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Matrix theory in a constant C field background

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Abstract

D0-branes moving in a constant antisymmetric C field are found to be described by quantum mechanics of the supersymmetric matrix model with a similarity transformation. Sometimes this similarity transformation is singular or ill-defined and cannot be ignored. As an example, when there are nonvanishing C_{-ij} components, we obtain the theory for Dp -branes which is effectively the noncommutative super Yang–Mills theory. We also briefly discuss the effects of other nonvanishing components such as C_{+ij} and C_{ijk} . © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

A background independent, nonperturbative M/string theory remains one of the eminent problems in this ambitious program. There exists a conjectured formulation of nonperturbative M theory in a flat background, in the infinite momentum frame [1]. This formulation makes heavy use of intuitions from the D0-brane physics. As a first step toward generalizing this formulation to a background independent one, one may consider D0-branes on a curved background. However, it turns out that D0-brane physics in this case resists a general understanding.

When none of the maximal supersymmetry is broken, such as compactifications on tori of dimensions less than 6, the problem is not so difficult (for reviews, see [2,3]). In a somewhat seemingly simpler situation where there is a general background of constant bosonic fields, a formulation has not been proposed. One naturally divides bosonic fields

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into two sets. The first consists of constant metric. This problem is more or less trivial, since with a linear coordinates transformation, the metric can be put into the standard Minkowski form¹. The second set consists of constant antisymmetric tensor field $C_{\mu\nu\rho}$. In the presence of an interesting physical system which the generalized matrix theory is supposed to describe, one cannot always gauge away this constant background. In the case when only C_{-ij} are nonvanishing, and x^i, x^j , etc., are compactified, there exists a proposal by Connes, Douglas and Schwarz [4]. In this proposal, one replaces the super Yang–Mills on the torus by the noncommutative super Yang–Mills (NCSYM), with the noncommutative moduli given by $\theta_{ij} = RC_{-ij}$, where R is the radius of the longitudinal circle. This proposal was later justified by considerations in string theory [5–10], where the kind of Sen–Seiberg’s argument [11,12] in the decoupling limit is employed.

In Section 2 we will start from the membrane action with the coupling to the C field, and discretize it to obtain the D0-brane Hamiltonian. The resulting quantum mechanics differs from the standard one only by a similarity transformation. But this transformation may be singular or ill-defined in various situations. Two such circumstances are discussed. In particular, when one considers compactification or orbifolding, different matrix models may result. As an example, in Section 3 we will derive the NCYM for a brane solution directly using this transformation. We will show that the Connes–Douglas–Schwarz proposal can be directly derived in a perturbation expansion of the matrix theory without resorting to Sen–Seiberg’s argument, or to quantizing open strings at all. We also show in Section 4 how our similarity transformation can be related to the map of Seiberg and Witten [10] between the noncommutative fields and the commutative ones. Our approach is so general as to enable us to discuss the effects of turning on other C field components such as C_{ijk} , C_{+ij} , C_{+-i} in Section 5.

Much remains to be done to unravel the physical effects of switching on other components of the constant C field, in different situations. The simplest is the effect of C_{-ij} on D0-branes. Our discussion in the next section indicates that there are effects even without compactification. If matrix theory is correct, we expect that the spectrum of threshold bound states of D0-branes is not changed. The first thing in mind is then to calculate the Witten index again for the system of N D0-branes.

2. A similarity transformation

The Hamiltonian of multiple D0-branes can be derived by starting with the membrane action in the light-cone gauge, and replacing all physical variables, say $X^i(\sigma_1, \sigma_2)$, by matrices $X^i = \{X_{mn}^i\}$. Here we briefly review this procedure, leaving details to the original literature [13]. In the light-cone gauge, X^+ is identified with time τ , and X^- becomes an auxiliary field satisfying the constraints

$$\partial_a X^- + D_\tau X^i \partial_a X^i + \text{fermionic terms} = 0, \quad (2.1)$$

where $\partial_a = \partial_{\sigma_a}$ and $D_\tau = \partial_\tau + \{A_0, \cdot\}$ and the Poisson bracket $\{A, B\} = \epsilon^{ab} \partial_a A \partial_b B$ is defined with respect to the pair $\{\sigma_1, \sigma_2\}$. We will concentrate on the bosonic variables,

¹ From a background independent perspective, however, this is not a satisfactory solution, since one always has to put the constant metric as parameters into the Hamiltonian. The g_{-i} component in particular deserves more careful examination.

