Flux-to-Voltage Transfer Function of the Series-SQUID Array With Grain-Boundary Josephson Junctions

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Abstract—The factors affecting the flux-to-voltage transfer function of the series-SQUID array with grain-boundary Josephson junctions were investigated. By using the proposed critical-current-modulation model, it is found that the voltage swing of the series-SQUID array is suppressed greatly by the spread in critical currents of Josephson junctions. For the inductance $L = 80$ pH and the logarithmic average critical current $2I_{ave} = 20$ μ A of the component SQUID, the allowed spread in critical currents of Josephson junctions is $0.4I_{ave}$ to $2.2I_{ave}$ for achieving the low flux noise with the series SQUID array.

Index Terms—Critical-current spread, high-temperature superconductors, Josephson junctions, SQUID.

I. INTRODUCTION

THE flux noise of the N -dc-SQUIDs array is expected to be less than a single dc SQUID by a factor of $N^{-0.5}$ [1], [2]. This characteristic is appealing for the magnetometer application. For the N identical dc SQUID in series, the voltage modulation depth V_{pp} should be amplified by a factor of N . And the white voltage noise of the array is $S_{v,array} = NS_v$, where S_v is the voltage noise of each dc SQUID in the array. Hence, the resultant flux noise $S_{\Phi}^{1/2}$ of the SQUID array is [1], [2]

$$S_{\Phi,array}^{1/2} = \left(\frac{\Phi_0 S_v^{1/2}}{\frac{\partial V}{\partial \Phi}} \right) \cdot N^{-1/2} \quad (1)$$

in which $\partial V / \partial \Phi \approx \pi V_{pp} / \Phi_0$ is the flux-to-voltage transfer function of the component SQUID in the array. By using (1), the predicted field noise level of the SQUID-array magnetometer can be much less than that of the conventional dc SQUID. However, the experimental results [3], [4] did not show that the flux noise of the HTS series-dc-SQUID-array magnetometer could be less than that of the conventional HTS dc SQUID magnetometer. The possible reason for the discrepancy may be the spread of critical currents [4], [5] in the

paralleled grain-boundary Josephson junctions of the HTS dc SQUID. The details about the effect of critical-current spread on $\partial V / \partial \Phi$ of the HTS SQUID array are not yet clear. In this report, we investigate the effect of the critical-current spread of the grain-boundary Josephson junctions on the voltage swing of the HTS series-dc-SQUID array. The dispersion in critical currents as well as the asymmetry of the component SQUID is taken into account in calculating the voltage swing and the flux noise of the array. The allowed level of the spread in critical currents to achieve the low flux noise is predicted.

II. SIMULATION FOR HTS SERIES dc SQUID ARRAY

A. Assumptions for the HTS Series dc SQUID Array

In simulating the voltage-flux modulation curve of the HTS series dc SQUID array, we assumed that:

1. There is a small but nonnegligible spread in the effective area of the SQUID. The spread is a Gaussian distribution.
2. All of the SQUIDs are geometrically symmetric.
3. The critical current in the grain-boundary Josephson junctions has the logarithmic spread [5].
4. The critical current I_0 and the resistance R_0 of the grain-boundary Josephson junctions show the scaling behavior [6] $I_0 R_0 \propto (j_c)^{0.5}$, in which j_c is the critical current density of the junction.
5. The voltage-current ($V - I$) characteristic curve of the SQUID follows the Ambegaokar-Halperin's expression [7] while the critical current is modulated by the magnetic flux Φ .
6. The intrinsic critical current asymmetry results in a flux shift [8] in the voltage-flux ($V - \Phi$) curve by $\Delta\Phi = 0.5(\alpha_I \beta_L) \Phi_0$, where α_I is the critical current asymmetry parameter, $\beta_L = 2LI_0 / \Phi_0$ is the screening parameter, Φ_0 is the flux quantum, and L is the SQUID inductance.
7. The spread in the junction resistance is small, and the flux-shift due to the resistance asymmetry is negligible.
8. The interaction between the adjacent SQUIDs in the array owing to inductive coupling is negligible.
9. The white voltage noise of the dc SQUID follows that of the Josephson junctions in parallel [9].
10. The flux-to-voltage transfer function of the SQUID array is $\pi V_{pp,array} / \Phi_0$, where $V_{pp,array}$ is the maximum voltage swing of the SQUID array.

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Fig. 1. Schematics of the HTS dc SQUID array with co-planar input coils.

Note that with the assumption 5, it implies the capacitance of the Josephson junction is neglected, the current-phase relation is sinusoidal, and no excess supercurrent exists in the grain-boundary Josephson junctions.

