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Nuclear Physics B 590 (2000) 198–212

www.elsevier.nl/locate/npe

Large N expansion from fuzzy AdS_2

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Received 26 June 2000; accepted 21 August 2000

Abstract

We study the quantum analogue of primary fields and their descendants on fuzzy AdS_2 . Three-point vertices are calculated and shown to exhibit the conventional $1/N$ expansion as well as nonperturbative effects in large N , thus providing a strong consistency check of the fuzzy AdS_2 model. A few new physical motivations for this model are also presented. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 11.25.Mj; 11.25.Sq

1. Introduction

Quantum mechanics and Einstein's general relativity are two major achievements in physics in twentieth century. In the last thirty years, string theory has become the most promising theory unifying these two otherwise highly irreconcilable theories. Nevertheless in string theory, until quite recently, spacetime has been treated as a continuous platform upon which one builds up a perturbation theory. It was suspected by many that when gravitational interactions become strong, a continuous spacetime is doomed, and one has to replace it by something else. In other words, when nonperturbative effects are properly taken into account, spacetime in string theory will become fundamentally fuzzy.

A concrete model in which space and time are quantized was proposed very recently [1]. The classical counter-part is the two-dimensional anti-de Sitter space which one can obtain as the near horizon limit of an extremal black hole in 4 dimensions. Quantization of spacetime has a long history, it was first proposed in [2–4]. The motivation at that time is to cure the UV divergences in QED, and this is rendered unnecessary by the modern quantum field theory. However, very general arguments show that spacetime are fundamentally

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noncommutative in string theory [5–8]. And that some unusual physics required to explain quantum black holes also indicates that space and time are quantized simultaneously [9,10]. One piece of such unusual physics is teleology which was beautifully demonstrated in a recent paper [11] in one of the simplest models of noncommutative spacetime. Another manifestation of noncommutativity of spacetime is the UV/IR relation in the AdS/CFT correspondence [12,13], indeed one of our motivations to propose the fuzzy AdS_2 model is to derive holography from spacetime noncommutativity.

Note that quantum spheres in the AdS/CFT correspondence were first proposed in [14,15] to explain the stringy exclusion principle [16]. A remarkable physical mechanism for this phenomenon is proposed in [17], and subsequently it was shown in [18] that this mechanism is compatible with the stringy spacetime uncertainty relation, providing another indication of quantization of space and time. For even dimensional spheres, it is shown in [1] that the quantum geometry of the sphere is really a fuzzy sphere. In particular, for S^4 , the model already made its appearance in [19,20]. The fuzzy S^2 is determined using the dipole mechanism and the fuzzy AdS_2 is proposed based on analogy.

The fuzzy AdS_2 model resembles much the noncommutative horizon of a 4-dimensional Schwarzschild black hole proposed by 't Hooft [9,10]. Its study and clarification of many physics issues in this context will undoubtedly shed much light on the quantum physics of black holes which have been source of riddles as well as inspirations.

We shall make a first step in studying physics in the fuzzy AdS_2 in this paper. The uncertainty relation results from this model is unusual in that it is much larger than the one showing up for instance in [9,10]. This is not surprising to us, since this model is a descendant of the 4-dimensional extremal black hole whose quantum properties are unusual too. The most convincing check that our model is correct is the right pattern of $1/N$ expansion for the effective interaction vertices. We will in the next section review some discussion of [21,22] on the solution to the Laplace equation in AdS_2 , and in Section 3 review our definition of fuzzy AdS_2 . We fix the geometric data of the fuzzy AdS_2 in Section 4 and solve the quantum analogue of the Laplace equation. Section 5 computes the three-point vertices, and demonstrates $1/N$ expansion in these vertices. Remarkably, there are also instanton effects such as terms $\exp(-2\pi nN)$, similar to [23]. In the discussion section we present a shock-wave argument for fuzzy AdS_2 .

We leave many interesting questions, such as how to understand holography, to future work.

2. AdS_2/CFT_1 duality

In this section we review what has been discussed about the AdS_2/CFT_1 duality [21,22].

For a gravitational theory on AdS_2 , the isometry group $SL(2, \mathbf{R})$ is enhanced to half of the $(1+1)$ -dimensional conformal group. We denote the Killing vectors for the isometry group by L_0, L_1, L_{-1} .

For a scalar field Φ of mass M propagating in the bulk of AdS_2 , its wave equation is

$$\nabla^2 \Phi = M^2 \Phi. \quad (1)$$

The Laplacian ∇^2 is just $4/R^2$ times the Casimir of $SL(2, \mathbf{R})$. To get a primary solution of the wave equation, we impose the constraints

$$L_1 \Phi_h = 0, \quad L_0 \Phi_h = h \Phi_h \quad (2)$$

on Φ . Then the solution is easily found to be

$$\Phi_h = (e^{-2iu^+} - e^{-2iu^-})^h, \quad (3)$$

where we used the global coordinates

$$ds^2 = -\frac{4R^2}{\sin^2(u^+ - u^-)} du^+ du^-. \quad (4)$$

This solution is normalizable and has

$$M^2 = \frac{4h(h-1)}{R^2}. \quad (5)$$

The primary solution in the bulk corresponds to a primary operator on the boundary CFT.

