\Sigma^c) = \sum_{i=r+1}^{r+s} dz^{(i)}_V(\Sigma^c).
\]

2.5. Igusa schemes associated to \( U(V) \). In the later application of the pull-back formula, we have to consider the Igusa schemes attached to the unitary group \( U(V) \) as well. Let \( \mathbb{Z}^{ab} \) be the ring of integers of \( \mathbb{Q}^{ab} \) and \( \mathcal{O}^{ab} \) be the ring generated by \( 0 \) and \( \mathbb{Z}^{ab} \). Let \( CL_\mathbb{Q}(K) = \mathbb{Q} \backslash \mathbb{A}_f^G / \nu(K) \) be a set of representatives \( C_K \) of \( CL_\mathbb{Q}(K) \) in \( \mathbb{A}_f^G \). For \( c \in C_K \), we consider the functor \( I_{U(V)}(K^n; c) : SCH/\mathcal{O}^{ab} \to \text{SET} \)
\[
I_{U(V)}(K^n; c)(S) = \left\{ (A, \lambda, \eta^{(p)} K^{(p)}, j)_S \mid (A, \lambda, \eta^{(p)} K^{(p)}) \in \mathcal{E}^{(p)}_{K,n} \right\} / \approx,
\]
where \( \mathcal{O}^{(p)}_{K,n} \) is independent of the choice of \( \lambda \) in \( \overline{\lambda} \). Pick \( g_\mathcal{O} \in G(\mathbb{A}_f^G) \), \( \nu(g_\mathcal{O}) = c \) and let \( \mathcal{O}^{(p)}_{K} = g_\mathcal{O} K^n g_\mathcal{O}^{-1} \cap U(V)(\mathbb{A}_f) \). Then over \( \mathbb{C} \) we have an isomorphism
\[
M_{U(V)}(X^+, e^\mathcal{O}) \simeq I_{U(V)}(K^n; c)/\mathbb{C}.
\]
As explained in [Hid04, §4.2.1] for the Hilbert modular varieties, we have
\[
\bigcup_{c \in CL_\mathbb{Q}(K)} I_{U(V)}(K^n; c) = I_G(K^n).
\]
When \( c = 1 \), we write \( I_{U(V)}(K^n) \) for \( I_{U(V)}(K^n; c) \).

2.6. Morphisms between Igusa schemes. Let \( (W, \theta, L) \) and \( (V, \theta, L, M) \) be as before. Let \( L^{-1} = L_{\Sigma} := e^+(L \otimes \mathbb{Z}_p) \) and \( L^{0} = L_{\Sigma} := e^-(L \otimes \mathbb{Z}_p) \). Then \( \{ L^{-1}, L^{0} \} \) is a polarization of \( L_p \). Recall that the standard polarization of \( M_p \) in §1.6 is
\[
M^{-1} = \sum_{i=1}^s (\mathcal{O}_K \otimes \mathbb{Z}_p) y^i \oplus L^{-1} \quad \text{and} \quad M^0 = L^0 \oplus \sum_{i=1}^s (\mathcal{O}_K \otimes \mathbb{Z}_p) x^i.
\]
Let \( \mathcal{O} = (V, \theta, L, M^{-1} \oplus M^0) \) and \( -\mathcal{O} = (W, -\theta, L, L^0 \oplus L^{-1}) \). Let \( (\mathcal{W}, \nu_\mathcal{O}) = (V \oplus (-W), \theta, x \oplus (-\theta)) \). Define an \( \mathcal{O} \)-lattice \( \mathcal{L} = M \oplus L \) in \( \mathcal{W} \) and a polarization \( \mathcal{L}^{-1}, \mathcal{L}^0 = (M^{-1} \oplus L^0, M^0 \oplus L^{-1}) \) of \( \mathcal{L}_p \). Put
\[
\mathcal{W} = \mathcal{V} \oplus (-\mathcal{W}) = (W, \nu_\mathcal{O}, L, \mathcal{L}^{-1} \oplus L^0).
\]
Let \( G_1 = U(V) \), \( G_2 = U(W) \) and \( G_3 = U(W) \) be the associated unitary groups. For open compact subgroups \( K_1 \subset G_1(\mathbb{A}_f) \) such that \( K_1 \times K_2 \subset K_3 \) and a fixed \( c \in \mathbb{A}_f^{\infty} \), we write \( I^3(K_i(p^n)) \) for the Igusa schemes \( I_{G_i}(K_i(p^n), c) \) associated to unitary groups with additional data. Then we have a natural morphism
\[
i_{V,W} : I^3(K_1(p^n)) \otimes \mathcal{O}^{ab} \times I^3(K_2(p^n)) \otimes \mathcal{O}^{ab} \rightarrow I^3(K_3(p^n)) \otimes \mathcal{O}^{ab}
\]
defined by
\[
i_{V,W} ([A_1, \lambda_1, \nu_1; K_1, j_1]), ([A_2, \lambda_2, \nu_2; K_2, j_2]) = [(A_1 \times A_2, \lambda_1 \times \lambda_2, \nu_1 \times \nu_2; K_3, j_1 \times j_2)].
\]
Now we consider a different polarization of \( L_p \). Let \( w^{+,i} \) (resp. \( w^{-,i} \)) be the image of \( w^i \) in \( W \) (resp. \( -W \)) as a subspace in \( W \). We define a basis \( \{ y^i, x^i \}_{i=1}^s \) of \( W \) by \( y^i = y^i, x^i = x^i \) if \( 1 \leq i \leq s \) and

\[
y^i = \frac{1}{2} w^{+,i-s} - \frac{1}{2} w^{-,i-s},
x^i = w^{+,i-s} \theta^{-1} + w^{-,i-s} \theta^{-1}.
\]

if \( s < i \leq r \). We put \( Y = \sum_{i=1}^{2s} (\Theta_K \otimes \mathbb{Z}_p) y^i \) and \( X = \sum_{i=1}^{2s} (\Theta_K \otimes \mathbb{Z}_p) x^i \). Then \( \{ Y, X \} \) is another polarization of \( L_p = M_p \oplus (-L_p) \). We have

\[
L_\Sigma = M_\Sigma \oplus (-L)_\Sigma = Y_\Sigma \oplus X_\Sigma.
\]

We define \( \Upsilon \in U(W)(\mathbb{Q}_p) \cong GL(W_\Sigma) \cong GL_{2r}(\mathbb{Q}_p) \) by \( y^i \Upsilon = y^i, x^i \Upsilon = x^i \) if \( 1 \leq i \leq s \) and \( w^{+,i-s}_\Sigma \Upsilon = y^i_\Sigma \) and \( w^{-,i-s}_\Sigma \Upsilon = x^i_\Sigma \) if \( s < i \leq r \). Then \( M^0_\Sigma \Upsilon = X_\Sigma \) and \( M^{-1}_\Sigma \Upsilon = Y_\Sigma \). The matrix representation of \( \Upsilon \) according the basis \( y^i_\Sigma \) and \( x^i_\Sigma \) is

\[
\Upsilon = \begin{bmatrix}
1_s & 1_{r-s} & -\frac{1}{2} \cdot 1_{r-s} \\
\frac{1}{2} \cdot 1_{r-s} & 1_s & \theta^{-1} \\
\theta^{-1} & \theta^{-1} & 1_{r-s}
\end{bmatrix}.
\]

We give an explicit expression of the morphism in (2.11) in terms of the complex coordinates defined in (2.8). Notations are as in §2.4. By definition of \( C^{r,s} \) we can decompose

\[
\mathbb{C}^{r,s} = \mathbb{C}^s(\Sigma^c) \oplus \mathbb{C}^{-s}(\Sigma^c) \oplus \mathbb{C}^s(\Sigma) \oplus \mathbb{C}^{-s}(\Sigma) = \mathbb{C}^{r,s} \oplus \mathbb{C}^{0,r-s}.
\]

Then for \( \tau = \begin{bmatrix} x \\ y \end{bmatrix} \in X_{r,s} \), we let \( i_\tau \in GL(\mathbb{C}^{r,s}) \) be the matrix such that according to the above decomposition,

\[
i_\tau = \begin{bmatrix}
1_s & 1_{r-s} & 1_s \\
1_{r-s} & \theta^{-1} y & 1_{r-s} \\
\theta^{-1} y & 1_{r-s} & \theta^{-1}
\end{bmatrix}
\]

and put

\[
(2.12) Z_\tau = \begin{bmatrix} x & 0 \\ y & \varsigma \end{bmatrix} \in X_{r,r},
\]

where \( \varsigma = -2^{-1} \theta \). For \( g \in U(V) \) and \( h \in U(W) \), a straightforward computation shows that

\[
(L_{(g,h)}(Z_\tau)) i_\tau = M_g(\tau) \oplus L_h(x_0),
\]

hence \( i_\tau \) induces an isomorphism

\[
(2.13) A(W)_{(g,h)}(Z_\tau) = \frac{\mathbb{C}^{r,s}}{L_{(g,h)}(Z_\tau)} \cong \mathbb{C}^{r,s} \oplus \mathbb{C}^{0,r-s} = A(V)_g(\tau) \times A(W)_h(x_0).
\]

Note that

\[
(2.14) i_\tau^*(\omega_{V/\mathbb{C}}(\Sigma), \omega_{V/\mathbb{C}}(\Sigma^c) \wedge \omega_{W/\mathbb{C}}(\Sigma^c)) = (\omega_{W/\mathbb{C}}(\Sigma), \omega_{W/\mathbb{C}}(\Sigma^c)).
\]

Taking into account the change of the polarization, we can deduce that the morphism in (2.11) over \( \mathbb{C} \) is simply given by

\[
(2.15) i_{V,W}([\tau, g], [x_0, h]) = [Z_\tau, (g, h) \Upsilon].
\]

2.7. CM abelian varieties and periods.
2.7.1. We consider a special case where \( r = 2 \) and \( s = 0 \). Then \( G = GU(W) \) is a definite unitary group of degree two. It follows that \( S_G(K) \) is finite over \( \emptyset \) and \( S_G(K)(\mathbb{C}) = M_G(X^+, K) \) is a finite set. Let \( \mathbb{P} \) be the universal quadruple over \( S_G(K) \). Then \( \mathbb{P} = \bigcup_{[h] \in S_G(K)} \mathbb{P}_h \) and each \( \mathbb{P}_h \) is defined over a discrete valuation ring \( \mathfrak{p} \subset \mathcal{O}(p) \). We consider the quintuple \( (\mathbb{P}, j) \) over \( S_G(K^+)/\mathbb{C} \). Since \( \mathbb{P}_h \) is an abelian variety with CM by \( \mathcal{O}_K \) and \( p \) is split in \( K \), it follows that \( \mathbb{P}_h \otimes \overline{\mathbb{F}}_p \) is an ordinary abelian variety, and \( j_\mathcal{C} \) descends to a level \( p^\infty \)-structure over a finite unramified extension of \( \mathfrak{p} \). In short, we can enlarge \( \mathfrak{p} \) so that the quintuple \( (\mathbb{P}, j) \) is defined over \( \mathfrak{p} \). Note that by the complex uniformization constructed in \$2.4.2\$, we have

\[
(2.16) \quad \mathbb{P}_h(\mathbb{C}) = \frac{\mathbb{C}(\Sigma^c)}{L_{[h]}^c \mathfrak{p} \mathbb{C}}.
\]

2.7.2. We introduce the CM-period and its \( p \)-adic avatar for the CM-algebra \( K \oplus K \). Let \( E_{K} \) be an elliptic curve with CM by \( K \) together with a complex uniformization \( i : E_{K}(\mathbb{C}) \simeq \mathbb{C}/\Omega_{K} \). It is well known that \( E_{K} \) is defined over \( \overline{\mathbb{Q}} \), extends to an abelian scheme \( \mathcal{E} \) over \( \overline{\mathbb{Z}}(p) \). We choose a Néron differential \( \omega_{K} \in H^0(\mathcal{E}, \Omega_{\mathcal{E}/\overline{\mathbb{Z}}(p)}) \), such that \( H^0(\mathcal{E}, \overline{\mathcal{E}}_{\overline{\mathbb{Z}}(p)}) = \mathcal{Z}_{(p)}, \omega_{K} \). On the other hand over \( \mathbb{C} \) \( i \) induces a holomorphic one form of first kind \( dz \) on \( E_{K} \), so there exists a complex number \( \Omega_{K} \) such that

\[
\omega_{K} = \Omega_{K} dz.
\]

We call \( \Omega_{K} \) the CM period of \( K \) which is well-defined up to \( \mathcal{Z}_{(p)} \).

By the isomorphism \( (2.16) \), the holomorphic one forms \( d_{\mathcal{E}}(\Sigma) \) defined in \( (2.9) \) give an \( \mathcal{O}_{K} \)-basis of \( \Omega_{\mathbb{P}_h, K} \).

By the assumption on \( L \) and \( \theta \), \( \mathbb{P}_h \) is \( \mathbb{Z}_{(p)} \)-isogenous to \( E_{K} \times E_{K} \). Then it is easy to show that the top form \( \omega_{\mathbb{P}_h} := \Omega_{K} \cdot \omega_{\mathbb{C}/\mathcal{E}(\Sigma)} \) over \( \mathbb{C} \), induced by complex uniformization actually descend to \( \mathcal{Z}_{(p)} \).

On the other hand, the level \( p^\infty \)-level structure \( j : \mathbb{Z}_{m^2} \rightarrow \mathcal{P}_h \) induces an formal top form \( \omega(j) \) of \( \mathcal{P}_h \) as in \( (2.21) \) which can descend to a top form of \( \mathbb{P}_h \) still denoted by \( \omega(j) \). Hence there exists \( \Omega_{p} \in (\mathbb{Z}_{(p)})^\times \) such that

\[
(2.17) \quad \Omega_{p} \omega(j) = \omega_{\mathbb{P}_h} = \Omega_{K} \omega_{\mathbb{C}/\mathcal{E}(\Sigma)}.
\]

2.8. Siegel modular forms for unitary groups.

2.8.1. We introduce the notion of Siegel modular forms for unitary groups. For a \( R \)-quintuple \( x = (A, \overline{\lambda}, \pi, j) \in I_G(K^n) \). We say \( \omega \) is an \( \mathcal{O}_{K} \)-top form of \( x \) if \( \omega = (\omega(\Sigma), \omega(\Sigma'), \omega(\Sigma^c)) \), where \( \omega(\Sigma) \) (resp. \( \omega(\Sigma') \)) is a generator of \( \Omega_{\Sigma} = \bigwedge^r \mathfrak{a}^+ \mathfrak{d} \Omega_{\mathfrak{a}} \) (resp. \( \Omega_{\Sigma'} = \bigwedge^s \mathfrak{a}^- \Omega_{\mathfrak{a}} \)). Let \( H = GL_{r} \times GL_{s} \). For \( h \in H \), we write \( h = (h_1, h_2) \in GL_{r} \times GL_{s} \). For \( \xi = (a, b) \in \mathbb{Z} \), we let \( \rho_{\xi}(h) = \det(h_1)^{-a} \det(h_2)^{-b} \). We have the obvious left action of \( H(\mathbb{R}) \) on \( \omega \).

Definition 2.3 (Katz–Hida). A Siegel modular form of weight \( \xi = (a, b) \) is a rule \( f \) which assigns to a pair \( (x, \omega) \) a \( R \)-quintuple \( x \) in \( I_G(K^n) \) together with an \( \mathcal{O}_{K} \)-top form \( \omega \) of \( \text{Lie} A \), an element \( f(x, \omega) \in R \), such that the following three conditions are satisfied.

(1) \( f(x, \omega) \) depends only on the \( R \)-isomorphism class of the pair \( (x, \omega) \).
(2) For any \( h \in H(\mathbb{R}), f(x, h \omega) = \rho_{\xi}(h) f(x, \omega) \).
(3) The formation \( f(x, \omega) \) commutes with base change. Namely, for any \( \pi : R \rightarrow R', \pi(f(x, \omega)) = f(x_{R'}, \omega_{R'}) \).

Tautologically modular forms of \( \xi \) can be viewed as sections in the sheaf \( \omega_{\mathcal{S}} \) as follows.

\[
\omega_{\mathcal{S}}(S) = \{ f(x, \omega) \in \mathcal{O}_{S} \mid f(x, h \cdot \omega) = \rho_{\xi}(h)f(x, \omega), \forall h \in H(\mathcal{O}_{S}) \}.
\]

Set

\[
\mathcal{M}_{\xi}(K^n, R) := H^0(I_G(K^n)/R, \omega_{\mathcal{S}}).
\]

We call \( \mathcal{M}_{\xi}(K^n, R) \) the space of geometric modular forms.

2.8.2. Automorphic forms and modular forms over \( \mathbb{C} \). Let \( J : G(\mathbb{R})^+ \times X^+ \rightarrow GL(r, \mathbb{C}) \times GL(s, \mathbb{C}) \) be the standard cocycle defined by

\[
g_{\infty}p(\tau) = p(g_{\infty}\tau)J(g_{\infty}, \tau), \forall (g_{\infty}, \tau) \in G(\mathbb{R})^+ \times X^+.
\]

Fix a point \( i \in X^+ \) and let \( K_{\infty}^0 \) be the stabilizer of \( i \) in \( G(\mathbb{R}) \). Then \( J : K_{\infty}^0 \rightarrow H(\mathbb{C}), k_{\infty} \mapsto J(k_{\infty}, i) \) defines an algebraic representation of \( K_{\infty}^0 \).
Definition 2.4. Let $U$ an open compact subgroup in $G(\mathbb{A}_f)$ and let $\chi$ be a Hecke character of $\mathcal{K}$ with infinity type $(b, a)$. Let $A_k(G, U, \chi)$ be the space of automorphic forms of weight $k = (a, b)$ and level $U$ with central character $\chi$. In other words, $A_k(G, U, \chi)$ consists of smooth and slowly increasing functions $F : G(\mathbb{A}) \to \mathbb{C}$ such that for all $(\alpha, k, u, z) \in G(\mathbb{Q}) \times K_0 \times U \times \mathbb{A}(\mathbb{A}_f)$,

$$F(zqgk\infty u) = \rho_k(J(k, \infty))(F(g))\chi^{-1}(z).$$

Define the function $\overline{AM}(F)$ on $X^1 \times G(\mathbb{A}_f)$ associated to $F \in A_k(G, U, \chi)$ by

$$\overline{AM}(F)(\tau, g) := \chi_f(\nu(g))\rho_k(J(g, \infty))F((g, g_\infty)),$$

where $g_\infty \in G(\mathbb{R})^+$ such that $g_\infty i = \tau$. Then $\overline{AM}(F)$ is a well-defined function on $X^1 \times G(\mathbb{A}_f)/U$. We put

$$A^\text{Hol}_k(G, U, \chi) = \{ F \in A_k(G, U, \chi) \mid \overline{AM}(F) \text{ is holomorphic in } X^1 \}.$$

We denote by $M_k(G, U, \chi, \mathbb{C})$ the space of holomorphic functions $f$ on $X^1 \times G(\mathbb{A}_f)$ such that

$$f(\alpha, g) = \nu(\alpha)^{-1} \rho_k(J(\alpha, Z)^{-1})f(\tau, g), \quad Z \in X^1, \quad \alpha \in G(\mathbb{Q})$$

and

$$f(\tau, zg) = \chi(z)f(\tau, g), \quad u \in U, \quad z \in Z(\mathbb{A}_f).$$

$M_k(G, U, \chi, \mathbb{C})$ is the space of modular forms of weight $k$ and level $U$ with central character $\chi$. It is easy to see that $\overline{AM}$ induces an isomorphism

$$\overline{AM} : A^\text{Hol}_k(G, U, \chi) \sim M_k(G, U, \chi, \mathbb{C}).$$

By G.A.G.A we have an injection:

$$M_k(G, K^n, \chi, \mathbb{C}) \hookrightarrow M_k(K^n, \mathbb{C}) \quad f \mapsto f([\tau, a], 2\pi i\omega_{V/\mathbb{C}}) := f(\tau, a),$$

where $\omega_{V/\mathbb{C}} = (\omega_{V/\mathbb{C}}(\Sigma), \omega_{V/\mathbb{C}}(\Sigma^c))$ is the $\theta_K$-top form defined in (1.6).

2.8.3. Analytic Fourier-Jacobi expansion. The set of cusp labels for $S_G(K)$ is defined to be

$$C(K) := (GL_s(K) \times GU(W)(\mathbb{A}_f)N_p(\mathbb{A}_f)\backslash G(\mathbb{A}_f))/K.$$

For $g \in G(\mathbb{A}_f)$, we denote by $[g]$ the class in $C(K)$.

For $f \in M_k(G, U, \mathbb{C})$, as a function on $X^1 \times G(\mathbb{A}_f)$, $f$ has the following analytic Fourier-Jacobi expansion.

$$f\left(\begin{bmatrix} x & \theta \cr \psi & y \end{bmatrix}, g\right) = f\left(\begin{bmatrix} x & \theta \cr \psi & y \end{bmatrix}, g; 2\pi i\omega_{V/\mathbb{C}}\right) = \sum_{\beta \in H_s(\mathbb{Q})} a_{\beta}(y, g; f)e^{2\pi i\text{Tr}(\beta x)}, \quad \left(\begin{bmatrix} x & \theta \cr \psi & y \end{bmatrix}, g\right) \in X^1 \times G(\mathbb{A}_f),$$

where $H_s(\mathbb{Q})$ is the set of $s \times s$ Hermitian matrices in $M_s(K)$. We put

$$\mathcal{F}_{[g]}(f) = \sum_{\beta \in H_s(\mathbb{Q})} a_{\beta}(y, g; f)q^{\beta}.$$

The formal power series $\mathcal{F}_{[g]}(f)$ is called the Fourier-Jacobi expansion of $f$ at the cusp $[g]$.