The schematic diagram of the HTS dc SQUID array made from the single-layered superconductor film is shown in Fig. 1. Generally, the critical currents I_1 and I_2 as well as the resistance R_1 and R_2 of the Josephson junctions in the component SQUID are intrinsically asymmetric. The degrees of asymmetry in the critical currents and the resistances are represented by the asymmetric parameters α_I and α_R respectively as follows:

$$\begin{aligned} I_1 &= (1 + \alpha_I)I_0, & I_2 &= (1 - \alpha_I)I_0 \\ R_1 &= \frac{R_0}{(1 - \alpha_R)}, & R_2 &= \frac{R_0}{(1 + \alpha_R)}, \end{aligned} \quad (2)$$

in which $I_0 = (I_1 + I_2)/2$, and $R_0 = 2R_1R_2/(R_1 + R_2)$. The allowed values of α_I and α_R are from -1 to 1 . Since we have assumed the geometrical symmetry in assumption 2, the asymmetry in critical currents and resistance are respectively caused by the intrinsic asymmetry in the critical current density j_c and the contact resistance ρ_{contact} of the grain-boundary Josephson junctions [8].

B. Critical-Current-Modulation Model

When the self-induced flux is negligible, i.e. the screening parameter $\beta_L \ll 1$, the critical current modulation of the asymmetric dc SQUID can be written as follows [10]:

$$I_c(\Phi) = 2I_0 \sqrt{1 - (1 - \alpha_I^2) \sin^2 \left(\frac{\pi\Phi}{\Phi_0} \right)} \quad (3)$$

in which I_0 is the critical current of the Josephson junctions, α_I is the critical-current-asymmetry parameter, Φ_0 is the magnetic flux quantum, and Φ is the applied magnetic flux. For $\alpha_I = 0$, (3) reduces to the one for the symmetric dc SQUID $I_c(\Phi) = 2I_0 |\cos(\pi\Phi/\Phi_0)|$. When the SQUID inductance is large, (3) no longer holds as the self-induced flux is nonnegligible. The critical current modulation amplitude is further reduced from the prediction of (3) due to the self-screening effect. As the amount of the self-induced flux can not be solved analytically, there is no explicit expression for $I_c(\Phi)$ with the non-negligible SQUID inductance [10]. However, since (3) demonstrates the main feature of the critical-current modulation, it may be possible to adapt (3) for the approximation of $I_c(\Phi)$ in the large inductance case. To do this, we added the dimensionless screening factor f_s to the sinusoidal term in (3) as follows:

$$I_c(\Phi) = 2I_0 \sqrt{1 - (1 - \alpha_I^2) \sin^2 \left(\frac{\pi\Phi}{\Phi_0} \right) \cdot f_s} \quad (4)$$

For the consistency with (3), the value of f_s must be $f_s = 1$ in the small inductance case. For the very large inductance case, the modulation amplitude of the critical current reduces to zero, which sets the limit $f_s \geq 0$. Hence, the value of f_s should vary monotonically from 1 to 0 when the SQUID inductance increases from 0 to infinite. The screening factor f_s should be a function of the screening parameter β_L for the inductance dependence. In addition, it could be a function of the noise parameter Γ since the thermal noise may reduce the critical-current-modulation amplitude of the SQUID.

To find the appropriate expression for $f_s(\beta_L, \Gamma)$, one may check the validity of the critical current modulation function in (4). To do this, we assumed the $V - I$ characteristic curve of the SQUID follows that of the over-damped Josephson junction with thermal noise. The equation found by Ambegaokar and Halperin [7] was used to calculate the $V - I$ curve of the SQUID:

$$V(I) = 0.5I_cR_0\pi\Gamma_a \left\{ \left(e^{4\pi x/\Gamma_a} - 1 \right)^{-1} \left[\int_0^{2\pi} g(\theta) d\theta \right] \left[\int_0^{2\pi} \frac{d\theta'}{g(\theta')} \right] + \left[\int_0^{2\pi} d\theta \int_0^\theta \frac{g(\theta)}{g(\theta')} d\theta' \right] \right\}^{-1} \quad (5)$$

in which $x = I/I_c$, I is the current, and $\Gamma_a = 2\pi k_B T / I_c \Phi_0$ is the apparent noise parameter, k_B is the Boltzmann constant, and T is the temperature. In the integrands, the function $g(\theta) = \exp[2(x\theta + \cos\theta)/\Gamma_a]$, and θ, θ' are the dummy variables for numerical integration. The critical current I_c in (5) is modulated by the magnetic flux Φ as given in (4). By using the (4) and (5), the voltage modulation depth V_{pp} of the SQUID biased at $I = 2I_0$ is given as follows:

$$V_{pp} = V(2I_0; \Phi = 0.5\Phi_0) - V(2I_0; \Phi = 0) \quad (6)$$

To find the form of f_s in terms of β_L and Γ , we check the consistency of the modulation depth predicted by (6) with the known empirical formula for the symmetric dc SQUID reported by Enpuku *et al.* [11]

$$V_{pp} = \frac{4I_0R_0}{\pi(1 + \beta_L)} \exp(-2.75\Gamma\beta_L) \quad (7)$$

and the one by Koelle *et al.* [12]:

$$V_{pp} = \frac{I_0R_0(7.3\beta_L^{0.15})}{\pi(1 + \beta_L)[(80\Gamma\beta_L)^{0.4} + 0.35(4\Gamma\beta_L)^{2.5}]} \quad (8)$$