Other solutions of the wave equation with which Φ_h forms an irreducible representation of $SL(2, \mathbf{R})$ can be obtained by acting L_{-1} on Φ_h . Denote the normalized solutions obtained this way by $\tilde{\Psi}_{hn}$, where n is the number of times L_{-1} acts on Φ_h . This is a highest weight representation. Its complex conjugation gives a lowest weight representation. All together they form a complete basis with which one can expand a scalar field of mass $M = 2\sqrt{h(h-1)}/R$ as

$$\Phi = \sum_{n=0}^{\infty} a_n^\dagger \tilde{\Psi}_{hn}^\dagger + a_n \tilde{\Psi}_{hn}. \quad (6)$$

The canonical commutation relation is

$$[a_m, a_n^\dagger] = \delta_{mn}. \quad (7)$$

As h is roughly proportional to the mass of a scalar field, it was shown that there should exist a cutoff of h around the value

$$h \sim \frac{1}{G_N}, \quad (8)$$

where G_N is the Newton constant in AdS_2 .

3. Fuzzy $AdS_2 \times S^2$

Consider the near horizon limit of a 4-dimensional charged black hole in string theory [25]. For instance, by wrapping two sets of membranes and two sets of M5-branes in T^7 , one obtains a 4D charged, extremal black hole [26]. The brane configuration is as follows. Denote the coordinates of T^7 by x_i , $i = 1, \dots, 7$. A set of membranes is wrapped on (x_1, x_2) , another set is wrapped on (x_3, x_4) . A set of M5-branes is wrapped on $(x_1, x_3, x_5, x_6, x_7)$, the second set is wrapped on $(x_2, x_4, x_5, x_6, x_7)$. By setting all charges to be N , one finds the metric of $AdS_2 \times S^2$ for (x_0, x_8, x_9, x_{10}) :

$$ds^2 = l_p^2 \left(-\frac{r^2}{N^2} dt^2 + \frac{N^2}{r^2} dr^2 + N^2 d\Omega_2 \right), \quad (9)$$

$$F = -Nd\Omega_{1+1} - Nd\Omega_2, \quad (10)$$

where l_p is the 4-dimensional Planck length, $d\Omega_{1+1}$ and $d\Omega_2$ are the volume forms on AdS_2 and S^2 , respectively. The field F is just the linear combination of all anti-symmetric tensor fields involved. Note that here for simplicity, we consider the most symmetric case in which all the charges appearing in the harmonics $1 + Q_i l_p / r$ are just N which in turn is equal to the number of corresponding branes used to generate this potential. As a consequence, the tension of the branes compensates the volume of the complementary torus. This means that the size of each circle of T^7 is at the scale of the M-theory Planck length.

The same space $AdS_2 \times S^2$ can also be obtained by taking the near horizon limit of the 4-dimensional extremal Reissner–Nordström solution.

3.1. Fuzzy S^2

In [1] we proposed that the S^2 part of the $AdS_2 \times S^2$ space is a fuzzy S^2 [27,28] defined by

$$[Y^a, Y^b] = i\epsilon^{abc} Y^c, \quad (11)$$

where Y_a 's are the Cartesian coordinates of S^2 . (We use the unit system in which $l_p = 1$.) This algebra respects the classical $SO(3)$ invariance.

The commutation relations (11) are the same as the $SU(2)$ Lie algebra. For the $(2N + 1)$ -dimensional irreducible representation of $SU(2)$, the spectrum of Y_a is $\{-N, -(N - 1), \dots, (N - 1), N\}$, and its second Casimir is

$$\sum_{a=1}^3 (Y_a)^2 = N(N + 1). \quad (12)$$

Since the radius of the S^2 is Nl_p (in the leading power of N), we should realize the Y_a 's as $N \times N$ matrices on this irreducible representation of $SU(2)$.

An evidence for this proposal is the following. For a fractional membrane wrapped on (x_1, x_3) or (x_2, x_4) , it is charged under the F field generated by a set of M5-branes. Denote the polar and azimuthal angles of S^2 by (θ, ϕ) . The stable trajectories of the membrane with angular momentum M are discrete, and

$$\cos \theta = \frac{M}{N}. \quad (13)$$

Furthermore, they all have the same energy of $1/N$. It follows that, since M is conjugate to ϕ , $\cos \theta$ and ϕ do not commute with each other in the quantized theory. The resulting Poisson structure on S^2 is precisely that of the fuzzy sphere.

