Constant-Voltage Steps in Josephson Tunneling

Chong Der HU

Department of Physics, National Taiwan University, Taipei, Taiwan R.O.C.

(Received June 25, 1997)

We studied the *ac* Josephson effect. In the resistively shunted junction model (RSJ) the phase difference is expanded in Fourier series. The Fourier components can be evaluated by solving the equations of motion. Truncation is applied to deal with the infinite products and series of Bessel functions. We calculate the constant-voltage steps and find that they are complicate functions of the *ac* power input.

KEYWORDS: Josephson tunneling, nonlinear phenomena

§1. Introduction

The *ac* Josephson effect has been an active field of study¹⁾ since it was discovered by Shapiro.²⁾ The most prominent feature in the *I-V* characteristic curve is the constant-voltage steps. They appear at $2eV_{dc} = n\hbar\omega$ where ω is the frequency of the *ac* bias power. The step size varies with the square root of the *ac* bias power in an oscillatory manner. In some cases it has a Bessel function behavior. Subharmonic steps had also been found. Usually, one can compute the step sizes with the voltage-source model³⁾ and indeed, find that they vary with the *ac* bias voltage as Bessel functions. However, many experimental data do not show this kind of behavior.³⁾ This discrepancy is attributed to the fact that the voltage across the junction is not easily maintained. Alternatively, the resistively shunted junction model (RSJ)⁴⁾ can be used to analyze this phenomenon. Since it involves a nonlinear differential equation, simulation was often used⁵⁾ and obtained results consistent with experiments. Here, we attempt to provide a different point of view. We proposed an analytical solution and from which the step sizes are computed. It can be shown clearly how the step sizes are related to the *ac* power input, as well as other parameters such as the resistance, capacitance, frequency and supercurrent amplitude. We believe this method is able to provide additional insight to Josephson tunneling. Furthermore, since the equation of motion of the RSJ model is the same as that of a dissipative pendulum under external forces, our method is useful for studying nonlinear dynamics. In §2 we derive the formalism. In §3 the step sizes are computed. The conclusion is given in §4.

§2. Derivation of Phase

The RSJ model gives the following equations of motion for the Josephson junction with a current source:

$$\dot{\phi} = 2eV(t) \quad (1) \quad \text{and}$$

and

$$\frac{C}{2e}\ddot{\phi} + \frac{1}{2eR}\dot{\phi} + I_J \sin \phi = I_{dc} + I_{ac} \sin \omega t, \quad (2) \quad \text{we get}$$

where I_J is the supercurrent amplitude, C and R are the capacitance and resistance of the junction, $I_{dc(ac)}$ is the *dc(ac)* bias current, and $\hbar$ was set to 1. Equation (2) is a nonlinear equation. Up to now, most efforts of solving this equation use numerical simulation.⁶⁾ Here, we propose an analytical solution. Before doing so, we would like to outline some well-known mathematical properties of the solution⁷⁾ and thus, illustrate the motivation of formulating the solution below. The most obvious question to ask is whether there exists a periodic solution. If there is, then its frequency should be commensurable to ω and hence, can be expanded in a Fourier series. If such series is substituted into eq. (2), then, in the presence of I_{dc} , a term linear in t is required. Its coefficient gives a new frequency due to the nonlinearity. Thus, the following expression is proposed:

$$\begin{aligned} \phi = & \phi_0 + \omega_0 t + \sum_{k,l} a_{kl} \sin(k\omega + l\omega_0)t \\ & + \sum_{m,n} b_{mn} \cos(m\omega + n\omega_0)t \end{aligned} \quad (3)$$

where ϕ_0 is related to initial conditions and ω_0 is to be determined. In general, the solution in eq. (3) is not periodic. It has some similarity with the one proposed by Hasselberg *et al.*⁸⁾ and also ref. 9 for the voltage-biased case. However, the difference is profound. In the voltage-biased circuit, ω_0 is equal to $2eV_{dc}$ where V_{dc} is the applied *dc* voltage. In the present case, ω_0 , though related to I_{dc} , has to be computed. If ω_0 is commensurable to ω , then the solution is periodic. Higher harmonics and subharmonic components appear. As we shall see this is when the constant-voltage steps appear.

