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Green functions in large N_c QCD

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In the presence of the nontrivial QCD ground state or vacuum, nonlocal condensates are used to characterize the quark or gluon propagator, or other Green functions of higher order. We wish to show in this paper that, provided that we may use the large N_c approximation to treat condensates of much higher order, there is a natural way of setting up closed sets of differential equations which govern the inter-related Green functions to a given order. As a specific example, the leading-order equations for the nonlocal condensates appearing in the quark propagator are derived and explicit solutions are obtained. Some applications of our analytical results are briefly mentioned.

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The ground state, or the vacuum, of QCD is known to be nontrivial, in the sense that there are non-zero condensates, including gluon condensates, quark condensates, and perhaps infinitely many higher-order condensates. In such a theory, propagators, i.e. causal Green's functions, such as the quark propagator

$$iS_{ij}^{ab}(x) \equiv \langle 0 | T(q_i^a(x)\bar{q}_j^b(0)) | 0 \rangle, \quad (1)$$

carry all the difficulties inherent in the theory. Higher-order condensates, such as four-quark condensates, represent an infinite series of unknown parameters unless some useful ways for reduction can be obtained. As the vacuum, $|0\rangle$, is highly nontrivial, there is little reason to expect that Wick's theorem (of factorization), as obtained for free quantum field theories, is still valid in general. Thus, we must look for alternative methods in order to obtain useful results.

It is known that the equations for Green functions up to a certain order usually involve Green functions of even higher order, thereby making such hierarchy of equations often useless in practice. In this paper, however, we wish to show that, provided that we may use the large N_c approximation to treat condensates of much higher order, there is in fact a natural way of setting up closed sets of differential equations which govern the inter-related Green functions to a given order. We consider this as an important accomplishment, both because we can always go over to the next level of sophistication in order to improve the approximation and because the large N_c expansion has been shown to yield desirable results for describing hadron physics.

In light of the nontrivial QCD vacuum, we begin by considering the feasibility of working directly with the various matrix elements such as the quark propagator of Eq. (1). Useful

relations may be derived if we regard the equations for interacting fields [1],

$$\{i\gamma^\mu(\partial_\mu + ig\frac{\lambda^a}{2}A_\mu^a) - m\}\psi = 0; \quad (2)$$

$$\partial^\nu G_{\mu\nu}^a - 2gf^{abc}G_{\mu\nu}^b A_\nu^c + g\bar{\psi}\frac{\lambda^a}{2}\gamma_\mu\psi = 0, \quad (3)$$

as the equations of motion for *quantized* interacting fields, subject to the standard quantization rule that the equal-time commutators or anticommutators among these quantized interacting fields are identical to those among non-interacting ones. As our basic example, we allow the operator $\{i\gamma^\mu\partial_\mu - m\}$ to act on the matrix element defined by Eq. (1) and obtain

$$\{i\gamma^\mu\partial_\mu - m\}_{ik}iS_{kj}^{ab}(x) = i\delta^4(x)\delta^{ab}\delta_{ij} + \langle 0 | T(\{g\frac{\lambda^n}{2}A_\mu^n\gamma^\mu q(x)\}_i\bar{q}_j^b(0)) | 0 \rangle. \quad (4)$$

We should always keep in mind that the QCD vacuum $|0\rangle$ is a nontrivial ground state which is in general not annihilated by operating on it an annihilation operator.

Eq. (4) can be solved by splitting the propagator into a singular, perturbative part and a nonperturbative part:

$$iS_{ij}^{ab}(x) = iS_{ij}^{(0)ab}(x) + i\tilde{S}_{ij}^{ab}(x), \quad (5)$$

where

$$iS_{ij}^{(0)ab}(x) \equiv \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} iS_{ij}^{(0)ab}(p), \quad (6)$$

$$iS_{ij}^{(0)ab}(p) = \delta^{ab} \frac{i(\hat{p}+m)_{ij}}{p^2 - m^2 + i\epsilon}, \quad (7)$$

with $\hat{a} \equiv \gamma^\mu a_\mu$ for a four-vector a_μ . The nonperturbative part then satisfies the equation:

$$\{i\gamma^\mu\partial_\mu - m\}_{ik}i\tilde{S}_{kj}^{ab}(x) = \langle 0 | T(\{g\frac{\lambda^n}{2}A_\mu^n\gamma^\mu q(x)\}_i\bar{q}_j^b(0)) | 0 \rangle. \quad (8)$$

This would be pretty much the end of the story unless we could find some way to proceed. We may note that Eq. (4) may also be derived by making use of, e.g., the path-integral formulation, and the issue of how to define renormalized composite operators, i.e. products of field operators, is by no means trivial (and fortunately we need not worry about such problem for the sake of this paper).