In [24], it was proposed that in $2 + 1$ dimensions the spacetime coordinates are quantized according to

$$[x, y] = \frac{i}{\cos^2 \mu} L, \quad (14)$$

where $\cos \mu$ is related to the mass of the particle, and L is the angular momentum on the 2-dimensional space. To complete the algebra we also have

$$[L, x] = iy, \quad [L, y] = -ix. \quad (15)$$

This algebra is Lorentz invariant, and its $(3 + 1)$ -dimensional generalization was given by Yang [4] much earlier. Note that this algebra (14) was proposed based on general grounds for a gravitational theory in $2 + 1$ dimensions, and we are content with the fact that it is actually a consequence of the fuzzy S^2 (11) for massless particles ($\cos \mu = 1$), where Y_3 acts on Y_1 and Y_2 as the angular momentum operator.

3.2. Fuzzy AdS_2

In [1] we further proposed that the AdS_2 part is also quantized. Let X^{-1} , X^0 , X^1 be the Cartesian coordinates of AdS_2 . The algebra of fuzzy AdS_2 is

$$[X^{-1}, X^0] = -iX^1, \quad (16)$$

$$[X^0, X^1] = iX^{-1}, \quad (17)$$

$$[X^1, X^{-1}] = iX^0, \quad (18)$$

which is obtained from the fuzzy S^2 by a “Wick rotation” of the time directions X^0 , X^{-1} . The “radius” of AdS_2 is $R = Nl_p$, so

$$\eta_{\mu\nu} X^\mu X^\nu = (X^{-1})^2 + (X^0)^2 - (X^1)^2 = R^2, \quad (19)$$

where $\eta = \text{diag}(1, 1, -1)$. The isometry group $SL(2, \mathbf{R})$ of the classical AdS_2 is a symmetry of this algebra, and thus is also the isometry group of the fuzzy AdS_2 .

For later use, we define the raising and lowering operators

$$X_\pm \equiv X^{-1} \pm iX^0, \quad (20)$$

which satisfy

$$[X^1, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = -2X^1, \quad (21)$$

according to (16)–(18).

The radial coordinate r and the boundary time coordinate t are defined in terms of the X ’s as

$$r = X^{-1} + X^1, \quad t = \frac{R}{2}(r^{-1}X^0 + X^0r^{-1}), \quad (22)$$

where we symmetrized the products of r^{-1} and X^0 so that t is a Hermitian operator. The metric in terms of these coordinates assumes the form (9). It follows that the commutation relation for r and t is

$$[r, t] = -iRl_p. \quad (23)$$

The following simple heuristic argument also suggests this commutation relation. Consider a closed string in AdS_2 . (Since the space is one-dimensional, the closed string actually looks like an open string with twice the tension.) Take the Nambu–Goto action

for a fractional string of tension $1/N$ and take the static gauge $t = p_0\tau$. It follows that the action is

$$\begin{aligned} S &= \frac{1}{2\pi N\alpha'} \int_{-\infty}^{\infty} dt \int_0^{2\pi} d\sigma \sqrt{(p_0\dot{r})^2} \\ &= \frac{1}{\pi N\alpha'} \int dt \int_0^{\pi} p_0 |\dot{r}| \\ &= \frac{1}{N\alpha'} \int d\tau \dot{r} \frac{\partial t}{\partial \tau}, \end{aligned} \quad (24)$$

where we have assumed that $\dot{r} > 0$ for $0 < \sigma < \pi$ and $\dot{r} < 0$ for $\pi < \sigma < 2\pi$. The last line above shows clearly that r is the conjugate variable to t up to a factor of $N\alpha' = Rl_p$, and so the quantization of this fundamental string is (23). We will set $l_p = 1$ in the rest of the paper.

4. Properties of fuzzy AdS_2

One can realize the algebra (16)–(18), which is the same as the Lie algebra of $SL(2, \mathbf{R})$, on a unitary irreducible representation of $SL(2, \mathbf{R})$. The question is which representation is the correct choice. As we show in Appendix A, since the range of X_1 goes from $-\infty$ to ∞ for AdS_2 ; when $R > 1/2$, the proper choice is the principal continuous series, and when $R < 1/2$, it is the complementary series. Since we have $R = N > 1$ for our physical system, we should consider the principal continuous series only. A representation in this series is labeled by two parameters $j = 1/2 + is$ and α , where s, α are real numbers, and $0 \leq \alpha < 1$. The label j determines the second Casimir as

$$c_2 = \eta_{\mu\nu} X^\mu X^\nu. \quad (25)$$

It follows from (19) and $R = N$ that one should take

$$j = \frac{1}{2} + iN. \quad (26)$$

However, strictly speaking, we can only be sure of these relations in the leading power of N .

What is the physical interpretation of the other label α of these representations? We will argue in Section 5 that α corresponds to the VEV of axion when viewing this system as IIB string theory via duality. However, for most of the discussions below, we will focus on the representation with $(j = 1/2 + iN, \alpha = 0)$, to be denoted by $\mathcal{D}_N$. This is the case for which the reflection symmetry $X^1 \rightarrow -X^1$ is not broken.

