Radiatively induced Chern-Simons terms at finite temperature

Yeong-Chuan Kao

Department of Physics, National Taiwan University, Taipei, Taiwan

Min-Fong Yang

Department of Physics, National Tsing Hua University, Hsinchu, Taiwan

(Received 28 August 1992)

We study the one-loop contribution to the induced parity-odd part $\varepsilon_{\mu\nu\lambda} p_\lambda \pi^{(0)}(p)$ of the gauge field self-energy in $(2+1)$ -dimensional gauge theories at a finite temperature T . At $T=0$, $\pi^{(0)}(p_0=0, \mathbf{p}=0)$ is the coefficient of the induced local Chern-Simons term. It is pointed out that $\pi^{(0)}(p)$ is not an analytic function at zero frequency and momentum when $T \neq 0$. Therefore, at $T \neq 0$, there exist no induced local Chern-Simons terms other than those due to regularization.

PACS number(s): 11.15.Bt

In recent years, $(2+1)$ -dimensional field theories with a Chern-Simons term [1] (Abelian or non-Abelian) have received much attention because of the many wonders the Chern-Simons terms can perform [2]. One of the most interesting features of the Chern-Simons term is that it can be generated dynamically after the Dirac fermion field is integrated out in gauge theories with a parity-odd fermion mass term [3]. In particular, there are many studies [4] on the situation at finite temperature and density partly because of its potential application to two-dimensional condensed matter systems. It has been claimed that, at finite temperature, a local Chern-Simons term with a temperature-dependent coefficient is dynamically induced [4]. The conclusion is based upon the calculation of the one-loop contribution to the parity-odd part $-\varepsilon_{\mu\nu\lambda} p_\lambda \pi^{(0)}(p)$ of the gauge field self-energy $\pi_{\mu\nu}(p)$. Usually $\pi^{(0)}(p)$ in the limit $p_0=0, \mathbf{p} \rightarrow 0$ (the spatial components of p_μ go to zero after p_0 is put to zero) is evaluated and interpreted as the coefficient of the induced local Chern-Simons term. Most of the studies in the literature, however, have failed to mention that $\pi^{(0)}(p_0=0, \mathbf{p} \rightarrow 0)$ is not equal to $\pi^{(0)}(p_0 \rightarrow 0, \mathbf{p}=0)$ [$\pi^{(0)}(p)$ in the limit that $\mathbf{p}$ is set to zero before p_0 goes to zero]. Namely, $\pi^{(0)}(p)$ is not an analytic function at $p=(0,0)$ [5]. It is certainly well known [6] in the study of the finite-temperature QED and QCD that the parity-even part of the gauge field self-energy contains nonanalytic functions at zero frequency and momentum (one consequence is that the electric mass and the magnetic mass for the gauge fields are not the same). In this work, we show that the same is true for $\pi^{(0)}(p)$ and discuss the implications with regard to the quantization condition [7] on the coefficient of the non-Abelian Chern-Simons term and the higher-order corrections to $\pi^{(0)}(p)$.

Although a knowledge of $\pi_{\mu\nu}(p)$ will give us the complete nonlocal quadratic (in gauge fields) term in the effective action, in practice, one usually Taylor expands $\pi_{\mu\nu}(p)$ in p_μ around $p=(0,0)$ (higher-degree terms give higher-derivative terms) and obtains the effective action as a sum of local terms, as it is easier to work with local terms in extracting physical results and checking consistency. At zero temperature, when $\pi^{(0)}(p)$ is an analytic