With the aid of (7) and (8), the possible $f_s(\beta_L, \Gamma)$ is found to be

$$f_s(\beta_L, \Gamma) = \{1 + \beta_L [1 + (\Gamma\beta_L)^2]\}^{-1.5} \quad (9)$$

The voltage modulation depths of the symmetric SQUID calculated with (7), (8), and (6) are shown in Fig. 2. For large β_L , we found that all the results are similar. For small β_L , the predictions of the three different equations differ. Since the equation in (4) reduces to (3) for small β_L , our model should be valid in this range. Our model is also in agreement with the empirical formula for the V_{pp} of asymmetric dc SQUID reported by

Fig. 2. Comparison in the predicted V_{pp} of symmetric dc SQUID between the empirical formulas reported by Enpuku *et al.* (dashed curve), Koelle *et al.* (solid curve), and the critical current modulation model (solid diamonds).

Fig. 3. Comparison in the predicted V_{pp} of asymmetric dc SQUID between the empirical formula reported by Müller *et al.* (curves) and the critical-current-modulation model (symbols) for various α_I .

Müller *et al.* [8]. The comparisons in the predicted V_{pp} 's between their formula and our model are shown in Fig. 3.

C. Simulation of Series dc SQUID Array

The critical-current-modulation model described above was applied to simulate of $V - I$ and $V - \Phi$ curves the series-dc-SQUID array with spread in critical currents of Josephson junctions. It is reported [5] that the spread in critical currents of the grain boundary Josephson junctions follows the log-normal distribution:

$$P(\gamma) = \frac{1}{\sigma_{\ln I} \sqrt{2\pi}} \exp \left[-0.5 \left(\frac{\gamma}{\sigma_{\ln I}} \right)^2 \right] \quad (10)$$

in which $\sigma_{\ln I}$ is the logarithmic critical current spread [5] of the logarithmic critical current $\gamma = \ln(I_0/I_{ave})$, and $I_{ave} = \exp(\langle \ln I_0 \rangle)$ is the logarithmic average of the critical current I_0 of all the Josephson junctions. In our simulation for the SQUID array the junction resistance R_0 is assumed to have the spread similar to (10), but the logarithmic average of the resistance R_{ave} is determined by the scaling behavior $I_{ave} R_{ave} \propto (j_c)^{0.5}$. The critical current of the i 'th dc SQUID in terms of the logarithmic critical currents $\gamma_{1,i}$ and $\gamma_{2,i}$ of the Josephson junctions is

$$2I_{0,i} = I_{ave} [\exp(\gamma_{1,i}) + \exp(\gamma_{2,i})] \quad (11)$$

and the critical current asymmetry parameter $\alpha_{I,i}$ of the i 'th SQUID becomes

$$\alpha_{I,i} = \frac{\exp(\gamma_{1,i}) - \exp(\gamma_{2,i})}{\exp(\gamma_{1,i}) + \exp(\gamma_{2,i})} \quad (12)$$

Fig. 4. Simulation results of, (a) the maximum voltage swing, (b) the flux noise, of the series SQUID array at 77 K. The average critical current and resistance of the component SQUIDs are $2I_{ave} = 20 \mu A$ and $0.5R_{ave} = 5 \Omega$. The resistance spread is assumed to be $\sigma_{\ln R} = 0.1$.

The critical current asymmetry results in a flux shift in the $V - \Phi$ curve of the i 'th SQUID by $\Delta\Phi = 0.5(\alpha_{I,i}\beta_{L,i})\Phi_0$, where $\beta_{L,i}$ is the screening parameter of the i 'th SQUID. By taking all the effects described above into account, the $V - I$ and $V - \Phi$ curves of the series dc SQUID are obtained by summing up those of the component SQUIDs. Finally, the flux noise of the SQUID array is calculated with the following equation [9]:

$$S_{\Phi,array}^{1/2} = \frac{\sqrt{\sum_{i=1}^N \left[1 + 0.5 \left(\frac{I_b}{2I_{0,i}} \right)^2 \right] \left(\frac{4k_B T R_{d,i}^2}{0.5R_{0,i}} \right)}}{\left(\frac{\pi V_{pp,array}}{\Phi_0} \right)} \quad (13)$$

in which $I_b = 2I_{ave}$ is the bias current, and $V_{pp,array}$ is the maximum voltage swing of the SQUID array, $2I_{0,i}$ is the critical current, $0.5R_{0,i}$ is the normal state resistance, and $R_{d,i}$ is the dynamic resistance $(dV/dI)_i$ of the i 'th SQUID.