Substituting eq. (3) into eq. (2) and using the relations¹⁰⁾

$$\exp(iz \sin \theta) = \sum_m J_m(z) e^{im\theta} \quad (4)$$

$$\exp(iz \cos \theta) = \sum_n i^n J_n(z) e^{in\theta}, \quad (5)$$

$$\begin{aligned}
& (r\omega + s\omega_0)^2 a_{rs} + \frac{(r\omega + s\omega_0)}{RC} b_{rs} + \frac{2eI_{ac}}{C} \delta_{r1} \delta_{s0} \\
&= \frac{2eI_J}{C} \sum_{\alpha, \beta} \cos \left(\sum \beta_{mn} \pi/2 + \phi_0 \right) \left[\prod_{k,l} J_{\alpha_{kl}}(a_{kl}) \right] \left[\prod_{m,n} J_{\beta_{mn}}(b_{mn}) \right], \quad (6)
\end{aligned}$$

$$\begin{aligned}
& (r\omega + s\omega_0)^2 b_{rs} - \frac{(r\omega + s\omega_0)}{RC} a_{rs} \\
&= \frac{2eI_J}{C} \sum_{\alpha, \beta} \sin \left(\sum \beta_{mn} \pi/2 + \phi_0 \right) \left[\prod_{k,l} J_{\alpha_{kl}}(a_{kl}) \right] \left[\prod_{m,n} J_{\beta_{mn}}(b_{mn}) \right] \quad (7)
\end{aligned}$$

and

$$\frac{\omega_0}{2eR} = I_{dc} - I_J \sum_{\alpha, \beta} \sin \left(\sum \beta_{mn} \pi/2 + \phi_0 \right) \left[\prod_{k,l} J_{\alpha_{kl}}(a_{kl}) \right] \left[\prod_{m,n} J_{\beta_{mn}}(b_{mn}) \right]. \quad (8)$$

The summations of α and β mean to sum over all possible integer values. In eqs. (6)–(8) α and β are under following constraints:

$$\sum_{k,l,m,n} (\alpha_{kl}k + \beta_{mn}m) = r \quad (9)$$

and

$$\sum_{k,l,m,n} (\alpha_{kl}l + \beta_{mn}n) = s. \quad (10)$$

There are infinite series and products of Bessel functions in eqs. (6)–(8). We can truncate them by neglecting Bessel functions with large indices and arguments with subscripts large in magnitude. This approximation is based on the fact that a_{kl} and b_{mn} are small in magnitudes if any one of the subscripts is large in magnitude. More details were given in ref. 9. If only the first few terms were kept, we have

$$\begin{aligned}
a_{10} \simeq & -\frac{2eI_{ac}}{C\omega^2} - \frac{b_{10}}{RC\omega} - \frac{2eI_J}{C\omega^2} \left\{ \cos \phi_0 J_1(a_{-11}) \left[\prod_{(k,l) \neq (-1,1)} J_0(a_{kl}) \right] \left[\prod_{m,n} J_0(b_{mn}) \right] \right. \\
& \left. + \cos(\phi_0 - \pi/2) J_1(b_{-11}) \left[\prod_{(m,n) \neq (-1,1)} J_0(b_{mn}) \right] \left[\prod_{k,l} J_0(a_{kl}) \right] \right\}, \quad (11)
\end{aligned}$$

$$\begin{aligned}
b_{10} \simeq & \frac{a_{10}}{RC\omega} - \frac{2eI_J}{C\omega^2} \left\{ \sin \phi_0 J_1(a_{-11}) \left[\prod_{(k,l) \neq (-1,1)} J_0(a_{kl}) \right] \left[\prod_{m,n} J_0(b_{mn}) \right] \right. \\
& \left. + \sin(\phi_0 - \pi/2) J_1(b_{-11}) \left[\prod_{(m,n) \neq (-1,1)} J_0(b_{mn}) \right] \left[\prod_{k,l} J_0(a_{kl}) \right] \right\}, \quad (12)
\end{aligned}$$

$$a_{01} \simeq -\frac{b_{10}}{RC\omega_0} + \frac{2eI_J}{C\omega_0^2} \cos \phi_0 \left[\prod_{(k,l)} J_0(a_{kl}) \right] \left[\prod_{m,n} J_0(b_{mn}) \right], \quad (13)$$