2.8.4. $p$-adic modular forms. Let $n \geq m$ be positive integers. Write $\mathbb{Z}_m$ for $\mathbb{Z}/p^m\mathbb{Z}$. Let $T_{n, m} = I_G(K^n)/\mathbb{Z}_m$. Let $T_{n, m} = \lim_{\rightarrow n} T_{n, m}$ and $T_{\infty, \infty} = \lim_{\rightleftarrows n} T_{n, m}$ be Igusa towers on $S_G(K)/\mathbb{O}$. Then there is a natural action of $GL_r(\mathbb{Z}_p) \times GL_s(\mathbb{Z}_p)$ on $T_{\infty, \infty}$. We define $p$-adic modular forms following [Hid02].

Definition 2.5. Put

$$V_{n, m} := H^0(T_{n, m}, \mathcal{O}_{T_{n, m}}) \text{ and } V_{\infty, m} := H^0(T_{\infty, m}, \mathcal{O}_{T_{\infty, m}}) = \lim_{\leftarrow n} V_{n, m}.$$ 

Let $N = N_r(\mathbb{Z}_p) \times N_s(\mathbb{Z}_p)$. We define the space of $p$-adic modular forms $V_p(G, K)$ by

$$V_p(G, K) = \lim_{\leftarrow m} V_n^N.$$
Let $R$ be a $\mathbb{Z}_m$-algebra. For a $R$-quintuple $(A, j)$, the $p^n$-structure $j$ induces the following isomorphism as $\mathcal{O}_X$-modules.

\begin{equation}
(2.20) \quad j_* : M^0 \otimes_R R \xrightarrow{\sim} \text{Lie} \mathcal{A}[p^n] = \text{Lie} A.
\end{equation}

Then (2.20) induces a $p$-adic $\mathcal{O}_X$-top form $\omega(j) = (\omega(j)(\Sigma), \omega(j)(\Sigma^c)) \in H^0(A, \Omega_X^2) \times H^0(A, \Omega_X^2)$, where

\begin{equation}
(2.21) \quad \omega(j)(\Sigma) : \wedge^r \text{Lie} A_{\Sigma} \xrightarrow{\sim} \wedge^r (M^0_{X^c} \otimes R) \cong R, \quad \omega(j)(\Sigma^c) : \wedge^r \text{Lie} A_{\Sigma} \xrightarrow{\sim} \wedge^r (M^0_{X^c} \otimes R) \cong R.
\end{equation}

The last isomorphism depends on the choice of a basis of $M^0$. (2.21) induces the embedding

\begin{equation}
(2.22) \quad \beta : H^0(T_{n,m}, \omega) \longrightarrow V_m \quad \tilde{f} \mapsto \beta(f)(A, j) = f(A, j, \omega(j))
\end{equation}

which induces the following morphism

\begin{equation}
M_{\mathbb{K}}(K^1_n, \mathbb{Z}_p) \xrightarrow{\text{res}} H^0(T^n_{\infty, \infty}, \omega_K) \rightarrow V_p(G, K), \quad f \mapsto \tilde{f} = \beta(\text{res}(f)).
\end{equation}

We call $\tilde{f} \in V_p(G, K)$ the $p$-adic avatar of $f$.

### 3. Siegel-Eisenstein series on $GU(n, n)$

#### 3.1. Let $n$ be a positive integer. In this section, we give a construction of $p$-adic Siegel-Eisenstein series on the quasi-split unitary group of degree $n$. We retain the notation in §2.6. Let $W$, $V$ and $\mathbf{W}$ be the quadruples defined in §2.6 with $r = n$ and $s = 1$. We further assume that the lattice $L$ is a $\mathbb{Z}$-maximal lattice with respect to the Hermitian form $2^{-1} \delta^{-1} \theta$ in the sense of Shimura [Shi97, Ch. I §4.7]. Let $G = GU(W, \eta_n)$ denote the group of unitary similitudes attached to $(W, \eta_n)$. For a $\mathbb{Q}$-algebra $R$, we identify $G(R)$ with

\begin{equation}
G(R) = \left\{ g \in M_{2n}(R \otimes \mathbb{Q} \mathcal{K}) \mid g \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix} g^* = \nu(g) \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix} \right\}
\end{equation}

according to the basis $\{y^i, x^i\}^n_{i=1}$. Thus $G$ is the standard quasi-split group of unitary similitudes of degree $n$. Let $P$ be the stabilizer of the flag $\{0\} \subset X \subset W$ in $G$. Then $P$ is the standard Siegel parabolic subgroup of $G$.

Recall that $X_{n,n}$ is the Hermitian symmetric domain associated to $G$ defined by

\[X_{n,n} = \{ Z \in M_n(\mathbb{C}) \mid i(Z^* - Z) > 0 \} \]

We choose a distinguished point $i$ in $X_{n,n}$ attached to $\theta$. Put

\[i = \begin{bmatrix} \sqrt{-1} \cdot 1_{n-1} & 0 \\ 0 & \varsigma \end{bmatrix}, \quad \varsigma = -\frac{\theta}{2} \]

Put

\[K^0_v = \{ g \in G(R) \mid gi = g \} \]

Let $K^0_v$ be the stabilizer of $L$ in $G(\mathbb{Q}_v)$ and let $K^0_f = \prod_{v \in h} K^0_v$ be a maximal open compact subgroup in $G(A_f)$. For $g \in G(A)$, we define

\[\delta(g) := \nu^{-n} \det(\tilde{d}d)^{-1}, \quad g = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}^r, \quad k, k \in K^0_{\infty}. K^0_f.
\]

We have $\delta_p(g) = \delta(g)^n$, where $\delta_p$ is the modular character of $P$.

#### 3.2. Eisenstein series.
3.2.2. Let \( \chi \) be a Hecke character of \( \mathcal{F} \) with infinity type \((k,0)\). Namely \( \chi : A_F^\times / \mathcal{F}^\times \to \mathbb{C} \) is a character such that \( \chi_{\infty}(z_{\infty}) = z_{\infty}^k \). Consider the induced representation \( I(\chi, s) = \text{Ind}_{P}^G(\chi, s) \), where \( \text{Ind}_{P}^G(\chi, s) \) is the unitary induced representation of \( \chi \) and \( s \in \mathbb{C} \) such that \( n(s - \frac{1}{2}) = s \). For a place \( v \) of \( \mathbb{Q} \), let \( I_v(\chi, s) \) be the induced representation attached to \( \chi_v := \chi_{(\chi \otimes \xi_{\mathbb{Q}_v})} \).

\[
I_v(\chi, s) = \left\{ \text{smooth functions } f : G(\mathbb{Q}_v) \to \mathbb{C} \mid f(pg) = \chi_v^{-1}(\det dp)\delta^s(p)f(g) \right\}.
\]

Then we have
\[
I(\chi, s) = \otimes_v I_v(\chi, s).
\]

Define the adelic Siegel-Eisenstein series associated to a smooth section \( \phi \in I(\chi, s) \) by
\[
E_\chi(g, \phi) := \sum_{\gamma \in \mathbb{P}(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{Q})} \phi(\gamma g),
\]
whenever the sum is convergent.

3.2.3. Denote by \( \psi \) the standard additive character of \( A / \mathbb{Q} \) such that \( \psi_{\infty}(x_{\infty}) = \exp(2\pi i x_{\infty}) \), \( x_{\infty} \in \mathbb{R} \). Let \( h \) be the set of finite places of \( Q \). If \( v \in h \), let \( dx_v \) be the Haar measure on \( Q_v \) such that \( \text{vol}(Z_v, dx_v) = 1 \). Let \( dX_v \) be the Tamagawa measure on the \( n \times n \) Hermitian matrices \( \mathcal{H}_n(Q_v) \) in \( M_n(K_v) \) such that \( \text{vol}(\mathcal{H}_n(Z_v), dX_v) = 1 \). If \( L \) is a lattice in \( \mathcal{H}_n(Q_v) \), put
\[
L^\vee = \left\{ x \in \mathcal{H}_n(Q_p) \mid \psi_v(\text{Tr}(xy)) = 1, \forall y \in L \right\}.
\]

If \( v \) is archimedean, let
\[
dX_v = \bigwedge_j dX_{jj} \bigwedge_{j < k} dX_{jk} \wedge d\overline{X_{jk}}.
\]

For \( \beta \in \mathcal{H}_n(K) \) and \( X \in \mathcal{H}_n(A) \), define \( \psi_{\beta}(X) = \psi \circ \text{Tr}_{X/Q}(\beta X) \). We choose the normalized measure \( dX \) on \( \mathcal{H}_n(A) \) so that \( \text{vol}(\mathcal{H}_n(Q) \backslash \mathcal{H}_n(A), dX) = 1 \). Then it is well-known that
\[
dX = C_{\mathcal{K}}(n) \cdot \otimes_v dX_v, \quad \text{where } C_{\mathcal{K}}(n) = 2^{\frac{n(n-1)}{2}} |D_{\mathcal{K}}|_{\mathbb{R}}^{-n(n-1)/4}.
\]

Let \( c^p(\chi) \) be the prime-to-\( p \) conductor of \( \chi \). We fix a prime-to-\( p \) integral ideal \( \mathfrak{c} \) of \( \mathbb{Z} \) such that
\[
(S1) \quad \mathfrak{c} \subset D_{\mathcal{K}} \cdot c^p(\chi).
\]

Let \( S = \{ v \in h \mid v|\mathfrak{c} \} \). Note that \( S \) is not empty.

3.2.3. Fourier expansion. Let \( N \) be the unipotent radical of \( P \) given by
\[
N = \{ n(X) = \begin{bmatrix} 1_n & X \\ 0 & 1_n \end{bmatrix} \mid X \in \mathcal{H}_n \}.
\]

Then Siegel-Eisenstein series \( E_{\chi} = E_{\chi}(g, \phi) \) has the Fourier expansion
\[
E_{\chi}(g, \phi) = \sum_{\beta \in \mathcal{H}_n(Q)} W_{\beta}(g, E_{\chi}), \quad \text{where } W_{\beta}(g, E_{\chi}) = \int_{\mathcal{H}_n(Q) \backslash \mathcal{H}_n(A)} E_{\chi}(n(X)g)\psi_{-\beta}(X)dX.
\]

Let \( w = \begin{bmatrix} 1_n & -1_n \\ 0 & 1_n \end{bmatrix} \). It is well-known that if \( \phi = \otimes \phi_v \) is decomposable and \( \text{supp} \phi_v \subset \mathbb{P}(Q_v)w \mathbb{P}(Q_v) \) for some \( v_0 \), then we have \( W_{\beta}(g, E_{\chi}) = C_{\mathcal{K}}(n)W_{\beta}(g, \phi) \) for \( g \in G(A) \) with \( g_{v_0} = 1 \), where
\[
W_{\beta}(g, \phi) = \prod_v \int_{\mathcal{H}_n(Q_v)} \phi_v(wn(X_v)g_v)\psi_{-\beta}(X_v)dX_v.
\]

Thus the Fourier expansion in (3.1) can be rephrased as
\[
C_{\mathcal{K}}(n)^{-1} E_{\chi}(g, \phi) = C_{\mathcal{K}}(n)^{-1} \phi(g) + M_w \phi(g) + \sum_{\beta \in \mathcal{H}_n(Q), \beta \neq 0} W_{\beta}(g, \phi),
\]
where \( M_w \phi(g) := W_0(g, \phi) \). Then \( M_w \phi \) is called the intertwining operator of \( w \), and we will write \( M \phi \) for \( M_w \phi \) in the sequel. Define the constant term \( E_P(g) \) along the parabolic subgroup \( P \) by
\[
E_P(g) = \int_{\mathcal{H}_n(Q) \backslash \mathcal{H}_n(A)} E_{\chi}(n(X)g)dX = \phi(g) + C_{\mathcal{K}}(n) \cdot M_w \phi(g).
\]
3.3. The local section at the archimedean place.

3.3.1. To define the desired Eisenstein series with good arithmetic and $p$-adic properties, we need to choose some special decomposable local section in $I(\chi, s)$. In this subsection, we give the choice of the local sections at the archimedean place.

For $g \in G(\mathbb{R})^+$ and $Z \in X_{n,n}$, we define the standard automorphy factors $J(g, Z)$ and $J'(g, Z)$ by

$$J(g, Z) := \det(c(g)Z + d_g) \text{ and } J'(g, Z) := \det(g)^{-1}J(g, Z)\nu(g)^n.$$  

Then we have

$$\delta(g) = |J(g, 1)J'(h, 1)\nu(g)^{-n}|^{-1} = |J(g, 1)|^{-2}|\det g|, g \in G(\mathbb{R}).$$

Let $J_{\mu, \lambda}(g, Z)$ denote the automorphy factor of weight $(\mu, \lambda) \in \mathbb{Z}^2$,

$$J_{\mu, \lambda}(g, Z) = J'(g, Z)^\mu J(g, Z)^\lambda.$$

Define the section of minimal $K^{(0, k)}_{\chi, \infty}$-type $(0, k)$ in $I_{\infty}(\chi, 0)$ by

$$\phi_{\chi, \infty}^h(g) = J(0, k)(g, Z) = J(0, 1)^{-k}.$$  

Set

$$\phi_{\chi, s, \infty}(g) = \phi_{\chi, \infty}^h(g)\delta(g)^s.$$  

3.3.2. Intertwining operator. The intertwining operator of $\phi_{\chi, s, \infty}$ can be computed by Gindikin-Karpelevič formula.

$$M_w\phi_{\chi, s, \infty}(g) = i^{-nk} \cdot 2^n \cdot \pi^{n^2} \cdot \frac{\Gamma_n(k - n - 2s)}{\Gamma_n(k + s)\Gamma_n(s)} \cdot \phi_{\chi, k - k - s, \infty}(g).$$

Note that $M_w\phi_{\chi, s, \infty}|_{s=0} = 0$ if $k > n$.

3.3.3. Fourier coefficients. We will compute the local Fourier coefficient of $\phi_{\chi, k, \infty}$. First of all, we recall several definitions and facts from [Shi82]. Put

$$H'_n = \{x + iy \in M_n(\mathbb{C}) \mid x, y \in \mathfrak{H}_n(\mathbb{C}) \text{ and } x > 0\}.$$  

We define the function $\xi(y, h; q_1, q_2)$ for $(y, h; q_1, q_2) \in H'_n \times \mathfrak{H}_n(\mathbb{C}) \times \mathbb{C}^2$ by

$$\xi(y, h; q_1, q_2) := \int_{\mathfrak{H}_n(\mathbb{C})} \det(y + ix)^{-q_1}\det(y - ix)^{-q_2}e^{-2\pi i\text{Tr}(hx)}dx$$

$$= i^{n(q_2-q_1)} \cdot 2^n \cdot \pi^{n^2} \cdot \frac{\Gamma_n(q_1)^{-1}\Gamma_n(q_2)^{-1}}{\Gamma_n(s)} \cdot \eta(2y, \pi h; q_1, q_2),$$

where

$$\Gamma_n(s) = \pi^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(s - j).$$

Then $\xi$ is well-defined when $\text{Re } q_1 \gg 0$ and $\text{Re } q_2 \gg 0$. Moreover $\xi$ has meromorphic continuation to the whole $H'_n \times \mathfrak{H}_n(\mathbb{R}) \times \mathbb{C}^2$.

Now we assume $h > 0$ and write $\pi h = AA^*$. Then we have

$$\eta(2y, \pi h; q_1, q_2) = \eta(2g, AA^*; q_1, q_2) = \det(\pi h)^{(q_1 + q_2 - n)} \cdot \eta(A^*2yA, 1; q_1, q_2)$$

and

$$\eta(g, 1; q_1, q_2) = e^{-\text{Tr}(g)2(q_1 + q_2 - n)}\Gamma_n(q_2)\det(2y)^{-q_2}\omega(2y; q_1, q_2),$$

where $\omega(z; p, q)$ is the function on $H'_n \times \mathbb{C}^2$ defined in [Shi82, p.281]. Shimura proves the following:

**Proposition 3.1** (Theorem 3.1 [Shi82]). $\omega(z, q_1, q_2)$ is a holomorphic function on $H'_n \times \mathbb{C}^2$ which satisfies

$$\omega(z; n - q_2, n - q_1) = \omega(z; q_1, q_2), \quad \omega(z, n, q_2) = \omega(z, q_1, 0) = 1.$$
Let $y = 1$ in (3.5). We have
\[
\eta(2, 2\pi h; q_1, q_2) = e^{-2\pi \Gamma(h) 2(q_1 + q_2 - n)n \Gamma_n(q_2) \det(4\pi h)^{-q_2} \omega(4\pi h; q_1, 1)}.
\]
Hence
\[
(3.6) \quad \xi(1, h; q_1, q_2) = i^{n(q_2 - q_1)} \cdot 2^{n(q_1 - q_2 - n + 1)} \cdot \pi^{nq_1} \cdot \Gamma_n(q_1)^{-1} \cdot \det(h)^{n-n} e^{-2\pi \Gamma(h) \omega(4\pi h; q_1, q_2)}.
\]
When $h = 0$, we have
\[
\eta(g, 0; q_1, q_2) = \Gamma_n(q_1 + q_2 - n) \det(g)^{q_2 - q_1}.
\]
Hence
\[
\xi(g, 0; q_1, q_2) = i^{n(q_2 - q_1)} \cdot 2^{n(k - 2n + 2)} \cdot \pi^{n^2} \cdot \Gamma_n(q_1)^{-1} \cdot \Gamma_n(q_2)^{-1} \cdot \Gamma_n(q_1 + q_2 - n) \det(g)^{n - q_2}.
\]
Given $Z \in X_{n,n}$, we can choose $p_\infty \in \mathcal{P}(\mathbb{R})$ such that $Z = p_\infty \cdot i$ and $\nu(p_\infty) = 1$. We write
\[
p_\infty = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{P}(\mathbb{R}), \quad a = d^{-*} \text{ and } Z = X_0 + iY_0
\]
with $X_0, Y_0 \in \mathcal{H}(\mathbb{C})$ and $Y_0 > 0$. From $p_\infty i = (a + b)d^{-1} = Z = X_0 + Y_0i$, we have $X_0 = bd^{-1}$ and $Y_0 = ad^{-1}$. Note that $\det(i) = \det(-i\epsilon)$ and $\det Y = \det(a^*a) \det(i\epsilon)$. Therefore
\[
\det(a)^{-k} J_{(0,k)}(p_\infty, i) = (\det Y_0)^{-k} \cdot \det(i\epsilon)^k.
\]
By the above formulae, a straightforward calculation shows that the local Fourier coefficient of $\phi = \phi_{X,s,\infty}$ is given by
\[
W_{\beta,\infty}(p_\infty, \phi_{X,s,\infty}) = \int_{\mathcal{H}(\mathbb{R})} \phi(wn(X_{\infty})p_\infty) \psi_\beta(X_{\infty})dX_{\infty}
= J_{(0,k)}(p_\infty, i)^{-1} (\det Y_0)^{n-k-s} \det(i\epsilon)^{-s} \psi_\beta(X_0) \xi(1, Y_0 \beta, k + s, s)
\]
By (3.4) and (3.5), the last equation equals
\[
(3.7) \quad J_{(0,k)}(p_\infty, i)^{-1} \cdot D_{p,q} \cdot \pi^{n(k+s)} \cdot \Gamma_n(k+s)^{-1} \det(\beta)^{k-n+s} \cdot 2\pi i \Gamma(h) \omega(4\pi Y_0 \beta; k + s, s),
\]
where
\[
D_{k,s} = i^{-nk} \cdot 2^{n(k-n+1)} \det(i\epsilon)^{-s}.
\]
Evaluating $\phi_{X,s,\infty}$ at $s = 0$, by Prop. 3.1 we obtain the local Fourier coefficient at the archimedean place.