since introduction of fermionic variables is straightforward. The Hamiltonian is written as

$$H = \int d^2\sigma \left(\frac{P^+}{2} (D_\tau X^i)^2 + \frac{1}{4P^+} \{X^i, X^j\}^2 \right). \quad (2.2)$$

To get to the D0-brane Hamiltonian, we replace P^+ by N/R , the Poisson bracket $\{, \}$ by $\frac{N}{i}[,]$ ($1/N$ is treated as the Planck constant), and the integral $\int d^2\sigma$ by $\frac{1}{N}\text{tr}$. The D0-brane Hamiltonian thus obtained reads

$$H = \text{tr} \left(\frac{1}{2R} (D_\tau X^i)^2 - \frac{R}{4} [X^i, X^j]^2 \right). \quad (2.3)$$

We believe that the above procedure generalizes to the case when there is a constant C field. The coupling of the membrane to the C field is

$$S_1 = \frac{1}{6} \int C_{\mu\nu\rho} dX^\mu \wedge dX^\nu \wedge dX^\rho, \quad (2.4)$$

and is a total derivative when all the components of C are constant². Thus equations of motion as well as constraints derived from the new action are the same as before. However, one cannot ignore this total derivative term at the quantum mechanical level. For instance, if we are to compute the quantum propagation of membrane from a time t_1 to another time t_2 , the propagator is given by the path integral

$$\langle \Psi(t_2) | \Psi(t_1) \rangle = \int [DX] e^{i(S_0 + S_1)}, \quad (2.5)$$

where S_0 is the membrane action without the C field. Now since S_1 is a total derivative, it can be written as two boundary terms at time t_1 and t_2 ,

$$S_1 = \frac{1}{6} \left(\int d^2\sigma C_{\mu\nu\rho} X^\mu \{X^\nu, X^\rho\}(t_2) - \int d^2\sigma C_{\mu\nu\rho} X^\mu \{X^\nu, X^\rho\}(t_1) \right). \quad (2.6)$$

The interpretation of the two boundary terms in the path integral is straightforward: They simply “renormalize” the initial and final wave functions. The new wave function $\hat{\Psi}$ then becomes

$$\hat{\Psi} = U\Psi, \quad (2.7)$$

with

$$U = \exp \left(\frac{-i}{6} \int d^2\sigma C_{\mu\nu\rho} X^\mu \{X^\nu, X^\rho\} \right). \quad (2.8)$$

For D0-branes, this unitary operator U is

$$U = \exp \left(\frac{-1}{6} C_{\mu\nu\rho} \text{tr} X^\mu [X^\nu, X^\rho] \right). \quad (2.9)$$

It is clear that our argument is independent of the precise form of the functional integration measure in (2.5).

² We remark that our approach of treating the WZ term is different from that outlined in Refs. [4,14]. In particular, Ref. [14] does not have an action which is consistent with the Hamiltonian derived there.

Notice that if X^- is involved in U , we should employ the constraints (2.1), thus the operator will contain the canonical momenta P^i , and an ordering in the exponential in (2.9) must be chosen. We shall discuss this in the next section.

The equivalent of “renormalizing” the wave functions is to perform a similarity transformation on operators and to keep all wave functions intact. Given the operator $\mathcal{O}$, the new operator is $U^\dagger \mathcal{O} U$. If the similarity transformation operator U behaves in a reasonable way, the new theory obtained is identical to the original one. However, if the similarity transformation is singular, the new theory can be really a different theory. To see that our similarity transformation is sometimes singular, we consider two cases separately.

The 1st case

We first consider the case in which no X^- is involved. The exponential in U is cubic in X . A simpler example of this type is a single particle with a single coordinate. For example, if

$$U = \exp(iatx^2),$$

then a time-independent wave function will become time-dependent after this transformation, or equivalently, the Hamiltonian will become time-dependent in the Heisenberg picture. On the other hand, if

$$U = \exp(iax^3),$$

then there is no effect at all.

For the case in which X^- is involved, as we shall see in the next section, the exponential in U will take roughly the form $X^3 P$. Again consider the simpler case of a single particle with

$$U = \exp\left(-\left(\frac{a}{n-1}\right)x^n \partial_x\right),$$

for $n > 1$. This operator is just $\exp(a\partial_y)$, where $y = x^{-(n-1)}$. We start with a wave function $\Psi(y)$, and demand it be normalizable and vanish at $x = \infty$. Thus $\Psi(y = 0) = 0$. The transformed wave function is $U\Psi(y) = \Psi(y + a)$. Its value at $x = \infty$ no longer vanishes, instead it is $\Psi(a)$. Apparently this new wave function is no longer normalizable.