Functions on the fuzzy AdS_2 are functions of the X 's. They form representations of the isometry group $SL(2, \mathbf{R})$. The three generators $L_{\mu\nu}$ of the isometry group act on X as

$$[L_{\mu\nu}, X^\kappa] = i(\delta_\nu^\kappa X_\mu - \delta_\mu^\kappa X_\nu), \quad (27)$$

where $X_\mu = \eta_{\mu\nu} X^\nu$. A very interesting property of the algebra of fuzzy AdS_2 is that the action of the generators $L_{\mu\nu}$ is the same as the adjoint action of the X_κ . That is,

$$[L_{\mu\nu}, f(X)] = \epsilon_{\mu\nu\kappa} [X^\kappa, f(X)], \quad (28)$$

for an arbitrary function $f(X)$. The operators $L_{\mu\nu}$ act on the functions as differential operators.

The integration over the fuzzy AdS_2 is just the trace over the representation $\mathcal{D}_N$

$$\int f(X) \equiv c \text{Tr}(f(X)) = c \sum_{n \in \mathbf{Z}} \langle n | f(X) | n \rangle, \quad (29)$$

where c is a real number. This integration is invariant under $SL(2, \mathbf{R})$ transformations. In the large N limit, $c_2 \gg 1$, the trace can be calculated and its comparison with an ordinary integration on the classical AdS_2 with metric (9) shows that

$$c = 2\pi N, \quad (30)$$

in the leading power of N . The inner product of two functions, as well as the norm of a function, are defined by integration over the fuzzy AdS_2 in the usual way:

$$\langle f(X) | g(X) \rangle = \int f^\dagger(X) g(X), \quad \|f(X)\|^2 = \int f(X)^\dagger f(X). \quad (31)$$

In view of organizing the functions into representations of $SL(2, \mathbf{R})$, in order to describe the boundary CFT dual to the bulk theory on fuzzy AdS_2 via holography, we derived all functions corresponding to the lowest and highest weight states in the principal discrete series. The information about the underlying fuzzy AdS_2 , i.e., the value of N , is encoded in the precise expressions of these functions.

Denote the lowest weight state by Ψ_{jj} , or just Ψ_j . The states of higher weights Ψ_{jm} ($m > j$) in the same irreducible representation are obtained as

$$[X_+, [X_+, \dots [X_+, \Psi_j] \dots]], \quad (32)$$

where X_+ appears $(m - j)$ times. In Appendix B, we find the explicit expressions for the functions Ψ_j as

$$\Psi_j = \left(\frac{1}{X^1(X^1 - 1) + c_2} X_+ \right)^j. \quad (33)$$

In the large N limit, using $c_2 = R^2$ and the following coordinate transformation

$$X^1 = R \cot(u^+ - u^-), \quad X_\pm = \frac{R}{\sin(u^+ - u^-)} e^{\mp i(u^+ + u^-)}, \quad (34)$$

one finds

$$\Psi_j \rightarrow \left(\frac{e^{-2iu^+} - e^{-2iu^-}}{-2iR} \right)^j, \quad (35)$$

where u^+, u^- are the coordinates appearing in (4), in agreement with (3).

Let

$$I_{jm} \equiv \text{Tr}(\Psi_{jm}^\dagger \Psi_{jm}), \quad (36)$$

then the normalized states are

$$\tilde{\Psi}_{jm} \equiv \frac{1}{\sqrt{cI_{jm}}} \Psi_{jm}. \quad (37)$$

As a normalized basis of an $SL(2, \mathbf{R})$ representation, they satisfy

$$[X_+, \tilde{\Psi}_{jm}] = a_{j(m+1)} \tilde{\Psi}_{j(m+1)}, \quad (38)$$

$$[X_-, \tilde{\Psi}_{jm}] = a_{jm} \tilde{\Psi}_{j(m-1)}, \quad (39)$$

where

$$a_{jm} = \sqrt{m(m-1) - j(j-1)}. \quad (40)$$

The explicit expressions of $I_m \equiv I_{mm}$ are given in Appendix C. They will be used later when we try to extract physical information from the fuzziness of the AdS_2 .

For a field Φ in the bulk of the fuzzy AdS_2 , one can decompose it into the basis functions $\tilde{\Psi}_{jm}$ as

$$\Phi(X) = \sum_{jm} \phi_{jm} \tilde{\Psi}_{jm}(X), \quad (41)$$

where ϕ_{jm} are the creation/annihilation operators of the physical state with the wave function $\tilde{\Psi}(X)$. By holography, these states are identified with those in the boundary theory, which is a one-dimensional theory.

An interesting question for a wave function on noncommutative theory is how to define expectation values. For instance, should the expectation value of X_1 for a wave function Ψ be

$$(1) \int \Psi^\dagger X^1 \Psi, \quad (2) \int \Psi X^1 \Psi^\dagger, \quad \text{or} \quad (3) \frac{1}{2} \int (X^1 \Psi^\dagger \Psi + \Psi \Psi^\dagger X^1)? \quad (42)$$

The answer is that it depends on how one measures it. If one measures the X^1 location of the wave function according to its interaction with another field Φ under control in the experiment, and if the interaction is described in the action by a term like

$$\int \Psi^\dagger \Phi \Psi, \quad (43)$$

then we expect that the choice (1) is correct. But if the interaction is written differently, the definition of expectation value should be modified accordingly.