$$b_{01} \simeq \frac{a_{10}}{RC\omega_0} + \frac{2eI_J}{C\omega_0^2} \sin \phi_0 \left[\prod_{(k,l)} J_0(a_{kl}) \right] \left[\prod_{m,n} J_0(b_{mn}) \right], \quad (14)$$

$$\begin{aligned}
a_{-11} \simeq & -\frac{b_{10}}{RC(\omega_0 - \omega)} - \frac{2eI_J}{C(\omega_0 - \omega)^2} \left\{ \cos \phi_0 J_1(a_{10}) \left[\prod_{(k,l) \neq (1,0)} J_0(a_{kl}) \right] \left[\prod_{m,n} J_0(b_{mn}) \right] \right. \\
& \left. + \cos(\phi_0 - \pi/2) J_1(b_{10}) \left[\prod_{(m,n) \neq (1,0)} J_0(b_{mn}) \right] \left[\prod_{k,l} J_0(a_{kl}) \right] \right\}, \quad (15)
\end{aligned}$$

and

$$b_{-11} \simeq \frac{a_{10}}{RC(\omega_0 - \omega)} - \frac{2eI_J}{C(\omega_0 - \omega)^2} \left\{ \sin \phi_0 J_1(a_{10}) \left[\prod_{(k,l) \neq (1,0)} J_0(a_{kl}) \right] \left[\prod_{m,n} J_0(b_{mn}) \right] \right. \\ \left. + \sin(\phi_0 - \pi/2) J_1(b_{10}) \left[\prod_{(m,n) \neq (1,0)} J_0(b_{mn}) \right] \left[\prod_{k,l} J_0(a_{kl}) \right] \right\}. \quad (16)$$

Above Fourier components are in general the most important ones. They can be evaluated approximately by letting $J_0(a_{kl}) \simeq J_0(b_{mn}) \simeq 0$. Others can be computed analogously. We show in Fig. 1 how a 's vary with I_{ac} under the conditions $\omega = 10/RC = 2\omega_J = 2\sqrt{2eI_J/C}$. Readers can evaluate b_{rs} approximately by multiplying the corresponding a_{rs} by the factor $1/RC\omega$. The figure clearly shows that a_{10} which is roughly proportional to $-2eI_{ac}/C\omega^2$ is the largest in magnitude. Other components decrease rapidly with increasing subscript magnitudes.

ω_0 can also be solved for by collecting the time-independent terms. In fact it is easier to calculate I_{dc} for a given ω_0 . One easily finds that for small I_J , $\omega_0 \simeq 2eRI_{dc}$ which obeys the Ohm law. In summary, the form in eq. (3) is a fast converging series and the truncation method is a good approximation.

§3. Constant-Voltage Steps

Much interest has been concentrated on the constant-voltage steps 3. In this section we calculate the step sizes under harmonic and subharmonic conditions. They occur when ω and ω_0 are commensurable:

$$q\omega = p\omega_0 \quad (17)$$

where p and q are non-negative integers. Now $\dot{\phi}$ is periodic in time. The time-average of $\dot{\phi}$ is

$$\langle \dot{\phi} \rangle = 2eV_{dc} \quad (18)$$

where V_{dc} is the measured dc voltage. Different initial conditions (ϕ_0) give different values of supercurrent and the step size is equal to the amplitude of variation. For a given I_{dc} , the supercurrent is equal to

$$I_J \sin \phi = I_J \sum_{\alpha, \beta} \left[\prod_{(k,l)} J_{\alpha_{kl}}(a_{kl}) \right] \left[\prod_{m,n} J_{\beta_{mn}}(b_{mn}) \right]$$

$$\times \sin \left\{ \phi_0 + \sum_{m,n} \beta_{mn} \pi/2 \right. \\ \left. + \sum_{k,l,m,n} [(\alpha_{kl}k + \beta_{mn}m)\omega t \right. \\ \left. + (\alpha_{kl}l + \beta_{mn}n + 1)\omega_0 t] \right\}. \quad (19)$$

Therefore, the time-average or the dc part of the current requires

$$\sum_{k,l,m,n} (\alpha_{kl}k + \beta_{mn}m) = -Q, \quad (20)$$

$$\sum_{k,l,m,n} (\alpha_{kl}l + \beta_{mn}n) + 1 = P, \quad (21)$$

and

$$\frac{P}{Q} = \frac{p}{q}, \quad (22)$$

where P and Q are the multiples of p and q respectively. Since larger P and Q implies larger indices of Bessel functions and smaller magnitudes of super current, we discuss only the case $P = p$ and $Q = q$.