These examples are quite similar to the case of a charged particle on a circle when a constant gauge field is turned on. In this case the new wave function is not periodic any more, if the original one is. Similarly, the above demonstrations show that the boundary conditions for wave functions are changed under the action of U .

Although it would be very interesting to investigate further such situations, in this paper we will only elaborate on the following situation.

The 2nd case

Another situation in which the physics is changed by a similarity transformation is when there are further constraints which reduce the physical Hilbert space to a smaller

space on which the unitary operator is no longer well defined. In the next section we will show that the noncommutativity of D-brane worldvolume due to constant B field background can be understood in this way. Before we examine the D-brane case, let us consider a toy model as a warm-up.

Consider a matrix model of 2×2 matrices X_i and a unitary transformation of the matrix model by the operator $U = \exp(i \text{tr}(\alpha \Sigma_i P_i))$, where α is a constant Hermitian matrix and P_i is the conjugate momentum matrix of X_i . Obviously this unitary transformation produces a shift α to all the matrices X_i : $(X_i)_{ab} \rightarrow \hat{X}_i = (X_i)_{ab} + \alpha_{ab}$ and does not change the commutation relations among the matrix elements $(X_i)_{ab}$, $a, b = 1, 2$. Now suppose we are interested in the commutative limit and impose the constraints $(X_i)_{12} = (X_i)_{21} = 0$. The resulting X_i satisfies $[X_i, X_j] = 0$ and can be viewed as a function on an ordinary commutative space consisting of two points. Obviously the constraint kills some degree of freedoms of X^i and the similarity transformation is no longer well defined in the constrained matrix model. However, one can also perform the similarity transformation first and then impose the constraint, this way we obtain a new matrix model *different* from the original one, since $[\hat{X}_i, \hat{X}_j]$ is now nonvanishing for generic α . In the next section, we will see that the constraints effecting matrix model compactification is quite similar in nature to the simple constraint we considered here. It is therefore important to first perform the similarity transformation and then impose the compactification constraints. This simple example illustrates the same key reason why the similarity transformation (2.9) for $C_{-ij} \neq 0$ results in the noncommutativity on a D-brane.

It should be clear from this example that this consideration can be applied to orbifolds as well as compactifications.

3. Noncommutative Yang–Mills from similarity transformation

When the only nonvanishing components are C_{-ij} , we expect that a brane solution in matrix theory is described by the NCSYM, if all indices $i, j, \dots$ are tangent to the brane. Similarly, if $X^{i,j}$ are compactified, the NCSYM also emerges. The two cases differ only in the Yang–Mills coupling, whose correct value can be obtained by treating the operation tr properly in each case. Thus, we shall not distinguish explicitly between the two.

Before going over to D0-branes, the U operator (2.8) can be rewritten as

$$U = \exp\left(-\frac{i}{2} \int d^2\sigma C_{-ij} \{X^-, X^i\} X^j\right). \quad (3.1)$$

Using the constraints (2.1),

$$\{X^-, X^i\} = -D_\tau X^k \{X^k, X^i\},$$

we now replace the Poisson bracket by a commutator, the integral by trace, so the U operator for D0-branes is

$$U = \exp\left(\frac{1}{4} C_{-ij} \text{tr} \left[[X^k, X^i], X^j \right]_+ D_\tau X^k\right), \quad (3.2)$$

where $[\cdot, \cdot]_+$ is the anti-commutator and we have judiciously chosen an ordering in the trace. Identifying the conjugate momenta $P^\mu = \frac{1}{R} D_\tau X^\mu$, we have the final form of the operator as

$$U = \exp(-I), \quad I = \frac{1}{4} \theta_{ij} \text{tr} [[X^i, X^k], X^j]_+ P^k, \quad (3.3)$$

where $\theta_{ij} = RC_{-ij}$. If we take the fermion part of constraints (2.1) into account, U will contain a part involving fermionic fields.

Everything we said so far is classical. When we quantize the system, it is natural to adopt the Weyl ordering prescription to have: $F(X)P \rightarrow \frac{1}{2}(F(X)P + PF(X)) = F(X)P - i\frac{\partial}{\partial X}F$ and the I thus obtained will be different by an additional term of $I_b := \frac{1}{4} \theta_{ij} \text{tr} [X^i, X^j]$. In fact, in order for U to be a unitary operator, it is necessary to use the Weyl ordering. However, I_b can be nonvanishing only in the large N limit and is proportional to the conserved membrane charge. It does not modify the operator $\mathcal{O}$ at all and only modifies the wavefunction by a phase. So long as we consider states with the same membrane charge, these terms have no observable effects and hence we will drop them from now on and use (3.3) as our definition. But they cannot always be omitted if we consider membrane processes with charge transfer.