Fixing a definition of the expectation value of X_1 , such as case (1) in (42), one finds that for the same j , the larger m is, which means larger energy at the boundary, the larger is the expectation value of X_1 for the wave function Ψ_{jm} . This can also be viewed as a manifestation of the UV/IR relation in AdS space.

5. Interaction in fuzzy AdS_2

To see how the noncommutativity of the fuzzy AdS_2 incorporate physical data, presumably including the effect of string quantization on AdS_2 , we consider an interaction term in the action of the bulk theory of the form

$$S_I = \lambda \int \Phi_1^\dagger \Phi_2 \Phi_3, \quad (44)$$

where λ is the coupling constant for this three-point interaction.

Expanding all three Φ_i 's as (41) in the action, one obtains the vertex

$$c\lambda \text{Tr}(\tilde{\Psi}_{j_1 m_1}^\dagger \tilde{\Psi}_{j_2 m_2} \tilde{\Psi}_{j_3 m_3}), \quad (45)$$

for the three states $(\phi_1)_{j_1 m_1}$, $(\phi_2)_{j_2 m_2}$ and $(\phi_3)_{j_3 m_3}$. Obviously, due to the isometry, the vertex vanishes unless $m_1 = m_2 + m_3$.

For simplicity, consider the special case where all three states participating the interaction are lowest weight states. Then the vertex (45) in question is $\lambda V_{m_1 m_2 m_3}$, where (for $m_1 = m_2 + m_3$)

$$V_{m_1 m_2 m_3}^2 = \frac{1}{c} \frac{I_{m_1}}{I_{m_2} I_{m_3}}. \quad (46)$$

In Appendix C, we find that

$$I_m = \frac{(2m-2)!}{((m-1)!)^2} \left[\prod_{k=1}^{m-1} \frac{1}{k^2 - 1 + 4c_2} \right] I_1, \quad (47)$$

where

$$I_1 = \frac{\pi}{\sqrt{c_2 - 1/4}} \tanh(\pi \sqrt{c_2 - 1/4}). \quad (48)$$

We therefore have the large N expansion of the vertex (46). Using (26), one finds (for $m_1 = m_2 + m_3$)

$$\begin{aligned} V_{m_1 m_2 m_3}^2 &= \mathcal{N}_{m_2 m_3} \frac{[\prod_{j_2=1}^{m_2-1} (1 + j_2^2/4N^2)][\prod_{j_3=1}^{m_3-1} (1 + j_3^2/4N^2)]}{8\pi^2 N^2 [\prod_{j_1=1}^{m_1-1} (1 + j_1^2/4N^2)]} \\ &\quad \times \left(1 + 2 \sum_{n=1}^{\infty} e^{-2\pi n N} \right), \end{aligned} \quad (49)$$

where

$$\mathcal{N}_{m_2 m_3} = \frac{[2(m_1-1)!][(m_2-1)!]^2[(m_3-1)!]^2}{[(m_1-1)!]^2[2(m_2-1)!][2(m_3-1)!]}. \quad (50)$$

With the possibility of corrections to (26) of order $(1/N)^0 \sim 1$, the $1/N$ expansion of the vertex is of the form

$$V_{m_1 m_2 m_3} \sim \frac{K}{N} \left(1 + \sum_{n=1}^{\infty} \frac{a_n}{N^{2n}} + \sum_{n=1}^{\infty} e^{-2\pi n N} \sum_{k=0}^{\infty} \frac{b_{nk}}{N^{2k}} \right). \quad (51)$$

This expression is reminiscent of a correlation function in the case of type IIB strings on $AdS_5 \times S^5$ [23]. It calls for an analogous interpretation.

The $1/N^2$ expansion in (51) suggests that the coupling constant in AdS_2 is of order $1/N^2$. This is indeed the case. The 11-dimensional Newton constant is just 1 in Planck units. Compactifying on $S^2 \times T^7$ of size $4\pi N^2$ results in a dimensionless effective Newton constant of order $1/N^2$ in AdS_2 .

The overall factor of $1/N$ in (51) is what one expects for a three-point correlation function. The Newton constant is $1/N^2$ in AdS_2 as we just mentioned, on the other hand it is also g_s^2 from the viewpoint of a string theory on AdS_2 . This means that $g_s \sim 1/N$. While the coupling constant λ in the bulk theory should not depend on N since it should not depend on the geometry of the space it lives on, the vertex (45) should have the leading dependence on N of a three-point function for closed string states, which is of order $g_s \sim 1/N$. This is exactly the dependence we find for the vertex (51).

Finally, we identify the terms in (51) proportional to $\exp(-2\pi n N)$ as contributions from instantons. This implies that the action of a single instanton equals $2\pi N$. We have just argued that the string coupling constant g_s is of order $1/N$. Since the D-instanton action is $2\pi/g_s$, it is precisely $2\pi N$ as we wish.