**Proposition 3.2.** Let $\Lambda_{n,\infty}(s, \chi) = i^{-nk} \cdot 2^{-n(k-n+1)} \cdot \pi^{-n(s+k)} \cdot \Gamma_n(s+k)$. Then
\[
J_{(0,k)}(p_\infty, i) W_{\beta,\infty}(p_\infty, \phi_{X,s,\infty})|_{s=0} = \begin{cases} 
\Lambda_{n,\infty}(0, \chi)^{-1} \cdot \det(\beta)^{k-n} \cdot e^{2\pi i \Gamma(h) \omega(4\pi Y_0 \beta; k + s, s)} & \text{if } \beta > 0 \\
0 & \text{otherwise}.
\end{cases}
\]

**3.4. The local section at finite $v \neq p$.**

**3.4.1.** In this subsection, we give the choice of the local sections at finite places other than $p$. We first introduce some notation and definitions. For $n$ a positive integer and $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2n}(R)$ with $a, b, c, d \in M_n(R)$, we write $a = a_g$, $b = b_g$, $c = c_g$ and $d = d_g$. For $a$ a subset in $R$ and $x \in M_{n \times m}(R)$,
\[
x \prec a \iff x_{ij} \in a, \forall i,j.
\]
Put $F = \mathbb{Q}_v$ and $E = \mathcal{K} \otimes \mathbb{Q}_v$ for $v \in \mathcal{H}$. Let $O = \mathcal{O}_v$ and $\mathcal{R} = \mathcal{O}_k \otimes \mathbb{Z}_v$. Put
\[
\mathcal{D}_1[\mathfrak{c}_v] = \left\{ x \in \text{GL}_{2n}(E) \mid \det x \in \mathbb{R}^x, x \prec \begin{bmatrix} \mathcal{R} \\ \mathcal{R} \mathfrak{c}_v 1 \end{bmatrix} \right\}
\]
and $\mathcal{D}(\mathfrak{c}_v) = \mathcal{D}_1[\mathfrak{c}_v] \cap G(F)$. We define an open-compact subgroup $K$ of $G(\mathbb{A}_f)$ by
\[
(3.8) \quad K = \prod_{v \in S} \mathcal{D}(\mathfrak{c}_v) \times \prod_{v \in S} \mathbf{K}_v^0.
\]
Definition 3.3. The local sections and
\[
\text{we can find } \sigma \text{ and } \phi
\]
\[
\text{Proposition 3.4 (Shi97, Prop. 19.2).}
\]
\[
\text{Fourier coefficients at } v
\]
\[
\text{The local section at } v
\]
\[
\text{where } V = \mathbb{R}^{n+1}_-; \text{ is the standard spherical section in } \mathbb{R}^{n+1}. \text{ We let } f_\nu^\circ \text{ denote the unique section in } I_\nu(\chi, s) \text{ which is invariant by } K_\nu^0. \]

Definition 3.3. The local sections \( \phi_{\chi,s,v} \) for \( v \neq p \) are defined as follows.
\[
\phi_{\chi,s,v} = \begin{cases} f_\nu^\circ(gS^{-1}) & v \notin S \cup \{ p \} \\ f_\nu^\circ(gwS^{-1}) & v \in S \end{cases}
\]

Put \( u = \begin{bmatrix} 1 \\ \zeta \sigma \ast^* \end{bmatrix} \). Let \( \mathcal{L}_v = \mathcal{H}(F) \cap (uM_n(2\epsilon_v)u^*) \) be a lattice in \( \mathcal{H}(E) \). Then one checks easily that \( \phi_{\chi,s,v} \) for \( \nu \in S \) is the unique section such that
\[
supp \phi_{\chi,s,v} = P(F)\mathcal{W}(\mathcal{L}_v); \phi_{\chi,s,v}(\nu(u)) = \chi_v^{-1}(\det u) |\det(u\nu)|^{-s} \nu, u \in \mathcal{L}_v.
\]

Note that \( \phi_{\chi,s,v} \) for \( v \in S \) is supported in the big cell.

We will define the local section at \( p \) in §3.5.

3.4.2. Fourier coefficients at \( v \notin S \). When \( v \notin S \cup \{p\} \), \( \chi_v \) is unramified and \( \phi_{\chi,s,v}(g) = f_\nu^\circ(gS^{-1}) \), where \( f_\nu^\circ \) is the standard spherical section in \( I_\nu(\chi, s) \). In this case, the Fourier coefficients have been calculated by Shimura.

Proposition 3.4 (Shi97, Prop. 19.2). Let \( \beta = r \). We have
\[
W_\beta(f_\nu^\circ, 1) = \int_{\mathcal{L}(F)} \mathcal{F}_{\chi,s,v}(\mathcal{W}(X_v)) \psi_{-\beta}(X_v) dX_v
\]
\[
= \Lambda_{n,v}(s, \chi)^{-1} \Lambda_{\nu,v}^{\mathbb{R}}(s, \chi) R_{\beta,v}(\chi_{-\nu,v}(\nu_{\nu,v})) |\nu_{\nu,v}|_{\nu,v}^{2s} \|\gamma_{\mathbb{R}}(Z_{\nu,v})^{(\beta)}
\]
where
\[
\Lambda_{n,v}(s, \chi) = \prod_{j=0}^{n-1} L_{\nu}(2s - j, \chi_{+\nu,v}),
\]
\[
\Lambda_{\nu,v}^{\mathbb{R}}(s, \chi) = \prod_{j=0}^{n-1} L_{\nu}(2s - n - j, \chi_{+\nu,v})
\]
and \( R_{\beta,v}(T) \) is a polynomial which only depends on \( \beta \) and equals to 1 for almost \( v \).

We only need this result when \( \beta > 0 \) and \( \Lambda_{n,v}^{\mathbb{R}}(s, \chi) = 1 \).

3.4.3. Fourier coefficients at \( v \in S \). Notations are as in §3.4. The local Fourier coefficient of \( \phi_{\chi,s,v} \) is
\[
W_\beta(\phi_v, 1) = \chi_v^{-1}(\det u) |\det(u\nu)|_{\nu,v}^{2s} \int_{\mathcal{H}(F)} \psi_{-\beta}(X_v) dX_v
\]
\[
= \chi_v^{-1}(\det u) |\det(u\nu)|_{\nu,v}^{2s} \int_{\mathcal{L}(F)} \psi(\beta) \text{ vol}(\mathcal{L}, dX_v)
\]

3.5. The local section at \( v = p \).
3.5.1. Gauss sum and \(\epsilon\)-factor. Let \(d^x x\) be the Haar measure on \(\mathbb{Q}_p^*\) such that \( \text{vol}(\mathbb{Z}_p^d, d^x x) = 1\). Let \(\mu\) be a character of \(\mathbb{Q}_p^*\). We let \(c_p(\mu) = |\epsilon(\mu)|^{-1}\) if \(\mu\) is ramified and \(c_p(\mu) = p\) if \(\mu\) is unramified. We define the Gauss sum \(G(\mu)\) by
\[
G(\mu) = \int_{\mathbb{Q}_p^*} \mu(x)\psi_p(x) d^x x.
\]
Then \(\epsilon(s, \mu) := G(\mu)|^s, \psi_p|^{-1}\) is Tate’s \(\epsilon\)-factor. It is easy to verify that \(\epsilon(s, \chi) = |N|^s \chi(N)\tau(\mu)\), where \(\tau(\mu)\) is the classical Gauss sum which only depends on \(\mu|_{\mathbb{Z}_p^*}\). Define a Bruhat-Schwartz function \(\Phi_\mu\) on \(\mathbb{Q}_p\) by \(\Phi_\mu(x) := \mu(x)\mathbb{I}_{\mathbb{Q}_p}(x)\). Then one can compute its Fourier transform easily:
\[
\hat{\Phi}_\mu(y) = \begin{cases} 
\chi^{-1}(y)G(\mu)\mathbb{I}_{|N|-1\mathbb{Z}_p}(y) & \text{if } \mu \text{ is ramified with conductor } N, \\
\mathbb{I}_{\mathbb{Z}_p - |p|^{-1}\mathbb{Z}_p} & \text{if } \mu \text{ is unramified}.
\end{cases}
\]
We now introduce the modified \(p\)-Euler factor \(E_p(s, \mu)\).

**Definition 3.5** (Modified \(p\)-Euler factor).

\[
E_p(s, \mu) := Z(s, \mu, \hat{\Phi}_\mu) = \int_{\mathbb{Q}_p} \mu(x)|x|^s \hat{\Phi}_\mu(x) d^x x.
\]

The following identity inspired by (3.2a)[HT91] is our key to the construction of the local section.

\[
E_p(s, \mu) = \frac{L_p(s, \mu)}{\epsilon_p(1 - s, \mu^{-1})\epsilon(s, \mu)}.
\]

3.5.2. Some Bruhat-Schwartz functions. We introduce some special Bruhat-Schwartz functions on \(M_3(\mathbb{Q}_p)\).

**Definition 3.6.** Let \(J, N\) and \(N_2\) be three \(p\)-power integers. Let \(\mathcal{X}_3(J)\) be the subset of \(M_6(\mathbb{Q}_p)\) of the form
\[
\begin{bmatrix}
p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
J\mathbb{Z}_p & \mathbb{Z}_p^\times & \mathbb{Z}_p \\
J\mathbb{Z}_p^\times & \mathbb{Z}_p^\times & \mathbb{Z}_p^\times
\end{bmatrix}.
\]
Let \(I_0(N, N_2)\) be the subset of \(M_6(\mathbb{Q}_p)\) such that \(I_0(N, N_2)^t\) the transpose of \(I_0(N, N_2)\) is of the form
\[
\begin{bmatrix}
\mathbb{Z}_p^\times & N\mathbb{Z}_p & N\mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p^\times & N_2\mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p^\times & \mathbb{Z}_p^\times
\end{bmatrix}.
\]

For \(\mu = (\mu_2, \mu_3)\), we let \(\Phi_1^{\mu, J}\) be the Bruhat-Schwartz function in \(M_3(\mathbb{Z}_p)\) such that
\[
\Phi_1^{\mu, J}(Z) = \mathbb{I}_{\mathcal{X}_3(J)}(Z)\mu_2(2Z_{22})\mu_3(p^{-1}Z_{31}), Z = (Z_{ij}).
\]

Set
\[
\Phi_2^{\nu, N}(Z) = I_{I_0(N, N_2)^t}(Z) \prod \nu_i(Z_{ii}).
\]

Given \(\mu\) and \(\nu\), we let \(N_2 = c_p(\nu_2)\) and choose \(J\) and \(N\) such that
\[
N \geq J > \text{max}\{c_p(\mu_1), c_p(\nu_1)\}.
\]
Then (3.14) implies that \(I_0(N, N_2)\) is a group. Let \(\Phi_1 = \Phi_1^{\mu, J}\) and \(\Phi_2 = \Phi_2^{\nu, N}\), which satisfy the following properties:

**Lemma 3.7.**

1. \(\Phi_1(tZ) = \mu_2(t_2)\mu_3(t_3)\Phi(Z)\) and \(\Phi_1(Zt) = \mu_2(t_2)\mu_3(t_3)\Phi_1(Z)\) for \(t = \text{diag}(t_1, t_2, t_3)\).
2. \(\Phi_2(Z + M_3(\mathbb{Z}_p)) = \Phi_2(Z)\) and \(\hat{\Phi}_2(Zt) = \hat{\Phi}_2(tZ) = \nu(t)^{-1}\hat{\Phi}_2(Z)\) for \(t \in T_3(\mathbb{Z}_p)\).
3. \(\Phi_1(Zn) = \Phi_1(Z)\) for \(n \in N_3(\mathbb{Z}_p)\).

**Proof.** Straightforward verification from the definitions and (3.14).
3.5.3. The local section at \( p \). In this subsection, we define the local Godement section at \( p \). Let \((\chi_1, \chi_2) = (\chi_p, \chi_p^2)\) and \(\chi_+ = \chi_1 \chi_2\). For \( z = (z_1, z_2) \) a character of \( W_2 = \mathbb{Z}_p^6 \times \mathbb{Z}_p^6 \) of finite order, we define \( \mu = (\mu_2, \mu_3) \) and \( \nu = (\nu_1, \nu_2, \nu_3) \) as follows

1. \( \nu_1 = \chi_+^1, \nu_2 = \chi_1 z_2 \) and \( \nu_3 = \chi_1 z_1 \),
2. \( \mu_2 = \chi_2^{-1} z_2 \) and \( \mu_3 = \chi_2^{-1} z_1 \).

Recall that the Fourier transform \( \hat{\Phi} \) for \( \Phi \in \mathcal{S}(M_3(\mathbb{Q}_p)) \) is defined by

\[
\hat{\Phi}(Z) = \int_{M_3(\mathbb{Q}_p)} \Phi(X) \psi(\text{Tr}(iXZ))dX.
\]

Define \( \Phi_{z2}^{J,N} \) a Bruhat-Schwartz function on \( M_{3 \times 6}(\mathbb{Q}_p) \) by

\[
\Phi_{z2}^{J,N}(X, Y) := \Phi_1(X) \hat{\Phi}_2(Y).
\]

We consider the Godement section \( f_\Phi \) associated to \( \Phi = \Phi_{z2}^{J,N} \).

\[
f_\Phi(g) := \chi_2(\det g) |\det g|^s \int_{\text{GL}(\mathbb{Q}_p)} \Phi((0, Z)g) \chi_1 \chi_2(\det Z) |\det Z|^{2s} d^\times Z.
\]

The section \( f_\Phi \) has the following properties.

**Lemma 3.8.**

(1) \( f_\Phi(gt) = (z_1, z_2, x_2, x_1, x_2 x_1^{-1} z_2^{-1}, x_2 x_1^{-1} z_1^{-1}) f_\Phi(g) \) for \( t \in T_6(\mathbb{Z}_p) \).

(2) \( f_\Phi \) is \( N_6(\mathbb{Z}_p) \)-invariant.

**Proof.** (1) follows from Lemma 3.7. As for (2), for \( A, D \in N_3(\mathbb{Z}_p) \) and \( B \in M_3(\mathbb{Z}_p) \), we have \( \Phi_1(XA) = \Phi_1(X) \) since \( J \geq c_p(\mu_2) \). If \( X \in \text{supp} \Phi_1 \), we have \( XB \in M_3(\mathbb{Z}_p) \). Therefore by Lemma 3.7 (1) and (2), we have

\[
\Phi(X, Y) \begin{bmatrix} A & B \\ D & 0 \end{bmatrix} = \Phi_1(X) \hat{\Phi}_2(YD + XB) = \Phi_1(X) \hat{\Phi}_2(YD) = \Phi(X, Y)
\]

3.5.4. Local Fourier coefficient at \( p \). We compute the local Fourier coefficients of \( f_\Phi \). To emphasize the dependence of the local Fourier coefficient on \( z \), \( \chi \) and \( \beta \), we put

\[
W_\beta(\beta; \chi, z) := W_\beta(f_\Phi, 1).
\]

**Proposition 3.9.**

\[
W_\beta(\beta; \chi, z) = |p^{-1}] \text{vol}(I_0(N, N_2), d^\times Z) \chi_+(\det \beta) |\det \beta|^{2s-3} \cdot \ll_{\chi_3}(\beta)H(\beta),
\]

where

\[
H(\beta) = z_1 z_2^{-1}(p^{-1} \beta_{31}) \cdot \chi_2^{-1} z_2 (p^{-1} \det \begin{bmatrix} \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix}).
\]

In particular, \( W_\beta(\beta; \chi, z) = 0 \) if \( \det \beta = 0 \).

**Proof.** By definition,

\[
W_\beta(\beta; \chi, z) = \int_{M_5(\mathbb{Q}_p)} \int_{\text{GL}_3(\mathbb{Q}_p)} \Phi((0, Z) \begin{bmatrix} 0 & -1 \\ 1 & X \end{bmatrix}) \chi_+(\det Z) |\det Z|^{2s} d^\times Z d\psi^{-}(X) dx
\]

\[
= \int_{\text{GL}_3(\mathbb{Q}_p)} \Phi_1(Z) \Phi_2(t^Z^{-1} \beta) \chi_+(\det Z) |\det Z|^{2s-3} d^\times Z.
\]

As \( \Phi_2 \) is supported in non-degenerate matrices, we see \( W_\beta(\beta; \chi, z) = 0 \) if \( \det \beta = 0 \). If \( \det \beta \neq 0 \), we make a change of variable \( Z \mapsto Z \beta \). Then (17) equals

\[
\chi_+(\det \beta) |\det \beta|^{2s-3} \int_{\text{GL}_3(\mathbb{Q}_p)} \Phi_1(Z \beta) \Phi_2(t^Z^{-1} \beta) \chi_+(\det Z) |\det Z|^{2s} d^\times Z
\]

\[
= \chi_+(\det \beta) |\det \beta|^{2s-3} \int_{I_0(N, N_2)} \Phi_1(Z \beta) \Phi_2(t^Z^{-1} \beta) \chi_+(\det Z) dx
\]
For $Z \in I_0(N, N_2)$, we write $Z = u n_{23}(x)$, where
\[ u \in \begin{bmatrix} 1 & Z_p & Z_p \\ NZ_p & 1 & 0 \\ NZ_p & 0 & 1 \end{bmatrix}, t \in T_3(Z_p), \text{ and } n_{23}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}. \]

Then by Lemma 3.7 we have $\Phi_1(Z_\beta) = \mu(t) \Phi_1(n_{23}(x_\beta))$ and $\Phi_2(Z^{-1}) = \nu(t)^{-1}$. Since $\nu^{-1} t \chi_+ = 1$, we have
\[
W_p(\beta; Z, \beta) = \chi_+(\det \beta) |\det \beta|^{2x-3} \vol(I_0(N, N_2), d^x Z) \int_{Z_p} \Phi_1(n_{23}(x_\beta)) dx.
\]

We use the following lemma to compute (3.18).

**Lemma 3.10.** Let $\mathcal{X}_2(J) = \left[ JZ_p \atop \Z_p \atop \Z_p \atop \Z_p \atop \Z_p \atop \Z_p \atop \Z_p \atop \Z_p \right]$ and define the function $Q_{\lambda_1, \lambda_2}$ on $M_4(\Z_p)$ by
\[
Q_{\lambda_1, \lambda_2}(g) = \|\mathcal{X}_2(J)(g)\lambda_1(b)\lambda_2(p^{-1}c), g = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right].
\]

Then we have
\[
\int_{Z_p} Q_{\lambda_1, \lambda_2}(\left[ \begin{array}{c} 1 \\ 0 \\ x \\ 0 \end{array} \right]) dx = |p^{-1}J| \mathcal{I}_{\mathcal{X}_2(J)}(g)\lambda_1^{-1}\lambda_2(p^{-1}c)\lambda_1(-p^{-1}\det g).
\]

**Proof.** Write $Q = Q_{\lambda_1, \lambda_2}$. Then
\[
\begin{align*}
\int_{Z_p} Q_{\lambda_1, \lambda_2}(g) dx &= \int_{Z_p} Q_{\lambda}(\left[ \begin{array}{cc} a + xc & b + xd \\ c & d \end{array} \right]) dx \\
&= \lambda_2(p^{-1}c)\mathcal{I}_{\mathcal{X}_2(J)}(c)\mathcal{I}_{\mathcal{X}_2(J)}(d) \int_{Z_p} \mathcal{I}_{\mathcal{X}_2(J)}(a + xc)\lambda_1(b + xd) dx \\
&= \lambda_2(p^{-1}c)\mathcal{I}_{\mathcal{X}_2(J)}(c)\mathcal{I}_{\mathcal{X}_2(J)}(d) \int_{Z_p} \mathcal{I}_{\mathcal{X}_2(J)}(b)\lambda_1(b)\mathcal{I}_{\mathcal{X}_2(J)}(p^{-1}c + xp)\lambda_1(1 + xb^{-1}d) dx \\
&= \lambda_2(p^{-1}c)\mathcal{I}_{\mathcal{X}_2(J)}(c)\mathcal{I}_{\mathcal{X}_2(J)}(d) \int_{Z_p} \mathcal{I}_{\mathcal{X}_2(J)}(b)\lambda_1(b) \mathcal{I}_{\mathcal{X}_2(J)}(p^{-1}c + xp)\lambda_1(1 + xb^{-1}d) dx.
\end{align*}
\]

Since $b^{-1}dp^{-1}JZ_p \subset JZ_p$, the integral $\int_{Z_p} \mathcal{I}_{\mathcal{X}_2(J)}(p^{-1}c + xp)\lambda_1(1 + xb^{-1}d) dx$ equals
\[
\int_{-c^{-1}a + p^{-1}JZ_p} \lambda_1(1 - c^{-1}ab^{-1}d) dx = |p^{-1}J| \lambda_1(p^{-1}b^{-1}) \lambda_1(-p^{-1}\det \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]).
\]

To proceed the computation of (3.18), we note that
\[
\Phi_1(n(x)(\beta)) = Q_{\mu_2, \alpha_3}(\left[ \begin{array}{c} 1 \\ 1 \end{array} \right] \beta).
\]

Therefore the formula follows from Lemma 3.10 and the fact that $\mu_3\mu_2^{-1} = z_1z_2^{-1}$.