The approach which we suggest here [2] is based upon two key elements, namely, the set of interacting field equations *plus* the rule of canonical quantization (for interacting fields). The equations which we obtain, such as Eq. (8), are much the same as the set of Schwinger-Dyson equations (for the matrix elements). An important aspect in our derivation is that nontriviality of the vacuum $|0\rangle$ is observed at every step – a central issue in relation to QCD.

In what follows, we wish to focus on the quark propagator, as specified by Eqs. (1) and (4)–(7). We write

$$i\tilde{S}_{ij}^{ab}(x) = \delta^{ab}\{\delta_{ij}f(x^2) + i\hat{x}_{ij}g(x^2)\}, \quad (9)$$

with $x^2 \equiv x_0^2 - \vec{x}^2$. $f(x^2)$ and $g(x^2)$ are what we refer to as “nonlocal condensates” in connection with the quark propagator. For example, we have

$$\langle : \bar{q}(x) q(0) : \rangle = -12f(x^2),$$

which is the nonlocal quark condensate in the standard sense. To proceed further, we choose to work only with the leading term in the fixed-point gauge and introduce

$$\begin{aligned} <: \{gG_{\mu\nu}q(x)\}_i^a \bar{q}_j^b(0) :> = \delta^{ab} \{(\gamma_\mu x_\nu - \gamma_\nu x_\mu)A(x^2) + i\sigma_{\mu\nu}B(x^2) \\ + (\gamma_\mu x_\nu - \gamma_\nu x_\mu)\hat{x}C(x^2) + i\sigma_{\mu\nu}\hat{x}D(x^2)\}, \end{aligned} \quad (10)$$

with $G_{\mu\nu} \equiv (\lambda^n/2)G_{\mu\nu}^n$, an antisymmetric operator. The invariant functions $A(x^2)$, $B(x^2)$, $C(x^2)$, and $D(x^2)$ are additional nonlocal condensates which we have to deal with explicitly in this paper.

Under the assumption that we keep only the leading term in the fixed-point gauge (a simplifying assumption which can be removed if necessary), we have

$$\begin{aligned} & \{i\gamma^\alpha\partial_\alpha - m\}_{ik} <: \{gG_{\mu\nu}q(x)\}_k^a \bar{q}_j^b(0) :> = -\frac{1}{2}x^\beta <: \{g^2G_{\mu\nu}G_{\alpha\beta}\gamma^\alpha q(x)\}_i^a \bar{q}_j^b(0) :> \\ = & -\frac{1}{2} \cdot \frac{1}{96} \cdot \frac{4}{3} \cdot <g^2G^2> <: \{(\gamma_\mu x_\nu - \gamma_\nu x_\mu)q(x)\}_i^a \bar{q}_j^b(0) :> \\ = & -\frac{1}{144} <g^2G^2> \delta^{ab}(\gamma_\mu x_\nu - \gamma_\nu x_\mu)\{f(x^2) + i\hat{x}g(x^2)\}. \end{aligned} \quad (11)$$

Here the first equality follows from the field equation and the second one is based on the large N_c approximation that the contribution in which $G_{\mu\nu}$ and $G_{\alpha\beta}$ do not couple to color-singlet is suppressed by a factor of $1/N_c^2$. Thus, the factorization in the present case is justified to order $1/N_c^2$.

Now, we may use Eqs. (8) and (11) and obtain a closed set of equations:

$$2f'(x^2) - mg(x^2) = -\frac{3}{2}i(B - x^2C), \quad (12)$$

$$2x^2g'(x^2) + 4g(x^2) + mf(x^2) = \frac{3}{2}x^2(A - D), \quad (13)$$

$$4iB' - 2iC - 2ix^2C' - mA = -\frac{1}{144} <g^2G^2> f(x^2), \quad (14)$$

$$2iA + 2ix^2D' - mB = 0, \quad (15)$$

$$-2iA' + 4iD' - mC = -\frac{i}{144} <g^2G^2> g(x^2), \quad (16)$$

$$2iB' + 2iC - mD = 0, \quad (17)$$

where the derivatives are with respect to the variable x^2 .