The more prominent steps are those with $p = 1$. According to eqs. (20) and (21), the most important term of the supercurrent is that with

$$\alpha_{kl} = \beta_{mn} = 0 \quad (23)$$

for all k, l, m and n except for a_{10} and b_{10} with the requirement

$$\alpha + \beta_{10} = -q. \quad (24)$$

The time-average of the supercurrent is

$$\langle I_J \sin \phi \rangle = (-1)^q I_J \sum_s \sin(\phi_0 + s\pi/2) J_s(b_{10}) J_{q-s}(a_{10}) \left[\prod_{(k,l) \neq (1,0)} J_0(a_{kl}) \right] \\ \times \left[\prod_{(m,n) \neq (1,0)} J_0(b_{mn}) \right]. \quad (25)$$

If $a_{10} \gg b_{10}$

$$\langle I_J \sin \phi \rangle \simeq (-1)^q I_J \sin \phi_0 J_q(a_{10}) \left[\prod_{(k,l) \neq (1,0)} J_0(a_{kl}) \right] \left[\prod_{(m,n)} J_0(b_{mn}) \right], \quad (26)$$

and if $a_{10} \ll b_{10}$

$$\langle I_J \sin \phi \rangle \simeq (-1)^q I_J \sin(\phi_0 + q\pi/2) J_q(b_{10}) \left[\prod_{(k,l)} J_0(a_{kl}) \right] \left[\prod_{(m,n) \neq (1,0)} J_0(b_{mn}) \right]. \quad (27)$$

The magnitude of the right hand side is the step size. Equation (25), in many cases, is not similar to the well-known Bessel function behavior in the voltage-source model

$$I_J J_q \left(\frac{2eV_{ac}}{\omega} \right)$$

which is given by substituting

$$\dot{\phi} = 2e(V_{dc} + V_{ac} \cos \omega t)$$

into $I_J \sin \phi$. There are two reasons. First of all, a_{10} does not vary linearly with V_{ac} or I_{ac} though the difference may be small (eq. (11)). More importantly, the factor $J_0(b_{10})$ also provides considerable variation with respect to I_{ac} because b_{10} may not be small (eq. (12)). One can clearly see that only when eqs. (26) and (27) are applicable the familiar Bessel function behavior appears. This can explain why experimental data and simulation results are not consistent with the simple Bessel function behavior. Our calculation also shows that the oscillation amplitudes of the step sizes decrease with increasing ω or ω_0 . This is because increasing frequencies imply smaller magnitudes of a_{rs} and b_{rs} ($(r, s) \neq (1, 0)$) and hence, smaller nonlinearity and steps.

We illustrate in Figs. 2 the functional relation of the steps sizes with the square root of the ac power input. Note that there is always a proportional constant difficult to be determined. It can be viewed as an ad-

justable parameter in data analysis. In our calculation the obvious choice of the variable for the case $RC\omega \gg 1$ is $2eI_{ac}/C[1 + (RC\omega)^{-2}] \simeq a_{10}$. Because in this case $|a_{10}| \gg |b_{10}|$ and hence, the variation of the averaged current comes mostly from that of a_{10} . If, however, $RC\omega \ll 1$, $|a_{10}| \ll |b_{10}|$, then the variation of b_{10} has more important impact and $2eI_{ac}/RC^2\omega[1 + (RC\omega)^{-2}] \simeq b_{10}$ should be chosen. In Fig. 2 we choose $\omega = 2\omega_J$ and $RC\omega = 10$ in Fig. 2(a) and $RC\omega = 1$ in Fig. 2(b). Three step sizes were shown, namely $q = 0$ in the solid lines, $q = 1$ in the

Fig. 1. a_{10} , a_{01} , a_{-11} and a_{02} were plotted against the square root of input ac power: $2eI_{ac}/C[1 + (RC\omega)^{-2}]$ with $\omega = 2\omega_J$ and $RC\omega = 10$. Note that a_{-11} is multiplied by a factor of 100 and a_{01} and a_{02} by 1000.