function, $\pi^{(0)}(0,0)$ is the coefficient of the induced local Chern-Simons term. We shall show below that $\pi^{(0)}(p_0=0, \mathbf{p} \rightarrow 0)$ is not equal to $\pi^{(0)}(p_0 \rightarrow 0, \mathbf{p}=0)$ at finite temperature as mentioned above. This nonanalyticity of $\pi^{(0)}(p)$ at $p=(0,0)$ indicates that we are not allowed to expand the parity-odd part of the perturbative effective action as a series of local terms; that is, we do not have a *bona fide* induced local Chern-Simons term. (Of course, regularization may still bring us a local Chern-Simons term with a temperature-independent coefficient. Such local counterterms are always possible.) In the non-Abelian case, the Chern-Simons term by itself is not invariant under large (topologically nontrivial) gauge transformations [7]. The coefficient of the Chern-Simons term must be quantized (it should be an integer in appropriate units) if the theory is to make sense unless the effective action contains other nonlocal terms that are able to compensate for the gauge noninvariance [8]. At finite temperature, irrespective of the existence of any compensating nonlocal (possibly nonperturbative) terms, neither $\pi^{(0)}(p_0=0, \mathbf{p} \rightarrow 0)$ nor $\pi^{(0)}(p_0 \rightarrow 0, \mathbf{p}=0)$ has to obey the quantization condition (they do not) because they are not the coefficient of a local Chern-Simons term [9]. How the topological constraints are reflected in the perturbative effective action is an interesting issue. Having made clear some subtle points unemphasized before, we shall present the main steps of the computations below.

Let us first consider the $(2+1)$ -dimensional QED at finite temperature and density defined by the following action in the imaginary-time (Euclidean) formulation [6]:

$$S = \int_0^\beta d\tau \int d^2x \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} (i \not{D} - e \not{A} - m + i \mu \gamma_0) \psi \right], \quad (1)$$

where $\beta=1/kT$, the inverse temperature, μ is the chemical potential, and the Dirac matrices γ_μ are 2×2 matrices obeying $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$. The fermion mass m is a parity-odd quantity. The standard imaginary-time Feynman rules [6] give us the one-fermion-loop contribution to the photon self-energy:

$$\pi_{\mu\nu}(p) = -\frac{e^2}{\beta} \sum_n \int \frac{d^2k}{(2\pi)^2} \frac{\text{Tr}[\gamma_\mu(\not{k}-m)\gamma_\nu(\not{k}-\not{p}-m)]}{(k^2+m^2)[(k-p)^2+m^2]}, \quad (2)$$

where $p=(p_0=2\pi l/\beta, \mathbf{p})$, $k_0=[(2n+1)\pi/\beta]-i\mu$, l and n being integers. Because of current conservation (the Ward identity) and rotational invariance [6], we write $\pi_{\mu\nu}(p)$ as

$$\pi_{\mu\nu} = (\delta_{\mu\nu}p^2 - p_\mu p_\nu)\pi^{(1)}(p) + \delta_{\mu i}\delta_{\nu j}(\delta_{ij}p^2 - p_i p_j)\pi^{(2)}(p) - \varepsilon_{\mu\nu\lambda}p_\lambda \pi^{(0)}(p), \quad (3)$$

where $i, j=1, 2$ refer to the spatial directions. The existence of the parity-odd part $-\varepsilon_{\mu\nu\lambda}p_\lambda \pi^{(0)}(p)$ is due to

$m \neq 0$ and $\text{tr}(\gamma_\mu \gamma_\nu \gamma_\lambda) = 2\varepsilon_{\mu\nu\lambda}$, and $\pi^{(0)}(p)$ is obtained as follows:

$$\begin{aligned} \pi^{(0)}(p) &= \frac{\varepsilon_{\mu\nu\lambda}p_\lambda}{-2p^2} \pi_{\mu\nu}(p) \\ &= 2me^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{\beta} \\ &\quad \times \sum_n \frac{1}{(k_0^2 + E_k^2)[(k_0 - p_0)^2 + E_{k-p}^2]}, \end{aligned} \quad (4)$$

where $E_k^2 = \mathbf{k}^2 + m^2$. The summation over n can be done by the contour-integral method and we have

$$\begin{aligned} \pi^{(0)}(p) &= me^2 \int \frac{d^2k}{(2\pi)^2} \left[\frac{1}{E_k[E_{k-p}^2 - (E_k + ip_0)^2]} \frac{1}{e^{-\beta(E_k + \mu)} + 1} - \frac{1}{E_k[E_{k-p}^2 - (E_k - ip_0)^2]} \frac{1}{e^{\beta(E_k - \mu)} + 1} \right. \\ &\quad + \frac{1}{E_{k-p}[E_k^2 - (E_{k-p} - ip_0)^2]} \frac{1}{e^{-\beta(E_{k-p} + \mu)} + 1} - \frac{1}{E_{k-p}[E_k^2 - (E_{k-p} + ip_0)^2]} \frac{1}{e^{\beta(E_{k-p} - \mu)} + 1} \left. \right]. \end{aligned} \quad (5)$$