**Lemma 3.11** (Trace computation). Let $u \in \GL_3(\Z_p)$. Then we have
\[
W_\beta(f_\phi, \left[ \begin{array}{c} 1 \\ u \\ 1 \end{array} \right]) = \chi_2^{-1}(\det u) |\det u|^{t+3} W_p(u^{-1}\beta; \chi_2). \quad (3.19)
\]

In particular if $x \in c_p(z_1z_2^{-1})Z_p$, then
\[
W_\beta(f_\phi, \left[ \begin{array}{c} 1 \\ u(x) \end{array} \right]) = W_\beta(f_\phi, 1), \quad u(x) = \left[ \begin{array}{ccc} 1 & 0 \\ 1 & x \end{array} \right]
\]
Proof. We have
\[ u \cdot \hat{\Phi}_2(Z) = \hat{\Phi}(Z u) = \int \Phi(Y) \psi((Y Z u) dY = \int \Phi(Y) \psi((Y^t u) Z) dY \]
\[ = \int \Phi(Y^t u^{-1}) \psi((Y Z) dY = \psi u^{-1} \Phi. \]

Following the same computation in Prop. 3.9, we have
\[ W_\beta(f_\Phi, \left[ \begin{array}{c} 1 \\ 3 \\ u \end{array} \right]) \]
\[ = \chi_2(\det u) \det u^* \cdot W_\beta((I_3 \otimes u) \cdot \Phi, 1) \]
\[ = \chi_2(\det u) \det u^* \cdot \chi_+(\det \beta) \det \beta^{2-3} \int_{GL_3(\mathbb{Q}_p)} \Phi_1(\beta Z \beta) u \Phi_2(\beta Z^{-1}) \chi_+(\det Z) \det Z^{2s} d^s Z \]
\[ = \chi_2(\det u) \det u^* \cdot \chi_+(\det \beta) \det \beta^{2-3} \int_{GL_3(\mathbb{Q}_p)} \Phi_1(\beta Z \beta) \Phi_2(\beta Z u^{-1}) \chi_+(\det Z) \det Z^{2s} d^s Z \]
\[ = \chi_2(\det u) \det u^{s+3} \cdot W_p(u^{-1} \beta; \chi, \zeta) \]

The second assertion follows from (3.16) directly. □

3.6. Normalization. In this subsection, we take a suitable normalization of the Siegel-Eisenstein series on \( GU(3, 3) \) attached to the section we have defined. Put
\[ NC(\chi, s) = C_\chi(3)^{-1} A_{3}^{S,U}(s, \chi) \cdot \frac{1}{|p-1| J_0 \cdot \text{vol}(I_0(N, N_2), d^s)} \]

Definition 3.12. Let \( \Phi = \Phi_{2, \chi, \zeta}^{N} \) be as in §3.5. Define the section \( \phi_{\chi, s} \in I(\chi, s) \) and its normalization \( \phi_{\chi, s}^* \) by
\[ \phi_{\chi, s} = \bigotimes_{v \neq p} \phi_{\chi, s, v} \otimes f_\Phi, \phi_{\chi, s}^* := NC(\chi, s) \cdot \phi_{\chi, s}. \]

Define the normalized \textit{adelic} Siegel-Eisenstein series by
\[ E_h(g, s, \chi, \zeta, \epsilon) = NC(\chi, s) \cdot E_h(g, \phi_{\chi, s}) = E_h(g, \phi_{\chi, s}^*) \]
\[ = \sum_{\gamma \in \mathbb{P}(\mathbb{Q}) \setminus \mathbb{G}(\mathbb{Q})} \phi_{\chi, s}^* (\gamma g), \ g \in \mathbb{G}(A). \]

When \( k > 3 \), \( E_h(g, s, \chi, \zeta, \epsilon) \) converges absolutely at \( s = 0 \) (cf. [Shi97] and [Har84]). Thus \( E_h(g, s, \chi, \zeta, \epsilon)|_{s=0} \) is an element in \( A_{h}^{\text{Hol}}(\mathbb{G}, K_1^\infty, \chi^{-3}) \), where \( K \) is defined in (3.8).

Let \( E(\chi, \zeta, \epsilon) \) be the associated holomorphic Siegel-Eisenstein series as in (2.18). For \( (Z, g) \in X_{3,3} \times \mathbb{G}(A_f) \), we have
\[ E(\chi, \zeta, \epsilon)(Z, g) = \text{Ann}(E_h(-, \phi_{\chi, s}^*)|_{s=0})(Z, g) \]
\[ = \chi^3(v(g)) \cdot I_{0}(\chi, g_{\infty}, i) E_h((g_{\infty}, g, s, \chi, \zeta, \epsilon)|_{s=0}, \]

where \( g_{\infty} \in \mathbb{G}(\mathbb{R})^+ \) with \( g_{\infty} i = Z \).

By the inspection on Fourier coefficients of \( E(\chi, \zeta, \epsilon) \), this normalized Eisenstein series in fact is independent of the choice of \( N = (N, N_2) \) and \( J \) (See Remark 3.6).

Proposition 3.13. Let \( g \in \mathbb{G}(A_f) \) such that \( g_{p} = g_v = 1 \) at some \( v \in S \). Then \( E(\chi, \zeta, \epsilon) \) has the following Fourier expansion:
\[ \mathcal{F}_\beta(E(\chi, \zeta, \epsilon)) = \sum_{0 < \beta \in X(1)(\mathbb{Q})} a_{\beta}(g, \chi, \zeta) q^{\beta}, \]

where \( a_{\beta}(g, \chi, \zeta) = a_{\beta}^h(g, \chi, \zeta) \det \beta^{h-3} \) and
\[ a_{\beta}^h(1, \chi, \zeta) = \chi_{+}^{\beta}(\det \beta) \cdot \det \beta^{h-3} H(\beta) \cdot \chi_{S}^{\beta}(\det \epsilon u^{-1}) \cdot D(\chi, \zeta) \]

\[ \times \prod_{v \notin S \cup \{ p \}, v \mid \det \beta} R_{\beta, v}(\chi_{v}(\varpi_v)) \cdot \mathbb{I}_{\zeta}(\beta), \]

(3.20)
\[ D(\chi, \mathcal{L}) = \chi^{S/\{p\}}(\det \sigma)(\det u)^{-1} \prod_{v \in S} \text{vol}(\mathcal{L}_v, dx_v) \]

and

\[ \Xi = \prod_{v \in S} \mathcal{L}_v' \times \mathcal{K}_\lambda(p) \times \prod_{v \in S \setminus \{p\}} \mathcal{H}_3(\mathcal{L}_v)' \]

is a compact subset in \( \mathcal{H}_3(\mathbb{A}_f) \) independent of \( \chi \) and \( z \). Moreover, given \( g = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}, A \in \text{GL}_3(\mathbb{A}_{\mathbb{K}, f}) \times \text{GL}_3(\mathbb{O}_{\mathbb{K}} \otimes \mathbb{Z}_p) \), we have

\[
(3.21) \quad a_\beta(g, \chi, z) = \chi^{-1}(\det A) |A^*A|^3 \cdot a_{A^*A_\beta A}(1, \chi, z) \det \beta^{k-3}.
\]

**Proof.** Since the support of \( \phi_{\chi, s, v} \) at \( v \in S \) is in the big cell, the \( \beta \)-th Fourier coefficient of \( E_3(g, \phi_{\chi, s}) \) is decomposed into a product of local coefficients if \( x_v = 1 \) at some \( v \in S \). Hence the formula (3.20) and (3.21) follows from

\[
(3.22) \quad W_\beta(\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}, \phi) = \chi^{-1}(\det A) |A^*A|^{s+n} W_{A^*A_\beta A}(1, \phi)
\]

together with Prop. 3.2, (3.11), Prop. 3.9 and Prop. 3.4. \( \square \)

**Remark.** By \( q \)-expansion principle over \( \mathbb{C} \) and the formulae of the Fourier coefficients, the Eisenstein series \( E(\chi, z, c) \) is independent of the choice of \( J \) and \( \sum \in \Phi^J_{\mathbb{Z}} \). Also, by Lemma 3.11 we have

\[ E(\chi, z, c)(Z, g \begin{bmatrix} 1 \\ u \end{bmatrix}) = E(\chi, z, c)(Z, g), u = \begin{bmatrix} 1 & 0 \\ c_\beta(z_1z_2^{-1}) & 1 \end{bmatrix}. \]

### 3.7. \( p \)-adic Siegel-Eisenstein series

In this subsection, we construct an Eisenstein measure with values in the space of \( p \)-adic Siegel-Eisenstein series. We need to introduce some notation. Let \( K(c^n) \) be the ray class field of \( K \) with conductor \( cp^n \) and let \( \mathfrak{S} = \lim_n \text{Gal}(K(c^n)/K) \). Then \( \mathfrak{S} \) is a \( \mathbb{Z}_p \)-module of rank two. Let \( C(\mathfrak{S}, \mathbb{C}_p) \) be the set of continuous \( \mathbb{C}_p \)-valued functions on \( \mathfrak{S} \). Let \( \mathfrak{U}_0 \) be the set of locally algebraic \( p \)-adic characters of \( \mathfrak{S} \) with infinity type \( (k, 0) \), \( k \geq 3 \). Then \( \mathfrak{U}_0 \) is a Zariski-dense subset in \( C(\mathfrak{S}, \mathbb{C}_p) \). We shall regard \( p \)-adic Galois characters as \( p \)-adic Hecke characters of \( K \) by geometrically normalized reciprocity law. Recall that to an algebraic Hecke character \( \chi \) of \( K \) we have associated \( \hat{\chi} \) its \( p \)-adic avatar in the introduction.

To construct a \( p \)-adic measure on \( \mathfrak{S} \), we recall the "Abstract Kummer congruences". ([Kat78, Prop. 4.0.6] or [HLS06, Lemma 3.4.1]).

**Lemma 3.14 (Abstract Kummer congruences).** Let \( \mathbb{V} \) be a \( p \)-adic Banach space. We consider measures on \( \mathfrak{S} \) with values in \( \mathbb{V} \). Let \( \chi \mapsto m_\chi \) a function from \( \mathfrak{U}_0 \) to \( \mathbb{V} \), and let \( \lambda(m) \) denote the corresponding \( \mathbb{V}[1/p] \)-valued measure. Then \( \lambda(m) \) extends a \( p \)-adic measure if and only if, for every integer \( m \) and for any finite sum \( \sum_j \alpha_j \chi_j \) with \( \alpha_j \in (\mathbb{Z}/p^n \mathbb{Z})^* \) and for any summation \( \lambda(m) \) such that \( \sum_j \alpha_j \chi_j(t) \in p^m R \) for all \( t \in \mathfrak{S} \), we have

\[ \sum_j \alpha_j m_\chi_j \in p^m \mathbb{V}. \]

Recall that \( V_p(\mathbb{G}, \mathbb{K}) \) is the space of \( p \)-adic modular forms for \( \mathbb{G} = GU(3, 3) \) as in \( \S 2.8.4 \). Let \( C(\mathbb{G}) = \mathbb{P}(\mathbb{K}_p^{\mathfrak{u}}) \times \mathbb{P}(\mathbb{Z}_p) \). For an \( \mathbb{O}_K \)-algebra \( R = R_{(p)} \subset \mathbb{C} \), by \( q \)-expansion principle we have

\[ M^p(\mathbb{K}_p^{\mathfrak{u}}, \xi, R) = \left\{ f \in M^p(\mathbb{K}_p^{\mathfrak{u}}, \xi, \mathbb{C}) \mid \mathcal{F}_g(f) \in \prod_{\beta \in \mathcal{U}_3(K)} R : q^3, \forall g \in C(\mathbb{G}) \right\}. \]

Let \( \hat{E}(\chi, z, c) \) denote the \( p \)-adic avatar of \( E(\chi, z, c) \). Note that a Siegel modular form \( f \) over \( \mathcal{O}_v \) of weight \( k = (a, b) \) with Fourier expansion \( \mathcal{F}_g(f) \) at \( g = (g^p, g_p) \), \( g_p = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \), the Fourier expansion of the \( p \)-adic avatar \( \hat{f} \) at \( g \) is

\[ \mathcal{F}_g(\hat{f}) = (\det A)^a (\det D)^{-b} c_p(\mathcal{F}_g(f)). \]
Theorem 3.15. Let \( W_2 = (\mathbb{Z}_p^*)^2 \). There exists an Eisenstein measure \( dE_{3,3} \) on \( G \times W_2 \) such that for \( \hat{\chi} \in \mathcal{W}_0 \) and \( \hat{z} = (z_1, z_2) \in W_2 \), the set of finite order characters of \( W_2 \), we have

\[
\int_{G \times W_2} (\chi, \hat{z}) dE_{3,3} = \hat{E}(\chi, \hat{z}, c) \in \mathcal{V}_p(G, \mathbb{K})
\]

Proof. For \( (\hat{\chi}, \hat{z}) \in \mathcal{W}_0 \times W_2 \), we define

\[
\int_{G \times W_2} (\chi, \hat{z}) dE_{3,3} = \hat{E}(\chi, \hat{z}, c).
\]

We verify the above indeed gives a well-defined measure by Lemma 3.14. Write \( \chi_1 = \chi_p \), \( \chi_2 = \chi_p \) and \( \chi_{+p} = \chi_1 \chi_2 \). For \( \beta \in \mathfrak{H}_3(\mathbb{Q}) \), the \( \beta \)-th Fourier expansion of \( \hat{E}(\chi, \hat{z}, c) \) at the infinity cusp (3.20) is given by

\[
t_p(a_\beta^b(1, \chi, \hat{z}) \det \beta^{k-3}) = D(\hat{\chi}, \mathcal{L}) \cdot \hat{\chi}_{+p}(\det \beta) \cdot \hat{\chi}_2(u_3) \prod_{v \in S^0, v \mid \det \beta} R_{\beta,v}((\hat{\chi}_+(w_v)) \cdot z_1(u_1)z_2(u_2)
\]

\[
(3.23)
\]

where \( u_1, u_2 \) and \( u_3 \) are some \( p \)-adic units which only depend on \( \beta \), and \( R_{\beta,v} \) is a polynomial with coefficients in \( \mathbb{Z} \). In general by (3.20) for \( a_1, a_2 \in \text{GL}_3(\mathbb{Z}_p) \), the \( a_1 a_2 \)-th coefficient

\[
t_p\left[a_{a_1a_2}^h(1, \chi, \hat{z})(\det \beta)^{k-3}\right] \det(a_1 a_2)^k
\]

also has the form

\[
(3.24)
\]

The \( \beta \)-th Fourier coefficient of \( \hat{E} \) at the cusp \( x = \left[ \begin{array}{c} a_1^1 \\ a_2^{-1} \end{array} \right] \in G(\mathbb{Z}_p) \subset G(\mathbb{A}_f) \) is given by

\[
\det(a_2^k) \cdot t_p(a_\beta(x, \chi, \hat{z})) = \det(a_2^k) \cdot t_p\left[\chi_1^{-1}(\det a_1)\chi_2^{-1}(\det a_2) |\det a_2a_1|_p^3 a_{a_2a_1}^h(1, \chi, \hat{z}) \det \beta^{k-3}\right]
\]

\[
=t_p\left[a_{a_2a_1}^h(1, \chi, \hat{z}) \det \beta^{k-3}\right] \det(a_2a_1)^k \cdot \hat{\chi}_1(\det a_1^{-1}) \hat{\chi}_2(\det a_2^{-1})
\]

which also has the form as in (3.24). Since \( \mathcal{W}_0 \times W_2 \) is a Zariski-dense subset in \( C(\mathfrak{G} \times W_2, \mathbb{C}_p) \), by \( q \)-expansion principle for unitary groups ([Hid04] and [Cha08]) and the abstract Kummer congruences, we obtain the desired measure. \( \square \)

4. Eisenstein series on \( GU(3,1) \) and the pull back formula

4.1. Let \( W \) and \( V \) be the quadruples as in §3.1 with \( n = 3 \). Let \( \mathfrak{a} \) be a positive integer. We define the skew-Hermitian form \( \theta \) by \( \left[ \begin{array}{c} \mathfrak{a}^T \mathfrak{b} & \mathbb{I} \\ \mathbb{I} & \mathbb{I} \end{array} \right] \) according to the basis \( w^1, w^2 \). Let \( N_P \) and \( M_P \) be subgroups of \( G = GU(V) \) given by

\[
N_P = \left\{ n(x, t) = \left[ \begin{array}{cc} 1 & x \\ \mathbb{I}_2 & -\theta x^* \end{array} \right] : x \in \mathbb{K}^2, t \in \mathbb{Q} \right\}
\]

and

\[
M_P = \left\{ m(a, h) = \left[ \begin{array}{cc} \nu(h)a^{-1} & h \\ a & h \end{array} \right] : a \in \mathbb{K}^\times, h \in GU(W) \right\}.
\]

Then \( P := M_P N_P \) is the standard parabolic subgroup in \( G \). Put

\[
w = \left[ \begin{array}{cc} -1 \mathbb{I}_2 & -1 \\ 1 & \mathbb{I} \end{array} \right].
\]
Then \( w \) is the unique nontrivial Weyl element with respect to \( P \). For \( g \in P(\mathbb{A}) \), we put
\[
\delta_P(g) = |\nu(g)(dd)^{-1}|_\mathbb{A}.
\]

4.2. Open compact subgroups. For \( v \in \mathfrak{h} \), put \( \theta_v = \sigma_v \sigma_v^* \). Define an open-compact subgroup \( D_W(c) = \prod_{v \in \mathfrak{h}} D_W(c)_v \) of \( U(W)(\mathbb{A}_f) \) by
\[
\begin{align*}
(4.1) & \quad D_W(c)_v = \{ g \in K(L_v) \mid \theta_v^{-1}(\sigma_v g \sigma_v^{-1} - 1) \leq c_v \} \text{ if } v \neq p \\
(4.2) & \quad D_W(c)_p = I_0(p) := \left\{ g_p \in GL_2(\mathbb{Z}_p) \mid g_p \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \pmod{p} \right\}.
\end{align*}
\]
Let \( X = U(W)(\mathbb{Q}) \setminus U(W)(\mathbb{A}_f) \) and \( X_W(c) = X/D_W(c) \). We assume that \( c_v \) at some \( v \in S \) is sufficient small so that the right \( D_W(c) \)-action on \( X \) is free ([Shi97, Lemma 24.3]). As \( U(W) \) is a definite unitary group, \( X_W(c) \) is a finite set. By the weak approximation, we can find a set of representatives \( \{b\} \) in \( X_W(c) \) such that \( b_v = 1 \) at \( v \in S \cup \{p\} \). We will identify \( X_W(c) \) with this set.

4.3. Automorphic representations and automorphic forms on \( GU(2) \).

4.3.1. Put
\[
B = \{ g \in M_2(K) \cong \text{End}_\mathbb{C}(W) \mid g \theta g^* = \det(g) \theta \}.
\]
Then it is well-known that \( B \) is a definite quaternion algebra over \( \mathbb{Q} \) with local invariants \( \text{inv}_v(B) = (-\mathfrak{s}, -D)_v \).
We denote by \( \mathcal{S}_2(B^\times, \mathbb{C}) \) the space of automorphic forms on \( B^\times \) of weight 2. Namely
\[
\mathcal{S}_2(B^\times, \mathbb{C}) = \{ \text{locally constant } \mathbb{C}-\text{valued functions on } B^\times \setminus B^\times(\mathbb{A}_f) \}.
\]
Let \( \xi \) be a Hecke character of \( K \) of finite order and let \( \pi^B \) be an irreducible automorphic representation in \( \mathcal{S}_2(B^\times, \mathbb{C}) \) with central character \( \xi_p := \xi|_\mathbb{A} \). By the choice of \( B \), we have \( GU(W) = B^\times \times_{\mathbb{Q}^\times} K^\times \). Hence \( \Pi := \pi^B \boxtimes \xi \) can be regarded an irreducible automorphic representation on \( GU(W) \). Let \( \mathfrak{n} \) be the conductor of \( \pi^B \). We make the following assumption:
\[
p^2 \not| \mathfrak{n}.
\]
In addition to (S1), we further suppose throughout the paper that the ideal \( \mathfrak{c} \) chosen in the beginning of §3.2 is sufficiently small so that
\[
(S2) \quad \mathfrak{c} \subset c(\xi)c(\pi^B).
\]

Definition 4.1. For a subring \( A \subset \mathbb{C} \), we put
\[
\mathcal{S}_2(B^\times, \mathfrak{n}, \xi_+, A) = \{ f \in \mathcal{S}_2(B^\times, \mathbb{C}) \mid f(zgu) = f(g)\xi_+(z), \forall u \in I_1(\mathfrak{n}) \},
\]
the space of weight two modular forms of of level \( \mathfrak{n} \) with nebentypus \( \xi_+ \) over \( A \).

In the remainder of this section, having fixed \( B^\times, \mathfrak{n} \) and \( \xi_+ \), we simply write \( \mathcal{S}_2(A) \) for this space. The isomorphism \( \iota : \mathbb{C} \cong \mathbb{C}_p \) induces \( \iota : \mathcal{S}_2(\mathbb{C}) \cong \mathcal{S}_2(\mathbb{C}_p) \). Then since \( \pi^B \) is ordinary at \( p \), there is an element up to scalar \( f \) in \( \mathcal{S}_2(\mathbb{C}) \) such that \( f \) is an eigenform of the \( U_p \)-operator and the eigenvalue is a \( p \)-unit with respect to \( \iota \). We can further manage \( \iota(f) \in \mathcal{S}_2(\mathbb{C}_p) \) to be \( p \)-primitive, which means that \( \iota(f) \) takes value in \( \mathbb{Z}_p \) and \( \iota(f) \neq 0 \pmod{p} \). We define a distinguished element \( \tau_\mathfrak{n} = \prod \tau_v \in B^\times(\mathbb{A}_f) \)
\[
\tau_v = \begin{bmatrix} 1 \mathfrak{n} \\ \mathfrak{n}_v & 0 \end{bmatrix}
\]
if \( v|\mathfrak{n} \); \( \tau_v = 1 \), \( v \not|\mathfrak{n} \).

Then the map
\[
f \mapsto f|\tau_\mathfrak{n}(g) = f(g\tau_\mathfrak{n}), f \in \mathcal{S}_2(A)
\]
defines an involution on \( \mathcal{S}_2(A) \).
4.3.2. Local representations at $p$. We have fixed a decomposition $\mathcal{K} \otimes \mathcal{Q}_p = \mathcal{Q}_p e^+ \oplus \mathcal{Q}_p e^-$ with the idempotents in §1.1. Then as a subgroup in $\text{GL}_2(\mathcal{K} \otimes \mathcal{Q}_p)$, we identify $\text{GU}(W)(\mathcal{Q}_p)$ with $\text{GL}_2(\mathcal{Q}_p) \times \mathcal{Q}_p^\times$ by $g \mapsto (e^+ g, \nu(g))$, so $B^\times(\mathcal{Q}_p)$ and $U(W)(\mathcal{Q}_p)$ as subgroups in $\text{GL}_2(\mathcal{Q}_p) \times \mathcal{Q}_p^\times$ are isomorphic to $\text{GL}_2(\mathcal{Q}_p)$. We make these isomorphisms precise as follows.

\[
\begin{align*}
\text{GL}_2(\mathcal{Q}_p) & \to B^\times(\mathcal{Q}_p), \ g \mapsto (g, \det(g)), \\
\text{GL}_2(\mathcal{Q}_p) & \to U(W)(\mathcal{Q}_p), \ g \mapsto (g, 1)
\end{align*}
\]

We regard the local representation $\pi_p^B$ as a representation of $\text{GL}_2(\mathcal{Q}_p)$ by the above identification. We further assume $\pi_p^B$ is ordinary at $p$. Namely $\pi_p^B = \pi(\delta_1, \delta_2)$ where $\delta_1$ and $\delta_2$ are two characters on $\mathcal{Q}_p^\times$. Moreover we may further assume that $\delta_2$ is unramified with $\nu_2(\delta_1(p)) = \frac{1}{2}$ and $\nu_2(\delta_2(p)) = -\frac{1}{2}$. Thus $\pi_p^B$ is isomorphic to a sub-representation of the unitary induced representation $I(\delta_1, \delta_2)$.