Fig. 2. The measured dc supercurrent versus $2eI_{ac}/C[1 + (RC\omega)^{-2}]$: the solid line denotes $\omega_0 = 2eV_{dc} = 0$, the short dashed line $\omega = \omega_0$ and the long dashed line $2\omega = \omega_0$ under the condition (a) $\omega = 2\omega_J$ and $RC\omega = 10$ and (b) $\omega = 2\omega_J$ and $RC\omega = 1$.

short-dashed lines and $q = 2$ in the long-dashed lines. One would observe that the curves in Fig. 2(a) are close to Bessel functions. In fact they can be approximated by

$$J_q(a_{10}) \simeq J_q \left(\frac{2eI_{ac}}{C[1 + (RC\omega)^{-2}]} \right),$$

as we have seen in eq. (26). Those in Fig. 2(b) are quite different from the Bessel functions. Now $a_{10} = b_{10}$, so the variation of b_{10} is significant. These curves are approximately

$$J_q^2 \left(\frac{2eI_{ac}}{C[1 + (RC\omega)^{-2}]} \right)$$

for $q = 1$ and

$$J_0 \left(\frac{2eI_{ac}}{C[1 + (RC\omega)^{-2}]} \right) J_q \left(\frac{2eI_{ac}}{C[1 + (RC\omega)^{-2}]} \right)$$

for $q = 2$. Analogously, for $RC\omega \ll 1$, the step size should be close to

$$J_q(b_{10}) \simeq J_q \left(\frac{2eI_{ac}}{RC^2\omega[1 + (RC\omega)^{-2}]} \right)$$

in view of eq. (27).

For $p \neq 1$, the subharmonic steps, the step size is considerably smaller. The most important terms of the supercurrent are

$$\begin{aligned} \langle I_J \sin \phi \rangle &= (-1)^q I_J \sum_{m,n} \sin[\phi_0 + \sum (n-m)\pi/2] \\ &\times J_{q-m}(a_{10}) J_m(b_{10}) J_{p-n-1}(a_{01}) J_n(b_{01}) \\ &\times \left[\prod'_{k,l} J_0(a_{kl}) \right] \left[\prod'_{m,n} J_0(b_{mn}) \right] \\ &+ J_{q+p-1-m}(a_{10}) J_m(b_{10}) \\ &\times J_{p-n-1}(a_{-11}) J_n(b_{-11}) \\ &\times \left[\prod''_{k,l} J_0(a_{kl}) \right] \left[\prod''_{m,n} J_0(b_{mn}) \right], \quad (28) \end{aligned}$$

where $\prod'$ and $\prod''$ mean that (k, l) and (m, n) are not equal to $(1, 0)$, $(0, 1)$ and $(1, 0)$, $(-1, 1)$ respectively. Since a_{-11} , b_{-11} and a_{01} , b_{01} are comparable in magnitude, there are two kinds of terms contributing to the dc supercurrent. Hence, the relation between the step sizes and I_{ac} are very complicated. In Fig. 3 under the conditions $\omega = 2\omega_J$ and $RC\omega = 10$, we show the steps labeled by (p, q) . There are still oscillations but the minima do not approach zero. Their positions are shifted slightly away from the zeroes of Bessel functions. Furthermore, our calculation shows that the larger the magnitude of p or q , the smaller the step size. This is expected. But our calculation also shows that when $|p - q| = 1$, the magnitudes of a_{-11} and b_{-11} are the largest, and hence, the step sizes are greater.