When $T=0$, $\pi^{(0)}(0, \mathbf{0})$ is the coefficient of the induced Abelian Chern-Simons term

$$\frac{i}{2} \int d^3x \varepsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda(x). \quad (6)$$

Calculating $\pi^{(0)}(p)$ in the limit $p_0=0$, $\mathbf{p} \rightarrow \mathbf{0}$ is quite straightforward. On the other hand, one must be more careful in approaching the limit $\mathbf{p}=\mathbf{0}$, $p_0 \rightarrow 0$, as p_0 takes discrete Matsubara frequency at $T \neq 0$. It is essential to keep the discreteness of p_0 in all the intermediate steps of the calculation. For example, $e^{i\beta p_0}$ is set equal to $1 (=e^{2\pi i l})$ in Eq. (5). Failing to do so would lead us to wrong answers. To describe particles with real frequency ω and make contact with the real-time formalism, we analytically extend our result from discrete imaginary $\omega (=ip_0=2\pi i l/\beta)$ to real ω . As Baym and Mermin [10] have shown, it is necessary to take $e^{i\beta p_0}=1$ before letting ip_0 be a real continuous variable ω . Thus, $\pi^{(0)}(p)$ [Eq. (4)], after analytic continuation and rearrangement, becomes

$$\pi^{(0)}(\omega, \mathbf{p}) = \frac{me^2}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{E_k E_{k-p}} \left[\frac{1-f_k^+ - f_{k-p}^-}{\omega + E_k + E_{k-p}} + \frac{-1+f_k^- + f_{k-p}^+}{\omega - E_k - E_{k-p}} + \frac{f_{k-p}^- - f_k^-}{\omega - E_k + E_{k-p}} + \frac{f_k^+ - f_{k-p}^+}{\omega + E_k - E_{k-p}} \right], \quad (7)$$

where $f_k^+ = 1/(e^{\beta(E_k + \mu)} + 1)$ and $f_k^- = 1/(e^{\beta(E_k - \mu)} + 1)$ are the Fermi-Dirac distribution functions of an antiparticle and particle, respectively. Therefore,

$$\begin{aligned} \pi^{(0)}(\omega=0, \mathbf{p} \rightarrow \mathbf{0}) &= \lim_{\mathbf{p} \rightarrow \mathbf{0}} \frac{me^2}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{E_k E_{k-p}} \left[\frac{1-f_k^+ - f_{k-p}^-}{E_k + E_{k-p}} + \frac{-1+f_k^- + f_{k-p}^+}{-E_k - E_{k-p}} + \frac{f_{k-p}^- - f_k^-}{-E_k + E_{k-p}} + \frac{f_k^+ - f_{k-p}^+}{+E_k - E_{k-p}} \right] \\ &= \frac{me^2}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{E_k^2} \left[\frac{1}{E_k} (1-f_k^+ - f_k^-) + \frac{\partial f_k^+}{\partial E_k} + \frac{\partial f_k^-}{\partial E_k} \right] \\ &= \frac{me^2}{4\pi} \int_{|m|}^{\infty} dE_k \frac{d}{dE_k} \left[\frac{1}{E_k} (f_k^+ + f_k^- - 1) \right] \\ &= \frac{e^2}{8\pi} \frac{m}{|m|} \left[\tanh \left[\frac{\beta}{2} (|m| + \mu) \right] + \tanh \left[\frac{\beta}{2} (|m| - \mu) \right] \right], \end{aligned} \quad (8)$$

while, for the reversed order of limits,

$$\begin{aligned} \pi^{(0)}(\omega \rightarrow 0, \mathbf{p}=\mathbf{0}) &= \lim_{\omega \rightarrow 0} \frac{me^2}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{E_k^2} \left[\frac{1-f_k^+ - f_k^-}{\omega + 2E_k} + \frac{-1+f_k^- + f_k^+}{\omega - 2E_k} \right] \\ &= \frac{me^2}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{E_k^3} (1-f_k^+ - f_k^-), \end{aligned} \quad (9)$$