The simulation results of the maximum voltage swing $V_{pp,array}$ and the flux noise of the SQUID arrays with 80-pH inductance for each element are shown in Fig. 4. The $V_{pp,array}$ decreases almost exponentially with the increased critical current spread $\sigma_{\ln I}$ as shown in Fig. 4(a). For the 40 SQUID array, the flux noise exhibits an exponential growth with the greater $\sigma_{\ln I}$ as shown in Fig. 4(b). Although the minimum flux noise of $0.17 \mu\Phi_0/\text{Hz}^{1/2}$ is achieved with $\sigma_{\ln I} = 0$, the flux noise grows with $\sigma_{\ln I}$ and reaches $1.0 \mu\Phi_0/\text{Hz}^{1/2}$ at $\sigma_{\ln I} = 1.2$, where it becomes the same as the single-SQUID curve. Since the voltage noise of the array was found to slightly increase by a factor of 1.3 from $\sigma_{\ln I} = 0$ to $\sigma_{\ln I} = 1.2$, the reduction in $V_{pp,array}$ should be responsible for the exponential growth in the flux noise.

As the critical-current spread is always nonzero, it is interesting whether the $\sigma_{\ln I}$ of the current junction technologies permit the flux-noise reduction with the HTS dc SQUID array. When the flux noise of the 40 SQUID array is half the noise of the single SQUID, the critical current spread is $\sigma_{\ln I} = 0.8$ according to Fig. 4(b). This level corresponds to the on-chip spread in critical currents of $0.4I_{ave}$ to $2.2I_{ave}$.

III. EXPERIMENTAL RESULTS

The above simulation technique was applied to fit our experimental results of the 40 SQUID series array, which was made from the single-layered $\text{YBa}_2\text{Cu}_3\text{O}_y$ superconductor film on

Fig. 5. (a) The observed voltage-flux curve of the 40 SQUID array fabricated on the bi-crystal substrate. The temperature was 55 K, and the inductance of the component SQUID was estimated to be 80 pH. (b) The simulated voltage-flux curve with the critical current spread $\sigma_{\ln I} = 1.43$, $I_{ave} = 0.5 \mu$ A. (c) Distribution of the critical currents of the Josephson junctions.

the bi-crystal SrTiO_3 substrate with $15^\circ/15^\circ$ mis-orientation angle. The hole-size of the component SQUID is $29 \mu\text{m} \times 30 \mu\text{m}$. The line-widths are $8 \mu\text{m}$ for the SQUID loop and $3 \mu\text{m}$ for the junctions. The estimated SQUID inductance is 80 pH. Four dummy SQUID loops are put into the ends of the array to avoid the nonuniform modulation field in the active component SQUID. The $V - \Phi$ curve of the array measured by using the uniform modulation field generated with a 50-cm Helmholtz coil is shown in Fig. 5(a). The simulated voltage-flux curve of the array and the distribution of the simulated critical currents of the Josephson junctions are shown Fig. 5(b) and (c) respectively. Although the measured $V - I$ curve was almost linear with the total resistance of $2.4 \text{ k}\Omega$, the $V - \Phi$ curve with a maximum modulation depth of $150 \mu\text{V}$ was observed with the bias current around $1 \mu\text{A}$. The best-fit spread in critical current was found to be $\sigma_{\ln I} = 1.43$ with $I_{ave} = 0.5 \mu\text{A}$. The range of critical current spread corresponds to the small β_L from 0.009 to 0.16. The small β_L avoided the deterioration of the voltage modulation depth of array caused by the flux shift in the component SQUID. The reduction in the modulation amplitude at large modulation fields was found dominated by the spread in the SQUID's effective area, similar to the result for the LTS SQUID array reported in [13]. The best-fit spread in the effective

area to simulate the reduced modulation field at large modulation field is 3.3%. According to the best-fit simulation result, the critical-current spread in our sample was too large to exhibit the low flux noise characteristic.

IV. CONCLUSION

We proposed the critical-current-modulation model and found that it fits well with the known empirical formulas for the voltage swing of a HTS dc SQUID. With the critical-current-modulation model, we found that the voltage swing of the array decreases about exponentially with the logarithmic critical current spread $\sigma_{\ln I}$, which results in the flux noise $S_\Phi^{1/2}$ being enhanced exponentially with $\sigma_{\ln I}$. For the 40 SQUID array at 77 K with $I_{ave} = 20 \mu\text{A}$, the allowed level of the on-chip critical-current spread in junctions is within $0.4I_{ave}$ to $2.2I_{ave}$ for achieving the low flux noise.

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