Write $\xi_p = (\xi_p, \xi_{\overline{p}})$. According to the identification (4.3), as a representation of $\text{GL}_2(\mathcal{Q}_p) \times \mathcal{Q}_p^\times$

\[
\Pi_p = I(\delta_1 \xi_{\overline{p}}^{-1}, \delta_2 \xi_{\overline{p}}^{-1}) \boxtimes \xi_p.
\]

Similarly, as a representation of $\text{GL}_2(\mathcal{Q}_p)$

\[
\Pi_p|_{U(W)(\mathcal{Q}_p)} = I(\xi_{\overline{p}}^{-1} \delta_1, \xi_{\overline{p}}^{-1} \delta_2).
\]

We will fix the choice of these models in the remainder of the paper.

4.4. The embedding $U(V) \times U(W) \hookrightarrow U(W)$.

4.4.1. Recall that $\varsigma = -\frac{2}{3}$. Let $i$ denote the diagonal embedding

\[
i : U(V) \times U(W) \to U(W),
\]

\[
i : (\alpha, \gamma) \mapsto (\alpha, \gamma) \in \text{GL}(V \oplus W) = \text{GL}(W).
\]

As an embedding between matrix groups $i$ can written as

\[
U(V) \times U(W) \hookrightarrow U(W)
\]

\[
(\alpha, \gamma) \mapsto i(\alpha, \gamma) = (\alpha, \gamma)_{\Delta} := \Delta^{-1} \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix} \Delta,
\]

where

\[
\Delta = \Delta(\theta) = \begin{bmatrix} 1 & 1_2 & -\varsigma \\ 1 & 1 & \varsigma^* \\ -1_2 & \varsigma^* & 1 \end{bmatrix}.
\]

4.4.2. Imbedding of Hermitian symmetric domains. Let $X_{3,1}$ and $X_{2,0}$ be the Hermitian symmetric domains as in §2.4.1. We have the following embedding of Hermitian symmetric domains:

\[
X_{3,1} \times X_{2,0} \hookrightarrow X_{3,3},
\]

\[
(\tau, \xi_0) \mapsto Z_{\tau},
\]

where $Z_{\tau} = \begin{bmatrix} x & 0 \\ y & \varsigma \end{bmatrix}$ for $\tau = \begin{bmatrix} x \\ y \end{bmatrix}$ is as in (2.12). We write for $\alpha \in G$

\[
\alpha = \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & d \end{bmatrix}.
\]

Let $i = \begin{bmatrix} i & 0 \\ 0 & \varsigma \end{bmatrix}$. Then the automorphy factors are given by

\[
J((\alpha, \gamma)_{\Delta}, i) = \det \begin{bmatrix} hi + d & 0 \\ gi + f & \varsigma \end{bmatrix} = j(\alpha, i) \det \gamma.
\]

The embedding defined in (4.4) at the archimedean places is compatible with the identification $\mathbf{G}(\mathbb{R})/K_\infty \cong X_{3,3}$ in an obvious sense.
4.4.3. We record some formulae for the future use. Put

$$\omega = \begin{bmatrix} \theta & -1 \\ 1 & -\theta \end{bmatrix}, \eta = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$  

Then \( \Delta \eta \Delta^* = \omega. \)

Lemma 4.2.

$$\delta \gamma = \begin{bmatrix} a & b & c \\ \frac{1}{2}g & \frac{1}{2}(e + \gamma) & \frac{1}{2}f \\ h & l & d \end{bmatrix} \begin{bmatrix} \frac{1}{2}b \theta \end{bmatrix} \begin{bmatrix} \theta^{-1}g & \theta^{-1}(e - \gamma) & \theta^{-1}f \\ \frac{1}{2}(\hat{e} + \hat{\gamma}) \end{bmatrix},$$

where \( \hat{x} = \theta^{-1}x\theta \)

Proof. It follows from straightforward computation. Put

$$Y = \begin{bmatrix} 1 & \xi \\ \xi & 1 \end{bmatrix}.$$ Then

$$\delta \gamma = Y \begin{bmatrix} a & b & c \\ \frac{1}{2}g & \frac{1}{2}(e + \gamma) & \frac{1}{2}f \\ h & l & d \end{bmatrix} \begin{bmatrix} \frac{1}{2}b \theta \end{bmatrix} Y^{-1} \begin{bmatrix} \theta^{-1}g & \theta^{-1}(e - \gamma) & \theta^{-1}f \\ \frac{1}{2}(\hat{e} + \hat{\gamma}) \end{bmatrix}.$$ 

Put

$$A = \begin{bmatrix} a & b \\ g & e \end{bmatrix}, B = \begin{bmatrix} c & 0 \\ f & 0 \end{bmatrix}, C = \begin{bmatrix} h & l \\ \theta^{-1}g & \theta^{-1}(e - \gamma) \end{bmatrix}, D = \begin{bmatrix} d & 0 \\ \theta^{-1}f & 0 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$ 

then

$$AL = \begin{bmatrix} 0 & -b \xi \\ 0 & -c \xi \end{bmatrix},$$

$$CL = \begin{bmatrix} 0 & -l \xi \\ 0 & -\theta^{-1}(e - \gamma) \xi \end{bmatrix},$$

$$LD = \begin{bmatrix} 0 & 0 \\ -\xi \theta^{-1}f & -\xi \hat{\gamma} \end{bmatrix},$$

$$LC = \begin{bmatrix} 0 & 0 \\ -\xi \theta^{-1}g & -\xi \theta^{-1}(e - \gamma) \end{bmatrix},$$

$$LCL = \begin{bmatrix} 0 & 0 \\ 0 & \xi \theta^{-1}(e - \gamma) \xi \end{bmatrix},$$

and

$$\delta \gamma = \begin{bmatrix} 1 & -L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A - LC & B + AL - LCL - LD \\ C & D + CL \end{bmatrix}.$$ 

Then

$$\begin{bmatrix} a & b \\ g + \xi \theta^{-1}g & e + \xi \theta^{-1}(e - \gamma) \\ h & l \\ \theta^{-1}g & \theta^{-1}(e - \gamma) \end{bmatrix},$$

$$\begin{bmatrix} c & -b \xi \\ f + \xi \theta^{-1}f & -c \xi - \xi \theta^{-1}(e - \gamma) \xi + \xi \hat{\gamma} \\ d & -l \xi \\ \theta^{-1}f & \theta^{-1}(e - \gamma) \xi \end{bmatrix}$$
In particular, when $\gamma = \pi(\alpha) := e$, we have
\[
(\alpha, \pi(\alpha))_\Delta = \begin{bmatrix} a & b & c & -b\alpha \\
\frac{3}{2}g & e & \frac{3}{2}f & 0 \\
h & l & d & -k\alpha \\
\theta^{-1}g & 0 & \theta^{-1}f & \theta^{-1}\hat{e} \end{bmatrix}.
\]

4.5. Eisenstein series on $GU(V)$ and the pull-back formula.

4.5.1. Induced representations on $GU(V)$ and the pull-back section. Let $\chi$ be the Hecke character of $K$ in \S 3.2. Given an automorphic representation $\Pi$ of $GU(W)$, we define the induced representation $I(\Pi, \chi, s)$ of $G$ by
\[
I(\Pi, \chi, s) = \{ \phi : G(A) \rightarrow \Pi \mid \phi(pg) = \chi^{-1}(d_P)\delta_P(p)^s\Pi(r_P)\phi(h). \}
\]
Let $\varphi = \otimes_v \varphi_v$ be a decomposable vector in $\pi^B = \otimes_v^B$. We identify $\varphi$ with the section $\varphi \otimes \xi$ in $\Pi$. Let $f_s = \otimes_v f_{s,v}$ be a decomposable section of $I(\chi, s)$. Put
\[
\varphi.f_s(g) := \int_{U(W)(\mathbb{Q}_v)} f_{s,v}((g, hh')_\Delta)\chi_v(\det(hh'))\Pi_v(hh')\varphi dh,
\]
where $g \in G(\mathbb{Q}_v)$ and $h'$ is any element in $GU(W)(\mathbb{Q}_v)$ such that $\nu(h') = \nu(g)$. We call $\varphi.f_s$ the local pull-back section of $f_s$ by $\varphi_v$. It is easy to see that $\varphi.f_s(h)$ is independent of the choice of $g'$ and $\varphi.f_{s,v}$ is an element in $I_v(\chi, s)$.

The global pull-back section $\varphi.f_s$ of $f_s$ by $\varphi$ is defined as follows. For $g = (g_v) \in G(A)$,
\[
\varphi.f_s(g) := \int_{U(W)(A)} f_s((g, hh')_\Delta)\chi(\det(hh'))\Pi(hh')\varphi dh = \otimes_v(\varphi.f_{s,v})(g_v).
\]
It will follows from Prop. 5.5 that the local and global pull-back sections $f_s \cdot \varphi$ converge absolutely when $Re s \gg 0$ and has meromorphic continuation to $\mathbb{C}$.

4.5.2. Eisenstein series on $GU(V)$. Fix a $B^\times(\mathbb{A})$-embedding
\[
A : \pi^B = \bigotimes_v \pi_v^B \hookrightarrow S_2(B^\times, \mathbb{C}).
\]
Let $A_{\varphi, \xi} = A(\varphi) \otimes \xi$ be an automorphic form for $GU(W)$. We observe that
\[
A(\varphi.f_s(g))(x) = \int_{U(W)(A_f)} f_s((g, hh')_\Delta)\chi(\det hh')A_{\varphi, \xi}(xhh')dh, \ x \in GU(W)(A_f).
\]
Define a $\mathbb{C}$-valued function $I(\varphi, f_s)$ on $M(\mathbb{Q})\mathbb{N}(A)\backslash G(A)$ by
\[
I(\varphi, f_s)(g) = A(\varphi.f_s(g))(1), \ g \in G(A).
\]
By the general theory for Eisenstein series in [MW95], to $\varphi.f_s$ we can associate an Eisenstein series $E_h(g, \varphi.f_s)$ for $G$ defined by
\[
E_h(g, \varphi.f_s) := \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} I(\varphi, f_s)(\gamma g).
\]

4.5.3. The pull-back formula. In [Shi97], Shimura proves the following pull back formula:

**Theorem 4.3.** Let $f_s$ be as above. Then
\[
E_h(g, \varphi.f_s) = \int_{U(W)(\mathbb{Q}) \backslash U(W)(A)} E_h((g, h)_\Delta, f_s)\chi(\det h)A_{\varphi, \xi}(h)dh.
\]
Recall that we have introduced $\delta_1$ and $\delta_2$ characters of $\mathbb{Q}^*_p$ in §4.3.2. Let $\bar{z} = (\xi_{p}^{-1} \delta_1, \xi_{q}^{-1} \delta_2)$ be a pair of characters of $\mathbb{Q}^*_p$, to which we attach $f_\bar{z} \in I_p(\chi, s)$ in §3.5.3 with auxiliary choices of integers $J, N$. Define the section $\phi^{\chi}_{X,s} \in I(\chi, s)$ by

\[
\phi^{\chi}_{X,s}(g) = \bigotimes_{v \neq p} \phi^{X,v}_{\chi} (g) \bigotimes_{v = p} f_\bar{z}(g \Upsilon), \ g \in G(\mathbb{A}) \text{ where } \Upsilon = \begin{bmatrix} 1 & \frac{1}{2} & - \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \Delta^{-1}.
\]

Let $f$ be a primitive ordinary form as in §4.3. Then $f = A(\varphi)$ for some $\varphi \in \pi^B$, and $\varphi = \otimes_v \varphi_v$ is decomposable. Now we define the Eisenstein series obtained by the pull-back of $\phi^{\chi}_{X,s}$ and $f|_{\tau_n}$ by

\[
E_h(g, s, \chi \mid f, \xi, c) = \frac{NC(\chi, s)}{\text{vol}(D_W(\xi), dh)} \cdot E_h(g, (\phi^{\chi}_{X,s})^{p\beta}, ) \chi^{s_j} := \phi^{\chi}_{X,s}|_{\tau_n \varphi}.
\]

Let $E(\chi \mid f, \xi, c)$ be the associated holomorphic Eisenstein series as in (2.18). For $(\tau, g) \in \mathcal{X}_{s, 1} \times G(\mathbb{A}_f)$, we have

\[
E(\chi \mid f, \xi, c)(\tau, g) = \text{AM}(E_h(-s, \chi \mid f, \xi, c))(\tau, g) = \chi^{(h(g))} j(g_{\infty, i}) k E_h((g_{\infty, g}, s, \chi | f, \xi, c)|_{s = 0},
\]

where $g_{\infty} \in G(\mathbb{R})^+$ such that $g_{\infty} i = \tau$. By definition, $E(\chi \mid f, \xi, c)$ is the holomorphic Eisenstein series attached to the section

\[
\frac{NC(\chi, s)}{\text{vol}(D_W(\xi), dh)} \cdot (\phi^{\chi}_{X,s})^{p\beta}|_{s = 0}.
\]

Define an automorphic form $F$ for $GU(W)$ by

\[
F = A(\tau_n, \varphi) \bmod \xi = f|_{\tau_n, \varphi} \xi.
\]

Applying the pull-back formula, we obtain

\[
E(\chi \mid f, \xi, c)(\tau, g) = \frac{1}{\text{vol}(D_W(\xi), dh)} \int_{U(W)(\mathbb{Q}) \setminus U(W)(A_f)} E(\chi, z, c)(Z, (g, h) \Delta \Upsilon) \chi(\text{det } h) F(h) dh
\]

(4.14)

where $|\Gamma_h|$ is the order of the group $U(W)(\mathbb{Q}) \cap h D_W(\xi) h^{-1}$. It is well-known that $|\Gamma_h| = 1$ if $c$ is sufficiently small.

On the other hand, we regard $E(\chi, z, c)$ as geometric modular form over $\mathbb{C}$, and thus by the discussion in §2.6 and (2.14), we get

\[
E(\chi, z, c)(i_V, (\tau, g), (\omega_{W/C}, \omega_{W/C}(\Sigma^c))) = E(\chi, z, c)(\tau, (g, h) \Delta \Upsilon, \omega_{W/C}) .\]

Therefore by (4.14)

\[
E(\chi \mid f, \xi, c)(\tau, g, \omega_{W/C}) = \sum_{h \in \mathcal{X}_W(\xi)} E(\chi, z, c)(i_V, (\tau, g), (\omega_{W/C}, \omega_{W/C}(\Sigma^c))) \chi(\text{det } h) F(h).
\]

(4.15)

**4.5.4. The measure attached to ordinary p-adic Eisenstein series.** Let $c$ be the ordinary projector on the space of modular forms for the unitary group $U(\mathbb{H})$ constructed by Hida in [Hid02] and let $E_{\text{ord}}(\chi \mid f, \xi, c) = e. E(\chi \mid f, \xi, c)$ be the ordinary projection of $E(\chi \mid f, \xi, c)$. Let $\bar{E}_{\text{ord}}(\chi \mid f, \xi, c)$ be the p-adic avatar of $E_{\text{ord}}(\chi \mid f, \xi, c)$. Then $\bar{E}_{\text{ord}}(\chi \mid f, \xi, c) = e. \bar{E}(\chi \mid f, \xi, c)$, and by (4.15) we can deduce that

\[
\bar{E}_{\text{ord}}(\chi \mid f, \xi, c)(A) = \sum_{h \in \mathcal{X}_W(\xi)} \bar{E}(\chi, z, c)(A \times \mathbb{Z}_h \tau) \cdot \chi(\text{det } h) F(h)
\]

(4.16)

for a test object $A \in I_G(K^*_A)$. Now we can easily prove the following theorem.
Theorem 4.4. There exists a measure $d\xi^{\text{ord}}_{\xi, \ell}$ on $\mathcal{G}$ such that for any algebraic $p$-adic character $\hat{\chi}$ of infinity type $(k, 0)$, we have

$$\int_{\mathcal{G}} \hat{\chi} d\xi^{\text{ord}}_{\xi, \ell} = \hat{E}^{\text{ord}}(\chi \xi^{-1} | f, \xi, \ell),$$

where $\hat{E}^{\text{ord}}(\chi \xi^{-1} | f, \xi, \ell)$ is an ordinary $p$-adic Eisenstein series of weight $(0, k)$.

Proof. Note that $\xi$ is of finite order, we see that $\hat{\chi} \in \mathcal{W}_0 \iff \hat{\chi} \xi^{-1} = \hat{\chi} \xi^{-1} \in \mathcal{W}_0$. For $\phi \in \mathcal{C}(\mathcal{G}, \mathbb{C}_p)$ and $h \in U(W)(\mathbb{A}_f)$, we put

$$\phi(h(x) = \phi(x \det h) \xi^{-1}(x \det h).$$

We define the measure $d\xi^{\text{ord}}_{\xi, \ell}$ by the following rule: for a test object $A \in I_{C}(K^n)$,

$$\int_{\mathcal{G}} \phi d\xi^{\text{ord}}_{\xi, \ell}(A) := \sum_{h \in \mathcal{X}_W(\ell)} e. \int_{\mathcal{G}} \phi(h, z) d\xi^{\text{ord}}_{\xi, \ell}(A) \cdot F(h, z) = (\delta_1 \xi^{-1}_p, \delta_2 \xi^{-1}_p).$$

It is clear that the theorem follows from the definition of $d\xi^{\text{ord}}_{\xi, \ell}$ and (4.16).

5. Constant term of the Eisenstein series

5.1. Constant term of modular forms and the $\Phi$-operator. Let $f$ be a holomorphic modular form on $G = GU(V)$ of weight $k = (0, k)$. For the remainder of this article, we shall identify $GU(W)$ with the image $G_P$ in $M_P$ by $h \mapsto m(1, h)$.

Definition 5.1. Let $Z_{N_P}$ be the center of $N_P$. For $g \in G(\mathbb{A}_f)$, the analytic Siegel $\Phi$-operator at $g$ is defined by

$$M_k(G, \mathbb{C}) \rightarrow S_2(G_P, \mathbb{C})

f \mapsto \Phi^\text{an}_{[g]}(f)(h) = \int_{Z_{N_P}(\mathbb{Q}) \setminus Z_{N_P}(\mathbb{A})} f(zh \cdot (\tau_0, g)) dz, h \in G_P(\mathbb{A}_f).$$

The measure $dz$ is normalized so that $\text{vol}(Z_{N_P}(\mathbb{Q}) \setminus Z_{N_P}(\mathbb{A}), dz) = 1$. It is well-known that this definition is independent of the choice of $\tau_0$ and $\Phi^\text{an}_{[g]}(f) = \Phi^\text{an}_{[\mathfrak{m}_n]}(f)$ for $n \in N_P(\mathbb{A})$ (See the discussion in [Har84, 2.2.1]).

On the other hand, the constant term for an adelic automorphic form $F$ is defined by

$$F_P(g) = \int_{N_P(\mathbb{Q}) \setminus N_P(\mathbb{A})} F(ng) \, dn.$$

The following lemma is evident.

Lemma 5.2. Let $F \in \mathcal{A}_k^{\text{hol}}(G)$ and $f = \mathcal{A}M(F) \in M_k(G, \mathbb{C})$ be the associated holomorphic modular form, where $\mathcal{A}M$ is defined in (2.18). Then

$$F_P(hg) = \Phi^\text{an}_{[g]}(f)(h).$$

It is well-known that the (adelic) constant term of $E = E(g, \phi)$ is

$$E_P(g) = \phi(g) + M_\omega(\phi)(g), w = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix},$$

where if $\phi = \bigotimes_v \phi_v$,

$$M_\omega(\phi)(g) = \prod_v M_\omega(\phi_v); M_\omega(\phi_v)(g) = \int_{N_P(\mathbb{Q}_v)} \phi(ng) \, dn.$$
5.2. The local pull-back section at $\infty$. We compute the local pull back section and the intertwining operator of the section in (4.11) at the archimedean place.

Proposition 5.3.

(1) $\phi_{X,s,\infty}(g) = j(g,1)^{-k}\delta^s(g) \cdot \varphi_\infty$.

(2) $M\phi_{X,s,\infty}|_{s=0} = 0$.