which is not equal to Eq. (8). Equations (8) and (9) are the basis of our claim explained above. Unfortunately we have not yet obtained an analytic expression for the integral in Eq. (9). To illustrate the difference, we give a numerical plot of Eqs. (8) and (9) in Fig. 1 with μ chosen to be $0.2|m|$. Both $\pi^{(0)}(p_0=0, \mathbf{p} \rightarrow 0)$ and $\pi^{(0)}(p_0 \rightarrow 0, \mathbf{p}=0)$ approach 0 at infinite temperature. They also approach $(e^2/4\pi)m/|m|$, the correct zero-temperature result, when $|m|\beta \rightarrow \infty$. This should be the case because both Eqs. (8) and (9) stem from Eq. (4) where one can easily take β to be infinite and recover the continuum limit. The results are still the same if we proceed from Eq. (4) and allow p_0 to approach 0 discretely. Our Eq. (8) is in agreement with the result found by Poppitz and Burgess [4] although they do not have our Eq. (9). One should not be too surprised by the nonanalyticity of $\pi^{(0)}(p)$ at zero frequency and momentum because of the zero-energy thermal pair (particle excitation-hole pair) elucidated by Matsumoto *et al.* [11]. A couple of remarks are in order here. First, there may be more contributions to $\pi^{(0)}(p_0=0, \mathbf{p} \rightarrow 0)$ and $\pi^{(0)}(p_0 \rightarrow 0, \mathbf{p}=0)$ due to regularization. For example, using the Pauli-Villars regularization scheme, we have to add $(e^2/4\pi)(M/|M|)$, [the common value of Eqs. (8) and (9) with m replaced by M and $|M| \rightarrow \infty$], where M is the regulator mass, to both $\pi^{(0)}(p_0=0, \mathbf{p} \rightarrow 0)$ and $\pi^{(0)}(p_0 \rightarrow 0, \mathbf{p}=0)$. Then the effective action will contain a local Chern-Simons term with a temperature-independent coefficient. We may not have such local terms in other (e.g., dimensional) regularization schemes [3]. A good discussion of the merits of various regularization schemes can be found in the paper by Niemi *et al.* [8]. Second, the nonanalyticities at $T \neq 0$ are still mysterious enough that there are efforts [12] to create new rules (e.g., avoiding setting $e^{i\beta p_0}=1$) to preserve analyticity. It has not been shown that these new rules are consistent ones.

It is easy to extend the above Abelian results to the non-Abelian case, when the parity-odd part of the gluon self-energy, $\pi_{\mu\nu}^{ab}$, is $-\epsilon_{\mu\nu\lambda} p_\lambda \delta^{ab} \pi^{(0)}(p)$, where a and b are color indices. We only have to attach the appropriate group factors to Eqs. (8) and (9). For example, for three-dimensional SU(2) QCD with fermions in the fundamental representation, the one-loop contribution to $\pi^{(0)}(p_0=0, \mathbf{p} \rightarrow 0)$ [$\pi^{(0)}(p_0 \rightarrow 0, \mathbf{p}=0)$] will be Eq. (8) [Eq. (9)] multiplied by $\frac{1}{2}$ (the one-flavor case) [3]. Hence, the temperature dependence and nonanalyticity property are exactly the same as in the Abelian case.

One may be curious about the higher-order corrections

FIG. 1. A plot of $\pi^{(0)}(p_0 \rightarrow 0, \mathbf{p}=0)$ (curve a) and $\pi^{(0)}(p_0=0, \mathbf{p} \rightarrow 0)$ (curve b) vs $\beta|m|$.