Similarly, we can consider n -point interaction vertex in the bulk theory on AdS_2 :

$$S_n = \lambda \int \Phi_1^\dagger \Phi_2 \cdots \Phi_n. \quad (52)$$

The leading dependence of the vertex will be $1/N^{n-2}$, which is exactly what it should be for an n -point function in string theory with coupling constant $g_s \sim 1/N$.

We conjecture that for M-theory compactified on $AdS_2 \times S^2$, the perturbative as well as nonperturbative effects of string quantization are encoded in the noncommutativity of the fuzzy $AdS_2 \times S^2$, in the sense that the low energy effective theory is most economically written as a field theory on this noncommutative space.

Finally we make a comment on the other label α for a representation in the principal continuous series. We have set $\alpha = 0$ in the above. When α is turned on, the expression of $a_{j(m+\alpha)}$ becomes

$$a_{j(m+\alpha)} = \sqrt{(m - 1/2 + \tau)(m - 1/2 + \bar{\tau})} \quad (53)$$

for $m \in \mathbf{Z}$, where

$$\tau = \alpha + iN \simeq \alpha + \frac{i}{g_s}. \quad (54)$$

The appearance of this combination of α with g_s suggests the identification of α with $\chi/2\pi$, where χ is the VEV of the axion field in IIB theory. While χ is only defined up to 2π , α is also defined only up to an integer.

6. Discussion

Although we already argued for the spacetime uncertainty in the fuzzy AdS_2 from the general stringy uncertainty, it should be interesting to compare the way 't Hooft introduces spacetime noncommutativity in his S-matrix ansatz [9,10,29]. The basic method is to study the effect of a shock-wave on a particle moving in the opposite direction.

Use the global coordinates (4). The metric induced by a left-moving shock-wave assumes the form

$$ds^2 = -e^{2\phi} du^+ du^- + h(du^+)^2, \quad (55)$$

where

$$e^{2\phi} = \frac{4R^2}{\sin^2(u^+ - u^-)}.$$

The scalar curvature is perturbed by a term

$$\frac{1}{2}e^{-2\phi}\partial_-(e^{-2\phi}\partial_-h), \quad (56)$$

and the Einstein equation with a constant negative curvature is solved provided

$$h = \cot(u^+ - u^-)g(u^+) + f(u^+). \quad (57)$$

In the full 4 dimensions, we expect another Einstein equation of the form $G_{++} = 8\pi G_4 T_{++}$, where G_4 is the 4D Newton constant. Now $G_{++} \sim h\mathcal{R}$, $\mathcal{R}$ is the scalar curvature. For a stress tensor T_{++} proportional to $\delta(u^+ - x^+)$, the only solution is $g(u^+) = 0$ and

$$h \sim l_p^2 \delta(u^+ - x^+). \quad (58)$$

The proportionality constant is determined by how the stress tensor is normalized. For a S-wave shock-wave smeared over S^2 , it is p_+/R^2 . However, the dipole mechanism of [1] seems to indicate that a shock-wave must be localized on a strip on S^2 whose area is proportional to Nl_p^2 . If true, we expect

$$\int du^+ T_{++} \sim \frac{p_+}{Rl_p}, \quad (59)$$

and this leads to

$$h \sim Rl_p p_+ \delta(u^+ - x^+). \quad (60)$$

Now the shift on u^- induced on a right-moving particle by the shock-wave is

$$\Delta u^- \sim \frac{p_+}{N} \sin^2(x^+ - u^-), \quad (61)$$

as can be computed using (55). This shift suggests a commutator

$$[u^+, u^-] \sim \frac{i}{N} \sin^2(u^+ - u^-) \quad (62)$$

the one that is compatible with our fuzzy AdS_2 model.

Acknowledgements

This work is supported in part by the National Science Council, Taiwan, and the Center for Theoretical Physics at National Taiwan University. The work of M.L. is also supported by a ‘‘Hundred People Project’’ grant.

Appendix A. Representations of $SL(2, \mathbf{R})$

The algebra of fuzzy AdS_2 can be realized on an irreducible representation of $SL(2, \mathbf{R})$ ($SU(1, 1)$). Similar to the case of $SU(2)$, a representation of $SL(2, \mathbf{R})$ can be easily constructed using the raising and lowering operators (20). The X 's act on the basis $\{|m\rangle\}$ as

$$X^1|m\rangle = m|m\rangle, \quad (63)$$

$$X_+|m\rangle = a_{m+1}|m+1\rangle, \quad (64)$$

$$X_-|m\rangle = a_m|m-1\rangle, \quad (65)$$

where

$$a_m = \sqrt{m(m-1) + c_2}. \quad (66)$$

The second Casimir is

$$c_2 = (X^{-1})^2 + (X^0)^2 - (X_1)^2 = -j(j-1), \quad (67)$$

which should be identified with $R^2 = N^2$.

However, not all such representations are unitary representations. They are unitary only in the following four situations.