Proof. Since $\pi_\infty$ is the trivial representation, we have

$$\phi_{X,s,\infty}(g) := \int_{G(\mathbb{R})} \phi_{X,s,\infty}((g,h)\Delta)\chi_\infty(h)\pi_\infty(h)\varphi_\infty dh = j(g,1)^{-k}\delta^s(g) \cdot \varphi_\infty.$$  

The first assertion follows. We proceed to show the second assertion. Write

$$M_{\phi_{X,s,\infty}} = \int_{N_P(\mathbb{R})} \phi_\infty(wn)dn$$

for

$$M_{\phi_{X,s,\infty}}(1) = \int_{C^2} \xi(1 + i[b], 0; s + k, s)db \wedge d\bar{b}, \quad [b] = \frac{1}{2}b\theta b^*$$

The last integral equals

$$\int_{C^2} (1 + i[b])^{-k-2s}db \wedge d\bar{b} = \det \left( \frac{i\theta}{2} \right)^{-2} \int_{C^2} (1 + bb^*)^{-1-2s}db \wedge d\bar{b}$$

$$= 4\pi^2(\det i\theta)^{-2} 4^2 \int_0^\infty (1 + x_1^2 + x_2^2)^{-1-k-2s}x_1x_2dx_1dx_2$$

$$= 2^4\pi^2(\det i\theta)^{-2} \frac{1}{(2s + k - 2)(2s + k - 3)}.$$  

Thus $M_{\phi_{X,s,\infty}}(1) = 0$ when $s = 0$. □

Corollary 5.4. The constant term of $E(\chi | f, \xi, \epsilon)$ is given by the section defined in (4.13).

Proof. This follows from the Prop. 5.3, (5.1) and (5.2). □

5.3. The local pull-back section at unramified places. For the remainder of this section, we put $F = \mathbb{Q}_v$ and $E = \mathcal{K} \otimes \mathbb{Q}_v$ for $v \in h$. Let $O$ and $R$ be the rings of integers of $F$ and $E$ respectively. In this subsection, we calculate the local pull-back section $\phi_{X,s,v}^p$ at $v \not\in S \cup \{p\}$. For $v \not\in S$, note that $U(W)(F) \cong U(1,1)(F)$ if $v$ is inert and $U(W)(F) \cong \text{GL}_2(F)$ if $v$ is split.

Let $f_{\varphi}^s$ be the unique spherical section in $L_v(\pi, \chi, s)$ such that $f_{\varphi}^s(1) = \varphi^\circ_v$.

Proposition 5.5. For $v \not\in S \cup \{p\}$

$$(f_{X,s,v}^s)^p = \frac{L_v(s - \frac{1}{2}, \pi, \chi)}{\Lambda_2(s, \chi^+)} f_{\varphi}^s.$$  

Proof. The is equivalent to

$$(f_{X,s,v}^s)^p(1) = \frac{L_v(s - \frac{1}{2}, \pi, \chi)}{\Lambda_2(s, \chi^+)} \varphi^\circ.$$  

The above local integrals have been computed by the doubling method in the following form.

Proposition 5.6. For $\varphi' \in \pi^\vee$, 

$$\langle f^p(1), \varphi' \rangle = \int_{U(W)(F)} \phi_{X,s,v}((1,h)\Delta)\chi(h)\varphi_\ell(h)\varphi' dh = \frac{L(s - \frac{1}{2}, \pi, \chi)}{\Lambda_2(s, \chi^+)} \langle \varphi, \varphi' \rangle,$$

where $\Lambda_2(s, \chi) = L(2s, \chi^+)(2s - 1, \chi, \tau_{K/Q})$.

Proof. This is the classical integral in the doubling method. We refer the computation to [GPSR87] and [Li92]. □
5.4. The local pull back section at \( v \in S \). In this subsection, we calculation the pull-back section of \( \phi_{\chi,s,v} \) when \( v \in S \). Let \( \theta_1 = \sigma^* \theta_\sigma \). Because \( L \) is an \( \mathcal{O} \)-maximal \( \mathcal{R} \)-lattice with respect to \( 2^{-1} \delta^{-1} \theta \) and \( \delta \) is chosen to be a generator of the different of \( K/\mathbb{Q} \), we have \( \theta_1 \prec 2 \mathcal{R} \) (cf. [Shi97, (20.6.1)])

\[
(5.3) \quad D_{\mathcal{V}}(\mathcal{C}) = \left\{ \alpha \in K_0^0 \mid \alpha \prec \begin{bmatrix} \mathcal{R} & \mathcal{R} & \mathcal{R} \\ \theta_1 \mathcal{C} & \mathcal{C} & \mathcal{C} \\ \mathcal{C} & \mathcal{C} & \mathcal{C} \end{bmatrix}, \theta_1^{-1} (e - 1) \prec \mathcal{C}, \ d - 1 \prec \mathcal{C} \right\}.
\]

If \( \alpha \in D_{\mathcal{V}}(\mathcal{C}) \), the relation \( \alpha \theta_{3,1} \alpha^* = \theta \) implies \( d^* a + f^* \theta_1 - c^* h = 1 \), hence \( a - 1 \prec \mathcal{C} \). From the identity \( \alpha^{-1} = \theta_{3,1} \alpha^* \theta_{3,1}^{-1} \in K_0^0 \), we see that

\[
(5.4) \quad \alpha^{-1} = \begin{bmatrix} -d^* & f^* \theta_1^{-1} - c^* \\ -\theta_1 f^* & \theta_1 a^* & \theta_1 b^* \\ -h^* & g^* \theta_1^{-1} - a^* \end{bmatrix} \prec \mathcal{R}.
\]

This shows \( D_{\mathcal{V}}(\mathcal{C}) \) is an open-compact subgroup in \( G(\mathcal{F}) \).

Let \( f_S(g) = f_{\mathcal{V},v}(gS^{-1}) \) be the section defined in (3.9). Now we compute the pull-back section \( f_S^{ph} \) of \( f_S \).

**Proposition 5.7.** \( f_S^{ph} \) is the unique section in \( I(\pi, \chi, s) \) such that

1. \( \text{supp} f_S^{ph} \subset P(\mathcal{F})D_{\mathcal{V}}(\mathcal{C}) \),
2. \( f_S^{ph}(\alpha) = \text{vol}(D_W(\mathcal{C}, dh) \hat{\varphi}_v) \), where \( D_W(\mathcal{C}) \) is the group defined in (4.1).

**Proof.** Let \( \alpha \) be in the support of \( f_S^{ph} \). To prove (1), we may assume \( \alpha \in K_0^0 \) by the Iwasawa decomposition.

By definition, we have

\[
(5.5) \quad \alpha_1 \prec \mathcal{R}, \theta_1^{-1} f \prec \mathcal{R} \quad \text{such that} \quad \frac{1}{2} f^* \theta_1 f = f^* \theta_1^{-1} \mathcal{R} \prec \mathcal{R}.
\]

Because \( f_S((\alpha, \gamma)\Delta) \neq 0 \) for some \( \gamma \in U(W(\mathcal{F})) \), we have \( S(\alpha, \gamma) \Delta S^{-1} = \Delta^{-1}(\alpha_1, \gamma_1) \Delta_1 \in P(F)D(\mathcal{C}) \), where \( \Delta_1 = S \Delta S^{-1} = \Delta(\theta_1) \) as in (4.4). By the formula (4.6), we find that

\[
\begin{bmatrix} 1 & \theta_1^{-1} \\ \theta_1^{-1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} d & h \\ f & e - \gamma_1 \end{bmatrix} \begin{bmatrix} \theta_1^{-1} & d^{-1} h \\ \theta_1^{-1} (g - fd^{-1} h) & \theta_1^{-1} (e - fd^{-1} l) - 1 \end{bmatrix} \prec \mathcal{C}.
\]

Observe that

\[
\begin{bmatrix} 1 & \gamma_1^{-1} \\ \gamma_1^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} f^* \theta_1^{-1} f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{a} & 0 & 0 \\ 0 & \hat{a} & 0 \\ \hat{b} & \hat{c} & \hat{d} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & d \end{bmatrix} \in D_{\mathcal{V}}(\mathcal{C}).
\]

Hence \( \alpha \in P(F)D_{\mathcal{V}}(\mathcal{C}) \). This completes the proof of (1).

We proceed to prove (2). If \( \alpha \in D_{\mathcal{V}}(\mathcal{C}) \), then by Lemma 4.2 and (5.4) together with the fact that \( \theta_1 \prec 2 \mathcal{R} \), we can deduce that

\[
S(\alpha, 1) \Delta S^{-1} \in D_0(\mathcal{C}) \quad \text{such that} \quad \frac{1}{2} (\theta_1^{-1} e \theta_1 + 1) \equiv 1 \mod \mathcal{C}.
\]

Because \( \mathcal{C} \) is sufficiently small as in (S2), we conclude that

\[
f_S((\alpha, 1)\Delta) = \chi^{-1} (\det \sigma^*).
\]

Now we use the following lemma.

**Lemma 5.8.** Let \( h \in U(W(\mathcal{F})) \), then

\[
(1, h) \Delta S^{-1} \in P(F)D(\mathcal{C}) \Leftrightarrow h \in D_W(\mathcal{C}).
\]
PROOF. Since
\[ Y_1 S(1, h_1) \Delta S^{-1} Y_1^{-1} = \begin{bmatrix} 1 & 1 \theta_1^{-1} \sigma h_1 \theta_1^{-1} \\ \theta_1^{-1} (1 - h_1) & 1 \end{bmatrix} \]
with \( h_1 = \sigma h \sigma^{-1} \), we have
\[
(1, h_1) \Delta \in P(F) D(\epsilon) \iff \theta_1^{-1} (h_1^{-1} - 1) = y \text{ for some } y \prec \epsilon \\
\iff h \in D_W(\epsilon)
\]
By Prop. 5.7,
\[
fs((1, h) \Delta) = \chi^{-1} (\text{det } h) \tilde{\varphi} dh
\]
Therefore
\[
f_S^p(\alpha) = \int_{U(W)(F)} fs((\alpha, 1) \Delta (1, h) \Delta) \chi(h) \pi(h) \cdot \tilde{\varphi} dh
\]
\[
= \chi^{-1} (\text{det } \sigma^*) \int_{D_W(\epsilon)} fs(1, h) \Delta \chi(h) \pi(h) \cdot \tilde{\varphi} dh
\]
\[
= \chi^{-1} (\text{det } \sigma^*) \text{vol}(D_W(\epsilon), dh) \tilde{\varphi}.
\]
This completes the proof of (2) \( \square \)

Prop. 5.7 shows that \( f_S^p \) is the unique section in \( I(\pi, \chi, s) \) such that
\[
supp f_S^p = P(F) D_W(\epsilon)
\]
\[
f_S^p(\alpha) = \chi^{-1} (\text{det } \sigma^*) \cdot \text{vol}(D_W(\epsilon), dh) \tilde{\varphi}, \alpha \in D_W(\epsilon).
\]

Now we are ready to determine \( \phi_{\chi, s, v}^p \). Note that \( \phi_{\chi, s, v}^p = fs|w', w' = (w, 1_2) \Delta \). Thus
\[
\phi_{\chi, s, v}^p = (fs|w')^p(g) = \int_{U(W)(F)} fs((g, h) \Delta w') \chi(\text{det } h) \pi(h) \tilde{\varphi} dh = f_S^p(gw), w = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}.
\]
Now the following proposition is straightforward.

**Proposition 5.9.**
\[
\phi_{\chi, s, v}^p = \text{vol}(D_W(\epsilon), dh_v) f_{\chi, s, v}^p,
\]
where \( f_{\chi, s, v}^p \) is the unique section supported in \( P(F) D_V(\epsilon) w^{-1} \) and
\[
f_{\chi, s, v}^p(uw^{-1}) = \chi_v^{-1} (\text{det } \sigma^*) \tilde{\varphi}_v, u \in D_V(\epsilon).
\]

6. **The ordinary projection of the local pull-back section**

6.1. We have computed the local pull back section \( \phi_{\chi, s, v}^p \) at places other than \( p \). In this section, we compute the ordinary projection \( e. \phi_{\chi, s, p} \) of the pull-back section \((f_{\beta}|Y)^p \) by using the ordinary linear functional for a regular principal series. This idea is inspired by Hida’s proof of multiplicity one theorem for ordinary vectors ([Hid04, Theorem 5.3]).

6.2. The ordinary linear functional.
6.2.1. In this subsection, we let $B$ denote the standard Borel subgroup of $\text{GL}_n$, $T$ and $N$ denote the diagonal matrices and the unipotent radical of $B$ respectively. We let $W$ be the Weyl group of $\text{GL}_n$ with respect to $T$. Consider the (unitary) induced representation $\mathcal{F} = \text{Ind}_{B}^{\text{GL}_n}(\lambda_1, \cdots, \lambda_n)$. We assume $\mathcal{F}$ is regular, which means the $p$-adic valuation of $\lambda_{i}(p)$ are distinct. We call these numbers $v_p(\lambda_{i}(p))$ $p$-adic weights.

Let $D = \{d \in T(\mathbb{Q}_p) \mid d^{-1}N(\mathbb{Z}_p) d \subset N(\mathbb{Z}_p)\} = \{\mu(p) \mid (\mu, R_{+}) \geq 0\}$. Let $\mathcal{A}_p := \mathbb{Z}_p[t_1, t_2, \cdots, t_n, t_n^{-1}]$ be the Atkin-Lehner ring of $G(\mathbb{Q}_p)$, where $t_i$ is defined by

$$t_i = [N(\mathbb{Z}_p)\alpha_i, N(\mathbb{Z}_p)], \quad \alpha_i = \left[1_{n-i}, p \cdot 1_i\right].$$

$t_i \in \mathcal{A}_p$ acts on $\mathcal{F}^{N(\mathbb{Z}_p)}$ by

$$v|t_i = \sum_{x \in N/\alpha_i^{-1} N} x_i \alpha_i^{-1} \cdot v.$$ We also define the action of $\mathcal{A}_p$ on the Jacquet module $J(\mathcal{F}) = \mathcal{F}^{N}$ of $\mathcal{F}$ by

$$v|t_i = \delta(\alpha_i)\alpha_i^{-1} \cdot \tilde{v}$$

so that the natural projection $\mathcal{F} \twoheadrightarrow J(\mathcal{F})$ is a $\mathcal{A}_p$-module homomorphism. Hida proves the following theorem in [Hid04].

**Theorem 6.1.** Let $\mathcal{F}^{\circ}$ be the maximal subspace of $\mathcal{F}^{N}$ on which the action of $\mathcal{A}_p$ is semisimple. Then the natural projection induces an isomorphism as $\mathcal{A}_p$-modules

$$\mathcal{F}^{\circ} \cong J(\mathcal{F}).$$

The linear functionals $l_w$ on $J(\mathcal{F})$ for $w \in W$ are defined by $l_w(f) := M_w(f)(1_n)$, the evaluation of the intertwining operator at the identity. It is well-known that $l_w$ enjoys the following properties:

$$l_w(u \cdot f) = l_w(f) \quad \text{and} \quad l_w(t \cdot f) = \lambda(wtw^{-1})\delta^\frac{1}{2}(t)l_w(f) \quad \text{for} \quad u \in U(\mathbb{Q}_p), \quad t \in T(\mathbb{Q}_p).$$

By definition,

$$l_w(f) = \int_{U_{w}^{-}\setminus U} f(wu) \, du, \quad \text{where} \quad U_{w}^{-} = \prod_{\alpha > 0, w_{\alpha} > 0} U_{\alpha}(\mathbb{Q}_p), \quad w_{\alpha}(t) := \alpha(w^{-1}tw)$$

whenever the integral is convergent. These $l_w$'s induce a $T(\mathbb{Q}_p)$-equivariant map

$$\bigoplus_{w \in W} l_w : J(\mathcal{F}) \longrightarrow \bigoplus_{w \in W} \mathbb{C} \cdot \lambda^{w} \delta^\frac{1}{2}, \lambda^{w} : = \lambda(wtw^{-1}).$$

Now we define the ordinary function $l_{w_3}$ as follows. Let $\mu_i \in X(\mathbb{Z})$ be a cocharacter such that

$$\mu_i(x) = \left[\begin{array}{c} 1_{i-1} \\ x \\ 1_{n-i} \end{array}\right] \in x \in \mathbb{Q}_p^N.$$ For $w \in W$, we put

$$a_i = v_p(\lambda(\mu_i(p))) \in \mathbb{Z}.$$ Since $\mathcal{F}$ is regular, these $a_i$ are distinct integers. Hence we can let $W$ act on $\{a_i\}_{i=1}^{n}$ by $a_{w_{i}} = v_p(\lambda(w(\mu_i(p))))$.

We let $w_3$ be the unique element in $W$ such that

$$a_{w_3,1} < a_{w_3,2} < \cdots < a_{w_3,n}.$$ Let $\alpha = \sum_{i=1}^{n} (i - 1) \mu_i$ and let $t_\alpha = [N(\mathbb{Z}_p)\alpha(p)N(\mathbb{Z}_p)] = t_1 t_2 \cdots t_{n-1}$. Then the normalized $u_p$-operator on $\mathcal{F}$ is defined by

$$v|t_\alpha := \delta(\alpha_i)^{-\frac{1}{2}} \lambda^{w_3}(\alpha_i) \cdot v$$

and $u_p \cdot v = v|t_\alpha$.

The Hida’s idempotent $e$ attached $u_p$ is defined by

$$e = \lim_{n \to \infty} (u_p)^{n!}.$$ Now we prove the following key lemma.
Lemma 6.2.

\[ l_w(c,v) = \begin{cases} l_{w_3}(v) & w = w_3, \\ 0 & w \neq w_3. \end{cases} \]

**Proof.** Put \( b_{w,i} = \sum_{j=i}^{n} a_{w,j}. \) Then

\[ v_p(\lambda(\alpha(p))) = \sum_{i=1}^{n} i \cdot a_{w,i} = \sum_{i=2}^{n} b_{w,i}. \]

It is easy to see that if \( w \neq w_3, \)

\[ b_{w_3,i} \geq b_{w,i} \text{ and } b_{w_3,i} > b_{w,i_0} \text{ for some } i_0. \]

Put \( D_w = \sum_i b_{w_3,i} - b_{w,i} \geq 0 \) and \( D_w = 0 \) only if \( w = w_3. \) From the identity

\[ l_w(v) = \delta(\alpha) l_w(\alpha^{-1}v) = \delta \lambda^w(\alpha)^{-1} l_w(v), \]

we see that \( l_w(u_p.v) = p^{\tau D_w} l_w(v), \) hence the assertion follows.

\[ \square \]

**Remark.** From this lemma together with Theorem 6.1, we see that

\[ \dim e .. \mathcal{F} \leq 1, \]

which has been proved in [Hid04, Theorem. 5.3] by essentially the same argument.

6.2.2. The functional \( l_{w_3}. \) We apply the above discussion to our case \( I(s) = I_p(\pi, \chi, s). \) Notations are as in §4.3.2. For brevity, let \( (\xi_p, \xi_p) = (\xi_1, \xi_2) \) and \( (\chi_1, \chi_2) = (\chi_p, \chi_p). \) Put

\[ \mathcal{F}(s) = I(\chi_2 | \xi_1^{-\frac{3}{2}}, \xi_2^{-\frac{1}{2}}, \delta_1, \xi_2^{-\frac{1}{2}}, \chi_1^{-1} | \xi_1^{-\frac{3}{2}}) \]

We identify these representations of \( GL_4(\mathbb{Q}_p) \times \mathbb{Q}_p^* \) with each other via the following map \( L: \)

\[ L : I_p(\pi, \chi, s) \cong I(\chi_2 | \xi_1^{-\frac{3}{2}}, \xi_2^{-\frac{1}{2}}, \delta_1, \xi_2^{-\frac{1}{2}}, \chi_1^{-1} | \xi_1^{-\frac{3}{2}}) \]

\[ f \mapsto g \mapsto L(f(g)) := f(g)(1_n) \]

Let \( \mathcal{F} := \mathcal{F}(0). \) Then the \( p \)-adic weights of \( \mathcal{F} \) are \( (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, k - \frac{3}{2}). \) Since \( k \geq 4, \mathcal{F} \) is a regular principal series. We put

\[ w_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad s_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad w_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \]

Let \( \phi^{w_3} \in I_p(\chi_1^{-\frac{3}{2}}, \chi_2^{-\frac{1}{2}}, \xi_2^{-\frac{1}{2}}, \delta_1) \) denote the unique section which is supported in \( Bw_1N(\mathbb{Z}_p) \) and invariant by \( N(\mathbb{Z}_p). \) Then we define \( \phi^{ord} := L^{-1}(M_{s_1} \phi^{w_3}) \in I(\pi, \chi_1, 0). \) Thanks to the following lemma, we can cut off the ordinary projection from a given section by using the ordinary functional.

Lemma 6.3. \( \phi^{ord} \) is an ordinary section in \( I_p(\Pi, \chi, 0) \) as in [Hid04], and

\[ e.f = l_{w_3}(f) \phi^{ord}, \quad \forall f \in I_p(\Pi, \chi_1, 0). \]

**Proof.** First we observe that the section \( \phi^{w_3} \) is supported in the big cell, and then it is an eigenvector of the \( u_p \)-operator. By (6.1),

\[ \phi^{ord}||t_1(x) = \delta(\alpha_1)^{-\frac{1}{2}} \lambda^{w_3}(\alpha_1) \sum_{w \in \mathcal{U}/U_{\alpha_1}} \phi(xuw^{-1}). \]

Straightforward computation shows \( \phi^{ord}||t_1 = \phi^{ord}, \) hence \( u_p \phi^{ord} = \phi^{ord} \) is an ordinary section. This proves the first assertion.