to $\pi^{(0)}(p_0=0, \mathbf{p} \rightarrow 0)$ and $\pi^{(0)}(p_0 \rightarrow 0, \mathbf{p}=0)$. At zero temperature, it has been proved by Coleman and Hill [13] that there are no more corrections to $\pi^{(0)}$ in the Abelian case [14]. This is another very interesting feature of the Chern-Simons term with many applications [15]. The Coleman-Hill proof makes use of the fact that the one-loop n -point Green's functions are analytic functions when the fermions are massive. The nonanalyticity shown in this work gives us a reason to doubt the validity of the nonrenormalization theorem at finite temperature—there may be higher-order corrections to $\pi^{(0)}(p_0=0, \mathbf{p} \rightarrow 0)$ and $\pi^{(0)}(p_0 \rightarrow 0, \mathbf{p}=0)$. The claim in Ref. [15] that the nonrenormalization theorem also holds at finite temperature is, in our opinion, premature. A better knowledge of the full (non-)analytic structure of the Green's functions at $T \neq 0$ is needed to determine whether there are more corrections. To summarize, with Eqs. (8) and (9), we have amended some previous studies on the parity-odd piece of the perturbative effective action. The implications with regard to quantization condition in the non-Abelian case and higher-order corrections are pointed out. We have only touched upon some of the very interesting issues that deserve to be further investigated.

One of us (M.-F. Yang) would like to thank E. Braaten for helpful correspondence. The work was supported by NSC Grant No. 81-0208-MM-002-62.

- [1] W. Siegel, Nucl. Phys. **B156**, 135 (1979); J. F. Schonfeld, *ibid.* **B185**, 157 (1981); R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981).
- [2] D. Arovas, J. R. Schrieffer, F. Wilczek, and A. Zee, Nucl. Phys. **B251**, 117 (1985); A. M. Polyakov, Mod. Phys. Lett. A **3**, 325 (1988); E. Witten, Commun. Math. Phys. **121**, 351 (1989); M. Luscher, Nucl. Phys. **B326**, 557 (1989).
- [3] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982); I. Affleck, J. Harvey, and E. Witten, Nucl. Phys. **B206**, 413 (1982); A. N. Redlich, Phys. Rev. D **29**,

2366 (1984).

- [4] A. Niemi, Nucl. Phys. **B251**, 155 (1985); K. Ishikawa and T. Matsuyama, *ibid.* **B280**, 523 (1987); R. D. Pisarski, Phys. Rev. D **35**, 664 (1987); K. S. Babu, A. Das, and P. Panigrahi, *ibid.* **36**, 3725 (1987); E. R. Poppitz, Phys. Lett. B **252**, 417 (1990); M. Burgess, Phys. Rev. D **44**, 2552 (1991).
- [5] In the context of anyon superconductivity, we have found a similar situation. See Y.-C. Kao and M.-F. Yang, Mod. Phys. Lett. B **4**, 1023 (1990).

- [6] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, *Rev. Mod. Phys.* **53**, 43 (1981); J. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989).
- [7] S. Deser, R. Jackiw, and S. Templeton, *Ann. Phys. (N.Y.)* **140**, 372 (1982).
- [8] L. Alvarez-Gaume, S. Della Pietra, and G. Moore, *Ann. Phys. (N.Y.)* **163**, 288 (1985); A. Niemi, G. W. Semenoff, and Y.-S. Wu, *Nucl. Phys.* **B276**, 173 (1986).
- [9] This is not to say that the perturbative effective action has been shown to be compatible with large gauge transformations.
- [10] G. Baym and N. D. Mermin, *J. Math. Phys.* **2**, 232 (1961).
- [11] H. Matsumoto, H. Umezawa, N. Yamamoto, and N. J. Papastamatiou, *Phys. Rev. D* **34**, 3217 (1986).
- [12] P. S. Gribosky and B. R. Holstein, *Z. Phys. C* **47**, 205 (1990), and references therein.
- [13] S. Coleman and B. Hill, *Phys. Lett.* **159B**, 184 (1985).
- [14] The non-Abelian case is subtler and unsolved. It is a challenge to understand the results found in some theories in which $\pi^{(0)}(0)$ is an off-shell quantity. See, for example, R. D. Pisarski and S. Rao, *Phys. Rev. D* **32**, 2081 (1985); Y.-C. Kao, J. Koller, and H. Yamagishi, *Phys. Rev. Lett.* **58**, 1077 (1987); Y.-C. Kao and M.-F. Yang, *Phys. Rev. D* **41**, 3267 (1990).
- [15] G. W. Semenoff, P. Sodano, and Y.-S. Wu, *Phys. Rev. Lett.* **62**, 715 (1989).