§4. Conclusion

We have proposed an analytical solution for the RSJ model. It shows Ohmic behavior in general. When ω and $\omega_0 = 2eV_{dc}$ are commensurable the solution becomes a

Fig. 3. The measured dc supercurrent versus $2eI_{ac}/C[1 + (RC\omega)^{-2}]$: the solid line denotes $\omega = 2\omega_0$ and the dashed line $3\omega = 2\omega_0$ under the condition $\omega = 2\omega_J$ and $RC\omega = 1$.

periodic function and produces constant-voltage steps. The step size is a function of the square root of the input ac power. For the harmonic case, i.e. $n\omega = \omega_0$, only if $RC\omega \ll 1$ and $RC\omega \gg 1$, the step size has the well-known Bessel function form. Otherwise, the magnitude of step size cannot be described by Bessel functions though there are clearly oscillations. For the subharmonic case, i.e. $p\omega = q\omega_0$, the step size still shows oscillations but without definite amplitude. The minima occur approximately at the zeroes of Bessel functions.

This work is supported in part by the NSC of Taiwan, R.O.C. under the contract number NSC86-2112-M-002-005.

Appendix

In this appendix we give some details of deriving eqs. (6) and (7). In view of eqs. (3)–(5), we found

$$\begin{aligned} e^{i\phi} &= e^{i(\phi_0 + \omega_0 t)} \left[\prod_{k,l} \sum_p J_p(a_{kl}) e^{ip(k\omega + l\omega_0)t} \right] \\ &\times \left[\prod_{m,n} \sum_q J_q(b_{mn}) e^{iq[(m\omega + n\omega_0)t + \pi/2]} \right] \quad (A.1) \end{aligned}$$

where p and q are integers. After some arrangement, we got

$$\begin{aligned} e^{i\phi} &= e^{i(\phi_0 + \omega_0 t)} \sum_{\alpha_{kl}} \left(\prod_{k,l} J_{\alpha_{kl}}(a_{kl}) \exp[i\alpha_{kl}(k\omega + l\omega_0)t] \right) \\ &\times \sum_{\beta_{mn}} \left(\prod_{m,n} J_{\beta_{mn}}(b_{mn}) \right. \\ &\times \left. \exp\{i\beta_{mn}[(m\omega + n\omega_0)t + \pi/2]\} \right) \quad (A.2) \end{aligned}$$

where α and β are also integers. Substituting eqs. (3) and (A.2) into eq. (2) and collecting the coefficients of

$\sin(r\omega t + s\omega_0 t)$ and $\cos(r\omega t + s\omega_0 t)$ separately, we obtain eqs. (6) and (7) with constraints eqs. (9) and (10). Equation (8) was obtained by requiring $r = s = 0$.

-
- 1) Recent works can be found in *Macroscopic Quantum Phenomena and Coherence in Superconducting Networks* (World Scientific, Singapore, 1995); *Proc. 2nd European Conf. on Applied Superconductivity (EUCAS), 1995* (IOP, Bristol UK, 1995). Other recent related works are A. Kenfack and T. C. Kofane: Phys. Rev. B **52** (1995) 10359; C. B. Whan and C. J. Lobb: Phys. Rev. B **53** (1996) 405 and L.-M. Kuang, Y. Wang and M.-L. Ge: Phys. Rev. B **53** (1996) 11764.
 - 2) S. Shapiro: Phys. Rev. Lett. **11** (1963) 80.
 - 3) For a review, see A. Barone and G. Paterno: *Physics and Applications of the Josephson Effect* (John Wiley and Sons, New York, 1982).
 - 4) D. E. McCumber: J. Appl. Phys. **39** (1968) 2503; *ibid.* 3113; W. C. Stewart: Appl. Phys. Lett. **12** (1968) 277.
 - 5) V. N. Belykh, N. F. Pedersen and O. H. Soerensen: Phys. Rev. B **16** (1977) 4853; *ibid.* 4860.
 - 6) One of the early important analytical works is A. Lee Patrick: J. Appl. Phys. **42** (1971) 325.
 - 7) V. A. Pliss: *Nonlocal Problems of the Theory of Oscillations* (Academic, New York, 1966).
 - 8) L.-E. Hasselberg, M. T. Levinsen and M. R. Samuelsen: Phys. Rev. B **9** (1974) 3757; J. Low Temp. Phys. **21** (1975) 567.
 - 9) C. D. Hu: Phys. Rev. B **54** (1996) 10065.
 - 10) P. M. Morse and H. Feshbach: *Methods of Theoretical Physics* (McGraw-Hill, New York, 1978) Chap. 5.
-