Now we prove the second assertion. First \( M_{s_1} \phi^{w_3} \in \mathcal{F} \) and \( l_{w_3}(M_{s_1} \phi^{w_3}) = M_{w_3}(\phi^{w_3})(1) = 1, \) so \( \phi^{ord} \) is a non-zero ordinary vector. The space of ordinary forms has at most one dimensional (Remark 6.2.1), so \( e.f = \beta \cdot \phi^{ord} \) for some \( \beta. \) By Lemma 6.2 we see \( \beta = l_{w_3}(f). \)

\[ \square \]
6.3. The ordinary section in $\pi^B_p$. We study the ordinary section in $\pi^B_p$. As in §4.3.2, $\pi^B_p$ is ordinary and is regarded as a sub-representation of $I(\delta_1, \delta_2)$.

**Proposition 6.4.** Then there exists $\varphi$ a unique section in $\pi^B_p \subset I(\delta_1, \delta_2)$ such that $\varphi$ is invariant by $I_1(p)$ and $\varphi|U_p = a_p(f)\varphi$, where $a_p(f) = \delta_2(p)^{-1}|p|^{-\frac{1}{2}}$. Moreover $\varphi(1) = 1$. $\varphi$ is called the ordinary section of $\pi^B_p$.

**Proof.** This is well-known. We recall the proof here, as we need the construction of $\varphi$ later. There are three cases:

- $\pi^B_p$ is a ramified principal series. In this case, $\delta_1$ is ramified with the conductor $p$. Thus $\varphi$ is in fact the new vector in $I(\delta_1, \delta_2)$. Namely, $\varphi$ is the unique function such that $\varphi(1) = 1$ and
  \[ \varphi(h \begin{bmatrix} a & b \\ pc & d \end{bmatrix}) = \delta_1(a)\varphi(h), \forall \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in I_0(p). \]

- $\pi^B_p$ is a unramified principal series. We have $I(\delta_1, \delta_2)I_0(p) = \mathbb{C}\delta^1 \oplus \mathbb{C}\delta^w$, where $\delta^1$ is the function with $\text{supp} \delta^1 = B(\mathbb{Q}_p/I_0(p))$ and $\delta^1(1) = 1$, and $\delta^w$ is the function with $\text{supp} \delta^w = B(\mathbb{Q}_p\mathbb{K}(\mathbb{Z}_p))$ and $\delta^w(1) = 1$. Let $\alpha = \delta_1 |p|^{\frac{1}{2}}$ and $\beta = \delta_2(p)|p|^{\frac{1}{4}}$. By a simple calculation, we find that
  \[ \delta^1|U_p = \beta^{-1}\delta^1 + (1-|p|)\beta^{-1}\delta^w \]
  \[ \delta^w|U_p = \alpha^{-1}\delta^w. \]

From the above, we can solve for $\varphi$ easily:

\[ \varphi = \delta^1 + \frac{1-|p|}{1-\delta_1(\delta_2(p))}\delta^w. \]

- $\pi^B_p$ is special. In this case, $\delta_2$ is unramified and $\delta_2^{-1} = |\cdot|$, and we have
  \[ 0 \rightarrow \pi^B_p \rightarrow I(\delta_1, \delta_2) \rightarrow \mathbb{C} \cdot |\cdot|^{\frac{1}{2}} \rightarrow 0. \]

Hence $\varphi$ must be of the form as in (6.5). As $U_p$ acts on $\mathbb{C} \cdot |\cdot|$ as a scalar $\alpha^{-1}$, we conclude that $\varphi$ is in $\pi^B_p$. In either of the above three case, we find that $\varphi(1) = 1$. □

When no confusion arises, we will identify $\varphi \otimes \xi_p$ the ordinary section of $\Pi_p = \pi^B_p \otimes \xi_p$, and its restriction to $U(W)(\mathbb{Q}_p)$ as a section in $\pi_p = \Pi_{U(W)}(\mathbb{Q}_p)$ is still denoted by $\varphi$. Note that when $\varphi_p$ is viewed as a section of $\pi_p = I(\delta_1, \delta_2) \otimes \xi_p^{-1}$, the eigenvalue of $U_p$-operator is $\xi_2(a_p(f))$.

The following lemma roughly says that the ordinary section is controlled by the evaluation at $w_3$.

**Lemma 6.5.** Let $W$ be the Weyl group of $GL_4(\mathbb{Q}_p)$. For $w \in W$, we have

\[ \phi^{ord}(w) = \varphi \text{ if } w = w_3, \text{ and } \phi^{ord}(w) = 0 \text{ for } w \neq w_3, \text{ if } \chi_+\chi_1\xi_2^{-1}\delta_1 \text{ are ramified}. \]

**Proof.** Define a function on $U(W)(\mathbb{Q}_p)$ by $f(h) = L(\phi^{ord}(m(1, h) s^{-1}w_1))$ for $h \in U(W)(\mathbb{Q}_p) \subset GL_2(\mathbb{Q}_p)$. Then $f \in I(\delta_1, \delta_2)$. We find that $w_3 = s_1^{-1}w_1$ and $f(1) = M_{s_1}\phi^{w_3}(s_1^{-1}w_1) = 1$. It is straightforward to verify that for $k = \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in I_0(p)$, $f(hk) = f(g)\xi_2^{-1}(\det k)\delta_1(a)$. Hence, if $\delta_1$ is ramified, then $f(h) = \varphi_p$. If $\delta_1$ is unramified, we can compute $f(w)$ as follows.

\[ f(w) = \phi^{ord}(s_1^{-1}w_1)(w) = \int_{U_{s_1}} \phi^{w_3}(s_1\mu(1, w) s_1^{-1}w_1)du \]

\[ = \int_{\mathbb{Q}_p} \phi^{w_3} \begin{bmatrix} 1 & 0 \\ 1 & a \end{bmatrix} w_1da \]

\[ = \int_{\mathbb{Q}_p} f(\begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix})da, \]

where

\[ f\begin{bmatrix} a & b \\ c & d \end{bmatrix} := \phi^{w_3} \begin{bmatrix} 1 & 0 \\ 1 & a \end{bmatrix} w_1. \]
Note that \( f \begin{bmatrix} \frac{1}{x} & 0 \\ 1 & 1 \end{bmatrix} = \mathbb{L}_p(x) \). By the identity
\[
\begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix} = \begin{bmatrix} -a^{-1} & 1 \\ 1 & a^{-1} \end{bmatrix},
\]
the last integral in (6.6) equals
\[
\int_{\mathbb{Q}_p^2} f \left( \begin{array}{cc} -a & 1 \\ a^{-1} & 1 \end{array} \right) |a|^{-1} d^x a = \sum_{n \geq 0} \int_{\mathbb{Q}_p^x} \delta_2 \delta_1^{-1}(a) d^x a
\]
\[
= \frac{1}{1 - \delta_1^{-1}(p)} (1 - |p|).
\]
Hence in the case where \( \delta_1 \) is unramified, we find that \( f(m) = \varphi(m) \) by (6.5).

Next we show the second assertion. Recall that \( z_1 = \xi_2 \delta_1^{-1} \) and \( z_2 = \xi_2 \delta_1^{-1} \). We identify with \( S_4 \) and \( w \) be an element in \( W \). Let \( \lambda' \) be the character of \( T_4(\mathbb{Z}_p) \) defined by \( \lambda'(diag(t_1, t_2, t_3, t_4)) = z_1(t_1)z_2(t_2)\chi_2(t_3)\chi_1^{-1}(t_4) \). Then we have \( \phi^{ord}(gt) = \lambda(t) \phi^{ord}(g) \) for \( t \in T_4(\mathbb{Z}_p) \).

Now we regard \( W \) the Weyl group of \( GL_4(\mathbb{Q}_p) \) as the permutation group \( S_4 \) acting on the standard basis of \( \mathbb{Q}_p^4 \). Suppose \( M_s, \phi^{w^i}(w) \neq 0 \) and \( w \neq (123) = w_2 \). Since \( \chi_1 \chi_2 \) are ramified and \( \phi^{ord}(wt) = \lambda(t) \phi^{ord}(w) = \lambda(wtw^{-1}) \phi^{ord}(w) \), it follows that \( w \) can only be \( (13) \) or \( (234) \). And \( s_1Bw \) contains the big cell, which implies the reduced decomposition of \( w \) contains \( (123) = (12)(23) \), so \( w = (13) = (12)(23)(12) \) or \( (1234) = (12)(34)(23) \). By direct computation, \( M_s, \phi^{w^i}((13)) = M_s, \phi^{w^i}((1243)) = 0 \). (since \( \chi_1 \chi_2 \) is ramified).

### 6.4. The computation of \( l_{w_3}(f_{\Phi \tau}^{ph}) \).

#### 6.4.1. We calculate the ordinary projection of the local pull back section at \( p \) in this subsection. Recall that
\[
(\phi_{\chi,s,p}^{\mathcal{I}})^{ph} = \chi_2(\det T) |\det T|^s f_{\Phi \tau}^{ph}
\]
for \( \Phi = \Phi_{L,\pi}^{\mathcal{I}} \) and \( z = (\xi_2^{-1} \delta_1, \xi_2^{-1} \delta_2) \). Then \( f_{\Phi \tau}^{ph} \) is \( N(\mathbb{Z}_p) \)-invariant by Lemma 3.8. It suffices to compute \( l_{w_3}(f_{\Phi \tau}) \) by Lemma 6.3. For \((x, y) \in \mathbb{Q}_p^2\), put
\[
u(x, y) = \begin{bmatrix} 1 & y \\ 1 & x \\ 1 & 1 \end{bmatrix}.
\]
We have \( U_{w_3} \setminus U = u(\mathbb{Q}_p^2) \). Recall that in (6.3) we identify \( I(s) = I_p(\pi, \chi, s) \) with \( \mathcal{I}(s) \) via \( L \), and we have the following commutative diagram when \( Re s \gg 0 \),
\[
\begin{array}{ccc}
I(s) & \xrightarrow{L} & \mathcal{I}(s) \\
{\downarrow} & & {\downarrow} \\
I_{w_3} & \xrightarrow{L} & I_{w_3} \\
{\downarrow} & & {\downarrow} \\
I(\pi_p) & \xrightarrow{L} & \mathbb{C},
\end{array}
\]
where \( l_{w_3} \) at left hand side is defined by
\[
l_{w_3}(f) := \int_{U_{w_3} \setminus U} \pi(wn) f \, dn,
\]
and the bottom map \( L \) is the evaluation at the identity \( 1_2 \). Put \( H = U(W) \). Then by definition,
\[
l_{w_3}(f_{\Phi \tau}^{ph}) = \int_{\mathbb{Q}_p^2} dx dy \int_{H(\mathbb{Q}_p)} dh f_{\Phi}(w_3 u(x, y), h)_{\Delta \chi_1^{-1}(\det h) \phi(h) \tau_{p \varphi}}.
\]
To compute $l_{w_3}(\rho_{\Phi_+})$, we begin with

$$
(0,Z)(w_3u(x,y),h)\Delta Y = (0,Z) \begin{bmatrix}
1 & \frac{1}{2} & -\frac{1}{2} \\
\theta^{-1} & 1 & \theta^{-1} \\
\theta^{-1} & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & t_y \\
v & C_{x,y} \\
1 & 0 & 0
\end{bmatrix} = (Z \begin{bmatrix}
0 & 0 & \theta^{-1} \\
\theta^{-1}v & \theta^{-1}C_{x,y} & 1 \\
0 & 0 & h
\end{bmatrix}, Z \begin{bmatrix}
1 & \theta^{-1} \\
& & & \\
& & & \\
& & & \\
& & & \\
& & &
\end{bmatrix}),
$$

where $C_{x,y} = \begin{bmatrix} 0 & y \\ 1 & x \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus

$$
l_{w_3}(\rho_{\Phi_+})
= \int dydx \int dh \int_{H(Q_p)} d^\times ZP(h)Q(Z)\Phi(Z \begin{bmatrix}
0 & 0 & \theta^{-1} \\
\theta^{-1}v & \theta^{-1}C_{x,y} & 1 \\
0 & 0 & h
\end{bmatrix})\pi(h)\tau_p\varphi,
$$

where $P(h) = \chi_1(\det h) |\det h|^s$ and $Q(Z) = \chi_1 \chi_2(\det Z) |\det Z|^{2s}$. We make change of variable

$$
Z \mapsto Z \begin{bmatrix}
1 & 0 \\
0 & C_{x,y}^{-1} \theta
\end{bmatrix}; h \mapsto C_{x,y}h.
$$

The integral (6.9) equals

$$
\chi_+(\det \theta) \int_{\mathbb{Q}_p^2} \chi_2^{-1}(-y)|y|^{-s}dydx \int_{H(Q_p)} P(h)\pi(C_{x,y}h)\tau_p\varphi dh
\times \int_{GL_3(Q_p)} Q(Z)\Phi_1(Z \begin{bmatrix}
0 & 0 \\
C_{x,y}^{-1}v & I_2 \\
0 & h
\end{bmatrix})\Phi_2(Z \begin{bmatrix}
1 & 0 \\
0 & h
\end{bmatrix})d^\times Z,
$$

We first compute the last integral. Write $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$. We find that

$$
\Phi_1(Z \begin{bmatrix}
0 & 0 \\
C_{x,y}^{-1}v & I_2 \\
0 & h
\end{bmatrix})\Phi_2(Z \begin{bmatrix}
1 & 0 \\
0 & h
\end{bmatrix}) = \Phi_1(\begin{bmatrix} Z_2C_{x,y}^{-1}v & Z_2 \\
Z_4C_{x,y}^{-1}v & Z_4 \end{bmatrix})\Phi_2(\begin{bmatrix} Z_1 & Z_2h \\ Z_3 & Z_4h \end{bmatrix}).
$$

Considering the support of $\Phi_1$ and $\Phi_2$, we see that

$$
Z_2 \in \mathbb{Z}_p^2, Z_4 \in I_0(p), Z_3 \in \mathbb{Z}_p^2.
$$

Since $Z_4 \in I_0(p)$, we can write

$$
Z = \begin{bmatrix} 1 & Y \\ 1 & Z_3 \end{bmatrix} \begin{bmatrix} a & 0 \\ Z_4 & Z_4 \end{bmatrix}, Y \in \mathbb{Z}_p^2, Z_4 \in I_0(p)
$$

and we have $d^\times Z = |a|^{-2} dY d^\times a d^\times Z_4dZ_3$. The last integral in (6.10) equals

$$
|N|^2 \int_{\mathbb{Q}_p^2} \chi_+\phi(a) |a|^{s-2} \Phi_1(\begin{bmatrix} a & 0 \\ Z_4 & Z_4 \end{bmatrix})\Phi_1(\begin{bmatrix} Z_2C_{x,y}^{-1}v & Z_2 \\
Z_4C_{x,y}^{-1}v & Z_4 \end{bmatrix})\Phi_2(\begin{bmatrix} Z_1 & Z_2h \\ Z_3 & Z_4h \end{bmatrix})d^\times Z_4
$$

Make change of variable $h \mapsto Z_4^{-1}h$. The integral (6.10) equals

$$
\chi_+\phi(\det \theta) |N|^2 E_p(\nu_1,2s-2) \int_{\mathbb{Q}_p^2} \chi_2^{-1}(-y)|y|^{-s}dydx \int_{I_0(p)} d^\times Z_4
\times \chi_2(\det Z_4)\Phi_1(\begin{bmatrix} 0 & 0 \\ Z_4C_{x,y}^{-1}v & Z_4 \end{bmatrix})\pi(C_{x,y}Z_4^{-1}) \int_{H(Q_p)} P(h)\Phi_2(\begin{bmatrix} Z_1 & Z_2h \\ Z_3 & Z_4h \end{bmatrix})\pi(h)\tau_p\varphi dh.
$$
Put

\[ R_{\nu_2, \nu_3} = \int_{H(Q_p)} P(h) \Phi_{\nu_2, \nu_3}(h) \pi(h) \tau_p \varphi \, dh \in \pi. \]

We claim

(6.13) \[ R_{\nu_2, \nu_3} = \text{vol}(I_0(N_2), dh) E_p(s - \frac{1}{2}, \nu_2) E_p(s - \frac{1}{2}, \nu_3) \tau_p \varphi. \]

We assume (6.13) first. Then the integral (6.12) can be rewritten as

\[ \chi_+ (\det \theta) |N|^2 E_p(\nu_1, 2s - 2) \text{vol}(I_0(N_2), dh) E_p(s - \frac{1}{2}, \nu_2) E_p(s - \frac{1}{2}, \nu_3) \]

\[ \times \text{vol}(I_0(p), d^\times Z_4) \int_{\tilde{\mathbb{Q}_p}^2} \chi_2^{-1}(-y) |y|^{-s} \pi(C_{x,y}) dy dx \int_{\mathbb{Z}_p} \Phi_1 \left( \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix} \right) \tau_p \varphi dz. \]

The last integral by Lemma 3.10 equals

\[ \int_{\tilde{\mathbb{Q}_p}^2} \chi_2^{-1}(-y) |y|^{-s} |p|^{-1} J \left[ \begin{bmatrix} -y_{-1} x & 1 \\ y_{-1} & 0 \end{bmatrix} \right] \mu_3(p^{-1} y^{-1}) \pi \left( \begin{bmatrix} 0 & y \\ 1 & x \end{bmatrix} \right) \tau_p \varphi \]

\[ = |p|^{-1} J |p|^{-s} \int_{\mathbb{Z}_p} dy \int_{\mathbb{Z}_p} dx \chi_2^{-1}(y) \mu_3(py) \pi \left( \begin{bmatrix} 1 & py \\ -1 & 1 \end{bmatrix} \right) \varphi \]

\[ = |p|^{-1} J |p|^{-s} \text{vol}(p^{-1} \mathbb{Z}_p^2, dy) \chi_2(p) \]

Combining the above equation with (6.14) and rearranging terms, we obtain finally

(6.14) \[ l_{\nu_2}(f_{\Phi, \tau}) = |N|^2 \text{vol}(I_0(p), d^\times Z_4) \text{vol}(I_0(N_2), dh) |p|^{-1} J |p - 1| \]

\[ \times E_p(2s - 2, \chi_+ \cdot E_p(s - \frac{1}{2}, \chi_1 \xi_{2}^{-1} \delta_1) \cdot E_p(s - \frac{1}{2}, \chi_1 \xi_{2}^{-1} \delta_2) \]

\[ \times \chi_+ (\det \theta) \cdot (\chi_2 | \cdot |^{-1})(-p) \cdot \varphi. \]

6.4.2. Computation of \( R_{\nu_2, \nu_3} \). It remains to do the calculation of \( R_{\nu_2, \nu_3} \) in (6.13).

**Lemma 6.6.** \( \varphi \) is the ordinary vector. Then

(6.16) \[ R_{\nu_2, \nu_3} = \text{vol}(I_0(N_2), dh) E_p(s - \frac{1}{2}, \nu_2) E_p(s - \frac{1}{2}, \nu_3) \tau_p \varphi. \]

**Proof.** There are two cases:

Case(1) \( \nu_2 \) and \( \nu_3 \) are ramified: In this case, since \( \nu_2 \nu_3^{-1} = \delta_1 \delta_2^{-1} \) and \( c_p(\delta_1 \delta_2^{-1}) = p, c_p(\nu_3) = c_p(\nu_2) = N_2 \).

For \( h \in \supp \Phi_{\nu_2, \nu_3}, \) we have \( h = N^{-1} u, \ u = \begin{bmatrix} a \\ -Nc \end{bmatrix} \in I_0(N). \) We recall that

(6.17) \[ \pi(u) \tau_p \varphi = \xi_{2}^{-1}(\det u) \delta_1(d) \tau_p \varphi \text{ for } u = \begin{bmatrix} a \\ cN \end{bmatrix} \in I_0(p), \]

and \( \Phi_{\nu_2, \nu_3}(hk) = \Phi_{\nu_2, \nu_3}(h) \nu(k), \) so we have

\[ R_{\nu_2, \nu_3} = \text{vol}(I_0(N), dh) \chi_1(N^2) |N|^{-2s} \Phi_{\nu_2, \nu_3} \left( \begin{bmatrix} N^{-1} \\ N^{-1} \end{bmatrix} \right) \xi_1 \xi_{2}^{-2}(N^{-1}) \tau_p \varphi \]

\[ = \text{vol}(I_0(N), dh) |N| G(\nu_2) G(\nu_3) |N|^{-2s} \tau_p \varphi \]

\[ = \text{vol}(I_0(N), dh) \epsilon(s - \frac{1}{2}, \nu_2)^{-1} \epsilon(s - \frac{1}{2}, \nu_3)^{-1} \tau_p \varphi \]

\[ = \text{vol}(I_0(N), dh) E_p(s - \frac{1}{2}, \nu_2) E_p(s - \frac{1}{2}, \nu_3) \tau_p \varphi. \]

Case(2) Either \( \nu_2 \) or \( \nu_3 \) is unramified: In this case, we have \( N_2 = p, \) and we can verify \( R_{\nu_2, \nu_3} \) also satisfies (6.17). Now we view \( \varphi \) as the ordinary section in the model \( I(\delta_1, \delta_2) \boxtimes \xi_{2}^{-1}. \) To prove the lemma, it suffices to show \( R_{\nu_2, \nu_3}(e) = \tau_p \varphi(e) \) and \( R_{\nu_2, \nu_3}(\tau_p^{-1}) = \tau_p \varphi(\tau_p^{-1}). \) We choose the measure

\[ dh = |a|^{-1} dy dx^* ad^* b dk \] for \( h = \begin{bmatrix} a \\ b \end{bmatrix} k, k \in \text{GL}_2(\mathbb{Z}_p). \)
We have
\[ R_{\nu_2,\nu_3}(e) = \int \int \int \chi_1(ab) |ab^*|a|^{-1} \tilde{\Phi}_{\nu_2,\nu_3}(\begin{bmatrix} a & y \\ b & y \end{bmatrix}) \tau_p \varphi(\begin{bmatrix} a \\ y/b \end{bmatrix}) \, dy \, dx \, db = \text{vol}(I_0(p), dh) E_p(s - \frac{1}{2}, \nu_2) E_p(s - \frac{1}{2}, \nu_3) \tau_p \varphi(e). \]

Similarly
\[ R_{\nu_2,\nu_3}(\tau_p^{-1}) = \int_{GL_2(G)} \chi_1|.|^{-1}(\det h) \tilde{\Phi}_{\nu_2,\nu_3}(h) \varphi(\tau_p^{-1} h \tau_p) dh \]
\[ = \int_{GL_2(G)} \chi_1|.|^{-1} \tilde{\Phi}_{\nu_2,\nu_3}(\tau_p h \tau_p^{-1}) \varphi(h) dh \]
\[ R_{\nu_2,\nu_3}(\tau_p^{-1}) = \text{vol}(I_0(p), dh) \int \int \chi_1(ab) |ab^*|a|^{-1} \tilde{\Phi}_{\nu_2,\nu_3}(\begin{bmatrix} b & y \\ py & a \end{bmatrix}) \varphi(\begin{bmatrix} a & y/b \end{bmatrix}) \, dy \, dx \, db = \text{vol}(I_0(p), dh) E_p(s - \frac{1}{2}, \nu_2) E_p(s - \frac{1}{2}, \nu_3) \tau_p \varphi(\tau_p^{-1}) \]

Now we summarize our calculations as the following proposition.