- (1) Principal discrete representations (lowest weight):
 $j \in \mathbf{R}$, $j > 0$, $\alpha = 0$ and $m = j, j+1, j+2, \dots$
- (2) Principal discrete representations (highest weight):
 $j \in \mathbf{R}$, $j > 0$, $\alpha = 0$ and $m = -j, -j-1, -j-2, \dots$
- (3) Principal continuous representations:
 $j = 1/2 + is$, where $s \in \mathbf{R}$, and without loss of generality, one can choose α such that $0 \leq \alpha < 1$. The eigenvalues of X^1 are $m = \alpha, \alpha \pm 1, \alpha \pm 2, \dots$
- (4) Complementary representations:
 $1/2 < j < 1$ and $(j-1/2) < |\alpha-1/2|$ ($j \in \mathbf{R}$), where again we assumed that $0 \leq \alpha < 1$. The eigenvalues of X^1 are again $m = \alpha, \alpha \pm 1, \alpha \pm 2, \dots$

All unitary irreducible representations of $SL(2, \mathbf{R})$ fit in these four cases, except the trivial identity representation. The two principal discrete representations correspond to a fuzzy version of the de Sitter space, which can be obtained as one of the two hypersurfaces in $(2+1)$ -dimensional Minkowski space by requiring that $(X^{-1})^2 + (X^0)^2 - (X^1)^2 = -R^2$. This can be seen by examining the spectrum of X^1 . On the other hand, the spectrum of X^1 for the other two series are both extending from $-\infty$ to ∞ . Furthermore, from (67) we see that for $R > 1/2$ we have to choose the principal continuous series, and for $R < 1/2$ the complementary series.

Appendix B. Functions on fuzzy AdS_2 as rep.s of $SL(2, \mathbf{R})$

A function Ψ on the fuzzy AdS_2 corresponds to a lowest weight state if

$$[X_-, \Psi]|n\rangle = 0, \quad \text{for all } n \in \mathbf{Z}. \quad (68)$$

Take the ansatz for the lowest weight state in a principal discrete representation with $j = m \in \mathbf{Z}$ as

$$\Psi_m = f_m(X^1) X_+^m. \quad (69)$$

One can solve the function $f_m(X^1)$ up to an overall normalization by imposing (68). The result is

$$\begin{aligned} f_m(X^1) &= \prod_{k=1}^m \frac{1}{(X^1 - k + 1)(X^1 - k) + c_2} \\ &= \frac{\Gamma(X^1 - m + (1 - \sigma)/2) \Gamma(X^1 - m + (1 + \sigma)/2)}{\Gamma(X^1 + (1 - \sigma)/2) \Gamma(X^1 + (1 + \sigma)/2)}, \end{aligned} \quad (70)$$

where $\sigma = \sqrt{1 - 4c_2}$. Inserting this back to (69), we find

$$\Psi_m = \left(\frac{1}{X^1(X^1 - 1) + c_2} X_+ \right)^m. \quad (71)$$

States in the same representation of higher weights can be obtained by taking commutators of X_+ with Ψ_m as in (32).

The highest weight representations in the principal discrete series can be obtained in a similar way, or by simply taking Hermitian conjugation of the functions in the lowest weight representations.

Appendix C. Calculation of I_m

From (71) and (36), we find

$$I_m(x) = \sum_{n=-\infty}^{\infty} \langle n | \Psi_m^\dagger \Psi_m | n \rangle = \sum_{n=-\infty}^{\infty} \prod_{k=1}^m \frac{1}{a_{m+k}^2(x)}, \quad (72)$$

where

$$a_m^2(x) = m(m - 1) + x. \quad (73)$$

In the expressions above we view I_m as a function of the Casimir c_2 and we have denoted c_2 by x .

The expression of $I_m(x)$ can be reduced algebraically to an expression involving only I_1 times a function of x . While generic I_m can be calculated similarly, we show here only the case of I_2 :

$$\begin{aligned} I_2(x) &= \sum_{n=-\infty}^{\infty} \frac{1}{(n(n - 1) + x)(n(n + 1) + x)} \\ &= \sum_n \frac{1}{2n} \left(\frac{1}{n(n - 1) + x} - \frac{1}{n(n + 1) + x} \right) \\ &= \sum_n \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n - 1} \right) \frac{1}{n(n - 1) + x} \end{aligned}$$

$$\begin{aligned}
&= \sum_n \frac{-1}{2} \frac{1}{n(n-1)(n(n-1)+x)} \\
&= \sum_n \frac{-1}{2x} \left(\frac{1}{n(n-1)} - \frac{1}{n(n-1)+x} \right) \\
&= \frac{-1}{2x} \left(\sum_n \left(\frac{1}{n-1} - \frac{1}{n} \right) - \sum_n \frac{1}{n(n-1)+x} \right) \\
&= \frac{1}{2x} I_1.
\end{aligned} \tag{74}$$

This derivation is not rigorous because terms involving $1/n$ appear and it diverges at $n = 0$. However it suffices to show the idea. It is easy to modify it to get a well-defined, rigorous derivation which leads to exactly the same final result. The general expression for I_m is

$$I_m(x) = \frac{(2m-2)!}{[(m-1)!]^2} \left[\prod_{k=1}^{m-1} \frac{1}{k^2 - 1 + 4c_2} \right] I_1(x). \tag{75}$$

It is just I_1 divided by a polynomial of x of order $(m-1)$. So our task is reduced to calculate $I_1(x)$.