To leading order in m , we obtain, by eliminating A , B , C , and D ,

$$x^2f_0''' + 3f_0'' - \xi_0^2x^2f_0' - 2\xi_0^2f_0 = 0, \quad (18)$$

$$(x^2)^3g_0''' + 5(x^2)^2g_0'' + \{2x^2 - \xi_0^2(x^2)^3\}g_0' - \{2 + 2\xi_0^2(x^2)^2\}g_0 = 0, \quad (19)$$

with $\xi_0^2 \equiv <g^2G^2> / 384$. Eq. (19) can be simplified considerably by introducing

$$g_0(x^2) \equiv (x^2)^{-2}\tilde{g}_0(x^2),$$

which leads to the equation:

$$x^2\tilde{g}_0''' - \tilde{g}_0'' - \xi_0^2x^2\tilde{g}_0' = 0. \quad (20)$$

Eqs. (18) and (20) can be solved by iteration, leading to the result:

$$f_0(t) = a_0\{1 + \frac{1}{1.3}(\xi_0 t)^2 + \frac{1}{1.3^2.3}(\xi_0 t)^4 + \cdots\} + a_1t\{1 + \frac{1}{2.4}(\xi_0 t)^2 + \frac{1}{2.4^2.6}(\xi_0 t)^4 + \cdots\}, \quad (21)$$

$$\tilde{g}_0(t) = c_2t^2\{1 + \frac{1}{2.4}(\xi_0 t)^2 + \frac{1}{2.4^2.6}(\xi_0 t)^4 + \cdots\}, \quad (22)$$

with $t \equiv x^2$ and

$$a_0 = -\frac{1}{12} \langle \bar{q}q \rangle, \quad a_1 = \frac{1}{192} \langle \bar{q}g_c\sigma \cdot Gq \rangle, \quad c_2 = -\frac{g_c^2 \langle \bar{q}q \rangle^2}{2^5 \cdot 3^4}. \quad (23)$$

Eq. (23) is obtained by comparing to the well-known series expansion for the quark propagator (see, e.g., [3]). Note that there are two integration constants, a_0 and a_1 , for $f(x^2)$ but there is only one permissible constant for $g(x^2)$ (and c_2 is in fact a four-quark condensate taken in the large N_c limit).

For a number of applications, it is useful to obtain analytic expressions for $f_0(t)$ and $g_0(t)$. This turns out to be possible by way of Laplace transforms.

$$\bar{f}_0(s) \equiv \int_0^\infty ds e^{-st} f_0(t), \quad \bar{g}'_0(s) \equiv \int_0^\infty ds e^{-st} \tilde{g}'_0(t). \quad (24)$$

We obtain

$$\bar{f}_0(s) = -\frac{2a_1}{\xi_0^2} - \frac{a_0}{\xi_0} \frac{s}{\sqrt{s^2 - \xi_0^2}} \sec^{-1} \frac{s}{\xi_0} + \frac{\gamma_0 s}{\sqrt{s^2 - \xi_0^2}}, \quad \bar{g}'_0(s) = \frac{2c_2}{(s^2 - \xi_0^2)^{3/2}}, \quad (25)$$

with $\gamma_0 = 2a_1/\xi_0^2$. Looking up the table for Laplace transforms, we find

$$\tilde{g}'_0 = \frac{2c_2}{\xi_0^3} \cdot \xi_0 t \cdot I_1(\xi_0 t), \quad (26)$$

with $I_1(z)$ the modified Bessel function of the first kind, to order one. It is straightforward to show that Eq. (26) yields the series expansion in Eq. (22). Also, the function $I_1(\xi_0 t)$ enters the second series in $f_0(t)$ as in Eq. (21).

As the first application, we note that that our analytical expressions for nonlocal condensates help to determine the induced condensates previously treated as new parameters in the the external-field QCD sum rule method, thereby making it more powerful than what it used to be. We also wish to mention that, making use of soft-pion theorems, the quark parton distributions for the Goldstone pions may also be evaluated in the massless limit.

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