**Proposition 6.7.** We have
\[ I_{\omega_1}(F_{\Phi, p}) = \text{vol}(I_0(p), dh) \text{vol}(I_0(N, N_2), d^\infty Z) \left| J^{-1} \right| (p - 1) \times E_p(2s - 2, \chi_+) \cdot E_p(s - \frac{1}{2}, \psi_{1}^{-1} \delta_1) \cdot E_p(s - \frac{1}{2}, \psi_{2}^{-1} \delta_2) \]
\[ \times \chi_+ (\det \theta) \cdot (\lambda_2|.)^{(p)}(p) \]

**Proof.** By (6.15) and note that
\[ |N|^2 \text{vol}(I_0(p), d^\infty Z_4) \text{vol}(I(N_2), dh) = \text{vol}(I_0(N, N_2), d^\infty Z) \text{vol}(I_0(N_2), dh) \left| pN_2^{-1} \right| \]
\[ = \text{vol}(I_0(N, N_2), d^\infty Z) \text{vol}(I_0(p), dh) \]

\[ \square \]

**7. Proof of the main result**

**7.1. Fourier-Jacobi expansion and \(\Phi\)-operators.** In this subsection, we give a brief discussion of arithmetic Fourier-Jacobi expansion of modular forms for the unitary group \(G = GU(3, 1)\). The purpose is to relate the constant term of our Eisenstein series \(E^{ord}(\chi | f, \xi, c)\) and that of its \(p\)-adic avatar.

**7.1.1. Local charts and Mumford families.** We begin with some notations. Let \(g \in G(k_f^{(p)})\) and write \(g^\gamma = kg^\gamma\) with \(\gamma \in G(\mathbb{Q})\) and \(k \in K\). Let \(Y_g = Y_g^\gamma\) and \(X_g^\gamma = X_g^\gamma\). Let \(X_g = \{ y \in I_v \mid \langle y, Y_g^- \rangle_{\lambda, 1} < \mathbb{Z} \}\) be the \(\mathbb{Z}\)-dual of \(X_g^\gamma\). Then we have the inclusion \(i : Y_g \rightarrow X_g\). We let \(I_g\) be the subgroup in \(X_g \otimes \mathbb{Z}\) generated by \(g' \otimes y - y \otimes y; xb \otimes y - x \otimes c(b)y, \forall, x \in X_g, y, y' \in Y_g, b \in \mathcal{O}_\mathcal{K}\).

Let \(\mathcal{S}_g = \text{S}(X_g \otimes \mathbb{Z})\) be the maximal free quotient of the group \(X_g \otimes \mathbb{Z}/I_g\). We have \(Y_g^\gamma = bg^\gamma\) and \(X_g^\gamma = a^x b\gamma\) for two fractional ideals \(a, b\) of \(\mathcal{O}_\mathcal{K}\). Then the dual of \(\mathcal{S}_g\) is the space of integral Hermitian forms on \(a \times b\) which is isomorphic to \((ab \mathcal{D}_\mathcal{K})^{-1} \cap \mathbb{Q}\), hence \(\mathcal{S}_g\) is the fractional ideal \(ab \mathcal{D}_\mathcal{K} \cap \mathbb{Q}\) of \(\mathbb{Q}\). Let \(\mathcal{S}_g^+ = \{ s \in c^{-1} \mathcal{S}_g \mid s > 0 \}\) and let \(\mathcal{S}_g^0 = \mathcal{S}_g^+ \cup \{0\}\).

Let \(S_g := S_{\mathcal{O}_\mathcal{K}, (K_g^\gamma)^0}\) and let \(\mathfrak{B} = (B, \overline{\mathcal{S}}_g, \nu_{y, \tau}, \tau_B)\) be the universal quadruple over \(S_g\). Let \(\mathcal{P}\) be the Poincaré line bundle over \(\mathfrak{B} \times \mathfrak{B}\). Define the group scheme \(\mathcal{Z}_g\) over \(S_g\) by
\[ \mathcal{Z}_g = \text{Hom}_{\mathcal{O}_\mathcal{K}}(X_g, \mathfrak{B}) \times \text{Hom}_{\mathcal{O}_\mathcal{K}}(Y_g, \nu_{y, \tau}) \text{Hom}_{\mathcal{O}_\mathcal{K}}(Y_g, \mathfrak{B}) \]
\[ = \{ (c, c') \in \text{Hom}_{\mathcal{O}_\mathcal{K}}(X_g, \mathfrak{B}) \times \text{Hom}_{\mathcal{O}_\mathcal{K}}(Y_g, \mathfrak{B}) \mid c(i(y)) = \lambda(c'(y)), y \in Y_g\} \]
Let $\mathcal{Z}^0_{[g]}$ be the connected component of $Z$. Each $\beta \in X \otimes Y$ induces a tautological map from $c(\beta) : \mathcal{Z}^0_{[g]} \rightarrow \mathbb{B} \times \mathbb{B}$. We let $L(\beta) = c(\beta)^*P$ be the line bundle over $\mathcal{Z}^0_{[g]}$ obtained by the pull back of $P$ via $c(\beta)$. By the symmetry of the polarization $\lambda$ and the ampltude of $P$, $L(\beta)$ only depends on the holomorphic image of $\beta$ in $\mathcal{J}_{[g]}$.

Let $K^\omega_{0,g} = gK^\omega_{0,g} \cap G_P(\mathbb{A}_f)$, $\bullet = 1$ or $0$. Let $I_{[g]}(K^\omega_0) := I_{G_P}(K^\omega_{0,g})$ be the Igusa scheme over $S_{[g]}$. In our simple case, $S_{[g]}$ and $I_{[g]}(K^\omega_0)$ are affine schemes of finite type over $\mathcal{O}$. Let $A_{[g]}$ (resp. $A^*_0_{[g]}$) be the coordinate ring of $S_{[g]}$ (resp. $I_{[g]}(K^\omega_0)$). Let $R_{[g]}$ be the $A^*_0_{[g]}$-algebra defined by

$$
R_{[g]} := \prod_{\beta \in \mathcal{J}_{[g]}} H^0(\mathcal{Z}^0_{[g]}, L(\beta))(\mathcal{O}).
$$

Let $I_+ \times I_+ \subset \mathcal{J}_{[g]}$ be the ideal of $\mathcal{J}_{[g]}$ generated by $\{q^3\}_{\beta \in I_+ \times I_+}$. By Mumford’s construction ([Cha85] and [FC90]), there exists a semi-abelian scheme $(M_{[g]}, \Omega_M)$ over Spec $R_{[g]}$ together with an $\mathcal{O}_K$-action $\iota : \mathcal{O}_K \rightarrow \End M \otimes \mathbb{Z}(p)$ such that $\mathcal{M} \otimes R_{[g]} = \tilde{\mathcal{M}}$, where $\tilde{\mathcal{M}}$ is the universal $\mathcal{O}_K$-Raynaud extension over $\mathcal{Z}^0_{[g]}$

$$
0 \rightarrow X'_g \otimes \mathbb{G}_m \rightarrow \tilde{\mathcal{M}} \rightarrow \mathcal{M} \rightarrow 0,
$$

and $\mathcal{M}$ is an abelian scheme over Spec $R_{[g]}[1/I_+]$. $(\mathcal{X}_B, K^\omega_{0,B})$ induces a natural polarization and level structures $(\mathcal{X}_M, \Omega_M)$ of $M$ over $R_{[g]}[1/I_+]$ (cf. [FC90, IV. 6.4, 6.5 and V. 2.5]). We define the $R_{[g]}[1/I_+]$-quadruple of level $K$ by

$$
\mathcal{M}_{[g]} = (M_{[g]}, \Omega_M, \mathcal{X}_M, \Omega_M, \Omega_M)_{R_{[g]}[1/I_+]}.
$$

We call $\mathcal{M}_{[g]}$ the Mumford quadruple at the cusp $[g]$. Moreover there exists a morphism $\varphi_{\mathcal{Z}_{[g]}} : \Spec R_{[g]}^\text{c} \rightarrow \mathcal{S}_G(K)$ such that $(\varphi_{\mathcal{Z}_{[g]}})^*G = \mathcal{M}_{[g]}$.

Similarly, let $\mathcal{Z}_{[g]}(k)$ be the universal quintuple over $I_{[g]}(K^n)$. Then $\mathcal{N}_{I_+}$ induces a canonical $p^n$-level structure $\mathcal{N}_{I_+}$ on $\mathcal{M}_{[g]}$ over $R_{[g]}$. Then $(\mathcal{M}_{[g]}, \mathcal{N}_{I_+})$ is the Mumford quintuple over $I_{[g]}(K^n)$.

### 7.1.2. Fourier-Jacobi expansion and the $\Phi$-operator

Let $R$ be an $\mathcal{O}$-algebra and let $f \in \mathcal{M}_{[g]}(K^n, R)$ be a modular form of weight $k = (0, k)$. We define the Fourier-Jacobi expansion $\mathcal{F}(f)$ of $f$ at the cusp $[g]$ as follows. Let $g \in G(\mathbb{A}^{(p)})$. Then $(7.2)$ induces the exact sequence of $\mathcal{O}_K$-modules

$$
0 \rightarrow \Omega_B \rightarrow \Omega_M \rightarrow \Omega_{\mathcal{J}_g} \rightarrow 0, \quad \mathcal{F}_g = X'_g \otimes \mathbb{G}_m.
$$

Let $d^*t$ be the canonical $\mathcal{O}_K$-basis of $\Omega_{\mathcal{J}_g}$. Choose a lifting $d^*t'_* \equiv d^*t$ in $\Omega_M$ and a canonical lifting $d^*t_{-}$ of $e^{-d^*t}$ in $\Omega_M$. Let $\omega_M = (d^*t'_* \wedge \omega_B, d^*t_{-})$ be a $\mathcal{O}_K$-top form of $\omega_{\mathcal{M}_{[g]}}$. Let $u \in H(Z_p) = GL_3(Z_p) \times GL_4(Z_p)$. We regard $u$ as an element in $U(V)(\mathbb{Q}_p)$ by the embedding $H(Z_p) \hookrightarrow GL_4(Z_p) \cong U(V)(\mathbb{Z}_p)$. If $f \in H^0(I_{G(K^n)}(\mathcal{J}_g)^e, \mathcal{O}_M)$, then evaluating $f$ at $(\mathcal{M}_{[g]}, u^{-1}j_M, \omega_M)$, we obtain the Fourier-Jacobi expansion of $f$

$$
f(\mathcal{M}_{[g]}, u^{-1}j_M, \omega_M) = \sum_{\beta} a^\beta_{[g]}(\beta, f)q^3 \in R_{[g]} \otimes \mathcal{O}_R.
$$

Suppose further that $u^{-1}N_{P(Z_p)}u \subset N_{P(Z_p)}$. Define the Siegel $\Phi$-operator:

$$\Phi^\beta_{[g]} : H^0(I_{G(K^n)}(\mathcal{J}_g)^e, \mathcal{O}_M) \rightarrow H^0(I_{[g]}(\mathcal{J}_g)^e, \mathcal{O}_M)$$

$$f \rightarrow \Phi^\beta_{[g]}(f) := a^\beta_{[g]}(0, f).$$

Now we consider the case $R = \mathbb{C}$. Recall that $I_{[g]}(K^n)(\mathbb{C}) = G_P(\mathbb{Q})^+ \cap G_P(K^n)$. Thus for every $h \in G_P(\mathbb{A}_f)$, the image $[h]$ of $h$ in $I_{[g]}(K^n)(\mathbb{C})$ gives rise to a classifying $\mathbb{C}$-algebra homomorphism $\varphi_{[h]} : A^*_0_{[g]} \rightarrow \mathbb{C}$. we have the following important comparison between analytic and algebraic Fourier-Jacobi expansion

$$
\mathcal{F}_{[h]}(f) = \varphi_{[h]} \left( f(\mathcal{M}_{[g]}, u^{-1}j_M, (d^*t'_* \wedge (2\pi i)^2 \omega_W)(\mathcal{J}_g)^e, d^*t_{-}) \right).
$$

7.2. The constant term of $E^{ord}(\chi \mid f, \xi, c)$. Let $E = E^{ord}(\chi \mid f, \xi, c)$ and let $\hat{E}$ be its $p$-adic avatar. We compare the constant terms of $E$ and $\hat{E}$. First, by (4.15) and the definition of $\hat{E}(\chi \mid f, \xi, c)$ together with the identity (2.17), we conclude that

$$\frac{1}{\Omega_p} \cdot \hat{E}$$

is the $p$-adic avatar of $\left(\frac{2\pi i}{\Omega_K}\right)^{2k} \cdot E$.

Let $\omega_M(\mathbb{C}) = (d^\times t_+ \wedge (2\pi i)^2 \omega_{W/C}(\Sigma), d^\times t_-)$ and $\omega_M(p) = (d^\times t_+ \wedge \omega_{B_n}(j_{B_n}), d^\times t_-)$. By (2.17) we have

$$\iota_{\infty}(E(M(g), J_M, \omega_M)) = \left(\frac{2\pi i}{\Omega_K}\right)^{2k} \cdot E(M(g), J_M, \omega_M(\mathbb{C}))$$

$$\iota_p(E(M(g), J_M, \omega_M)) = \frac{1}{\Omega_p} \cdot E(M(g), J_M, \omega_M(p)) = \hat{E}(M(g), J_M).$$

Comparing the constant terms on both sides in (7.3), we obtain

$$(7.4) \quad \left(\frac{2\pi i}{\Omega_K}\right)^{2k} \cdot \Phi^\mu_g(E) (\mathbb{P}^1, \omega_{W/C}(\Sigma)) = \frac{1}{\Omega_p} \cdot \Phi^\mu_g(\hat{E}) (\mathbb{P}^1).$$

Comparing the constant terms on both sides in (7.3), we obtain

$$(7.5) \quad \Phi^\mu_{\omega g}(E) (h) = \Phi^\mu_{\omega g}(\hat{E}) (\mathbb{P}^1, \omega_{W/C}(\Sigma)).$$

7.3. Proof of the main result. We are now in a position to prove the main result Theorem 1. Actually we will prove it in a more general setting. First of all we recall our set-up. We begin with a quadruple $(\psi, c, f, \xi)$, where

- $\psi$ is a Dirichlet Hecke character of $K^\times$ and $c$ is an integral ideal of $\mathbb{Z}$ which satisfies (S2).
- $f$ is an ordinary new GL$_2$-cusp form in $S_2(\Gamma_0(pN_f), \epsilon_f)$.
- $\xi$ is a Hecke character of $K^\times$ of finite order such that $\xi|_{Q^\times} = \xi = \epsilon_f^{-1}$.

We assume $N$ has a decomposition $N_f = N_+N_-$ with $(N_+, N_-) = 1$ and $N_-$ is product of an odd number of distinct primes. We choose a positive integer $s$ which is a $p$-unit and $(-s, -D)_v = -1$ exactly at $v \mid \infty N_-$. $B$ is the definite quaternion algebra defined in §4.3 with the above choice of $s$. Then $B$ is exactly ramified at the infinity and $N_-$, and

$$\pi_f = JL(p^B)$$

is in the image of Jacquet-Langlands of $B$.

Let $f$ be the corresponding $p$-primitive form of $f$ for $GU(W)$.

We give the definition of the normalized $L$-values.

Definition 7.1 (Normalized $L$-values). Recall that we have chosen $I(\delta_1, \delta_2)$ a model of $\pi_p$ such that $\delta_2|\cdot|^{\frac{1}{2}}(p)$ is a unit. Then the normalized $L$-values are defined by

$$(7.6) \quad L^{alg,S}_{\psi}(\delta)(\chi, \psi) = E_\psi(\delta)(\chi, \psi) \cdot \frac{\Gamma(k - 2)}{(2\pi i)^{k-2}} \cdot \frac{L^S_{\psi}(\delta)}{\Omega_{\mathbb{Q}}}.$$

and

$$(7.7) \quad L^{alg}_{\psi}(\delta)(\chi, \psi) = E_\psi(\delta)(\chi, \psi) \cdot \frac{\Gamma(k - 1)}{(2\pi i)^{k-1}} \cdot \frac{L^S_{\psi}(\delta)}{\Omega_{\mathbb{Q}}}.$$

where $\Omega_{\mathbb{Q}}$ is the CM period associated to $\psi$. We express (7.7) in terms of $L$-values attached to Galois representation $\rho = \rho_f$. Note that $\rho_f = \text{rec}(\pi_f^\vee(-\frac{1}{2}))$ and $\xi + = \epsilon_f^{-1}$, we see that (7.7) equals

$$(7.8) \quad L^{alg,S}_{\psi}(\delta)(\chi, \psi) = L^{alg,S}_{\psi}(0, \rho \otimes \chi \xi)$$

as defined in the introduction.

Remark. According to the recipe of J. Coates in [Coe91], a $p$-adic $L$-function has no $p$-Euler factor for the eigenspace of positive (geometric) Hodge-Tate weights, and we need to modify $p$-Euler factor for the eigenspace of negative Hodge-Tate weights.
Theorem 7.2. Let \((\psi, c, f, \xi)\) as above. Let \(\text{E}^{\text{ord}}_{\xi, c}(g)\) denote the \(\mathbb{C}_p\)-valued measure induced by the evaluation of constant term at \(g \in G(\mathbb{A}_f)\). Let \(g \in G(A_f(p))\) be a cusp defined by \(g_v = w^{-1}\) if \(v \in S\) and \(g_v = 1\) elsewhere and \(h \in G_P(A_f(p))\). We have
\[
\frac{1}{\Omega^p} \int \hat{\chi} \text{d}E^{\text{ord}}_{\xi, c}(hg) = |D_{\mathbb{C}}|^2 (p-1) \cdot \epsilon_p^{-1} \left[ L^{alg,S}_{\rho}(-2, \chi+c, \rho)L^{alg,S}_{\rho}(-1, \rho \otimes \chi) \right] 
\]
\[
\times (\hat{\chi}^{-1})_p(-p \cdot \det \theta)(F(h)).
\]
In other words, the ordinary \(p\)-adic modular form \(\Phi^{w_3}(\mathbb{E}^{ord}(\xi^{-1} \mid f, \xi, c))\) on \(G_P\) is given by
\[
\frac{1}{\Omega^p} \cdot \Phi^{w_3}(\mathbb{E}^{ord}(\xi^{-1} \mid f, \xi, c)) = L^S_p(-2, \chi+c, \rho)L^S_p(\rho \otimes \chi) \cdot \hat{F}
\]
Moreover the constant term at other cusps is a \(p\)-integral multiple of this element in \(\mathbb{Z}_p\). Therefore the ideal generated by the constant terms at all cusps is
\[
L^S_p(-2, \chi+c, \rho)L^S_p(\rho \otimes \chi).
\]

Proof. By Cor. 5.4 the constant term of \(E^{\text{ord}} := \text{E}^{\text{ord}}(\chi \mid f, \xi, c)\) is
\[
\Phi^{\text{fin}}_{w_3}(E^{\text{ord}})(h) = \frac{NC(s, \chi)}{\text{vol}(D_{\mathbb{W}}(c))} \left[ \bigotimes_{v \mid p} \phi^{pb}_{\xi, s, n} \bigotimes_{v \neq p} c_{\mathbb{F}_{\mathbb{V}}}^{eb} \right] (hw^3g)_{s=0} : \chi(\det(hg)).
\]
Therefore the first assertion follows from the computation in Prop. 5.3, Prop. 5.5, Prop. 5.9 and Prop. 6.7 together with (7.4) and (7.5). By [TU99, Lemma 4.3 and Cor. 4.1], the ordinary section is only supported in \(g \in G(\mathbb{A}_f)\) such that \(g_v\) has \(p\)-depth 0 (Def. 4.1 loc.cit.). Hence the second assertion follows from Lemma 6.5 and \(p\)-adic continuity.

References


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