Viewing $I_1(x)$ as a holomorphic function of $x \in \mathbb{C}$, one can convince him/herself of the following equality

$$I_1(x) = \frac{\pi}{\sqrt{x-1/4}} \tanh(\pi\sqrt{x-1/4}) \tag{76}$$

by checking that the two sides of the equal sign share the following properties:

- (1) All the poles they have are at $x = -n(n-1)$ with the residue 2 for all integers $n \in \mathbb{Z}$.
- (2) They have zeros at $x = -n(n-1)$ for any half-integer n except $1/2$.
- (3) They equal π^2 at $x = 1/4$.
- (4) They approach the function $\pi/\sqrt{x}$ as $x \rightarrow \infty$.

References

- [1] P.-M. Ho, M. Li, Fuzzy spheres in AdS/CFT correspondence and holography from noncommutativity, hep-th/0004072.
- [2] H.S. Snyder, Quantized space-time, Phys. Rev. 71 (1946) 38.
- [3] H.S. Snyder, The electromagnetic field in quantized space-time, Phys. Rev. 72 (1947) 68.
- [4] C.N. Yang, On quantized space-time, Phys. Rev. 72 (1947) 874.
- [5] T. Yoneya, in: K. Kawarabayashi, A. Ukawa (Eds.), Wandering in the Fields, World Scientific, 1987, p. 419.
- [6] N. Kawamoto, T. Kugo (Eds.), Quantum String Theory, Springer, Berlin, 1988, see p. 23.
- [7] M. Li, T. Yoneya, Phys. Rev. Lett. 78 (1997) 1219, hep-th/9611072.
- [8] M. Li, T. Yoneya, Short-distance space-time structure and black holes in string theory: a short review of the present status, J. Chaos, Solitons and Fractals 10 (1999) 423, hep-th/9806240.
- [9] G. 't Hooft, Physica Scripta T15 (1987) 143.
- [10] G. 't Hooft, Nucl. Phys. B 335 (1990) 138.

- [11] N. Seiberg, L. Susskind, N. Toumbas, The teleological behavior of rigid regge rods, hep-th/0005015.
- [12] L. Susskind, E. Witten, The holographic bound in anti-de Sitter space, hep-th/9805114.
- [13] A. Peet, J. Polchinski, UV/IR relations in AdS dynamics, hep-th/9809022.
- [14] A. Jevicki, S. Ramgoolam, Noncommutative gravity from AdS/CFT correspondence, JHEP 9904 (1999) 032, hep-th/9902059.
- [15] P.-M. Ho, S. Ramgoolam, R. Tatar, Quantum spacetimes and finite N effects in 4D Yang–Mills theories, hep-th/9907145.
- [16] J. Maldacena, A. Strominger, AdS_3 black holes and a stringy exclusion principle, JHEP 9812 (1998) 005, hep-th/9804085.
- [17] J. McGreevy, L. Susskind, N. Toumbas, Invasion of the giant gravitons from anti-de Sitter space, hep-th/0003075.
- [18] M. Li, Fuzzy gravitons from uncertain spacetime, hep-th/0003173.
- [19] J. Castelino, S. Lee, W. Taylor, Longitudinal 5-branes as 4-sphere in matrix theory, Nucl. Phys. B 526 (1998) 334, hep-th/9712105.
- [20] M. Berkooz, H. Verlinde, Matrix theory, AdS/CFT and Higgs–Coulumb equivalence, JHEP 9911 (1999) 037.
- [21] A. Strominger, AdS_2 quantum gravity and string theory, JHEP 9901 (1999) 007, hep-th/9809027.
- [22] J. Maldacena, J. Michelson, A. Strominger, Anti-de Sitter fragmentation, JHEP 9902 (1999) 011, hep-th/9809027.
- [23] T. Banks, M. Green, Nonperturbative effects in $AdS_5 \times S^5$ string theory and $d = 4$ SUSY Yang–Mills, JHEP 9805 (1998) 002, hep-th/9804170.
- [24] G. 't Hooft, Quantization of point particles in $(2 + 1)$ -dimensional gravity and space–time discreteness, gr-qc/9601014.
- [25] J. Maldacena, A. Strominger, Statistical entropy of four-dimensional black holes, Phys. Rev. Lett. 77 (1996) 428, hep-th/9603069.
- [26] I.R. Klebanov, A.A. Tseytlin, Intersecting M-branes as four-dimensional black holes, hep-th/9604166.
- [27] F. Berezin, General concept of quantization, Commun. Math. Phys. 40 (1975) 153.
- [28] J. Madore, Introduction to Noncommutative Differential Geometry and Its Physical Applications, 2nd edn., Cambridge Univ. Press, Cambridge, 1999.
- [29] G. 't Hooft, Transplanckian particles and the quantization of time, gr-qc/9805079.