The effect of adding the C_{-ij} field background in the matrix model is to replace every operator $\mathcal{O}$ by

$$\hat{\mathcal{O}} = U^\dagger \mathcal{O} U.$$

With the anticipation that on compactification X_j will be replaced by $iD_j = i\partial_j + A_j$, we will split X into $X^j = X_0^j + X_1^j$ with X_0^j corresponding to some constant background configuration that will be identified with $i\partial_j$ after compactification. I is split correspondingly into $I = I_0 + I_1$, with

$$\begin{aligned} I_0 &= -\frac{i}{4} \theta_{ij} \text{tr} [[X_0^i, X^\mu], X_0^j]_+ \frac{\delta}{\delta X^\mu}, \\ I_1 &= -\frac{i}{4} \theta_{ij} \text{tr} [[X^i, X^\mu], X_1^j]_+ \frac{\delta}{\delta X_\mu} - \frac{i}{4} \theta_{ij} \text{tr} [[X_1^i, X^\mu], X_0^j]_+ \frac{\delta}{\delta X^\mu}. \end{aligned} \quad (3.4)$$

One can separate the full operator U into two parts

$$U = U_1 U_0, \quad \text{with } U_0 = e^{-I_0}, \quad (3.5)$$

and $U_1 = e^{-I_1}(1 + \mathcal{O}(\theta^2))$. The omitted higher order terms $\mathcal{O}(\theta^2)$ are terms that can arise in U as I_0 and I_1 do not commute. The main reason for this separation (3.5) with the action of U_0 singled out explicitly is that, roughly speaking, with X_{0j} identified with $i\partial_j$, U_0 will result in the star product and U_1 will relate the noncommutative $U(1)$ fields to the commutative ones.

Decompose $\hat{\mathcal{O}}$ into

$$\hat{\mathcal{O}} = U_0^\dagger \tilde{\mathcal{O}} U_0, \quad \tilde{\mathcal{O}} = U_1^\dagger \mathcal{O} U_1. \quad (3.6)$$

Let us first consider only the effect of U_0 and ignore U_1 . This can be viewed as the 0th order calculation in a perturbative expansion in terms of X_1^μ . Denoting the matrix X^μ by Φ and using $P_\mu = -i\frac{\delta}{\delta X^\mu}$, we find

$$\begin{aligned} \hat{\Phi} &= \Phi + \left(\frac{-i}{2} \theta_{ij} \right) [X_0^i, \Phi] X_0^j + \cdots + \frac{1}{n!} \left(\frac{-i}{2} \theta_{i_1 j_1} \right) \cdots \left(\frac{-i}{2} \theta_{i_n j_n} \right) \\ &\quad \times [X_0^{i_1}, \dots, [X_0^{i_n}, \Phi] \cdots] X_0^{j_1} \cdots X_0^{j_n} + \cdots \end{aligned} \quad (3.7)$$

Now consider a solution representing a single dual Dp-brane whose longitudinal directions coincide with those along which θ_{ij} is nonvanishing. We use σ_i to denote these directions. The directions transverse to the brane will be denoted by X^a, X^b . For the longitudinal directions, X^i is replaced by $iD_i = i(\partial_i - iA_i)$ with A_i being the $U(1)$ gauge field and tr shall be replaced by $\int d\sigma^3$. Let $X_0^i = i\partial_i$, then

$$\hat{\Phi} = U_0^\dagger \Phi U_0 = \Phi \left(\sigma_i + \frac{i}{2} \theta_{ij} \partial_j \right) = \Phi(\hat{\sigma}_i), \quad (3.8)$$

where by definition

$$f(\hat{\sigma}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2} \theta_{i_1 j_1} \right) \cdots \left(\frac{i}{2} \theta_{i_n j_n} \right) (\partial_{i_1} \cdots \partial_{i_n} f(\sigma)) \partial_{j_1} \cdots \partial_{j_n}.$$

Thus we obtain a function whose arguments are $\hat{\sigma}_i = \sigma_i + \frac{i}{2} \theta_{ij} \partial_j$ and they no longer commute among themselves. It is

$$[\hat{\sigma}_i, \hat{\sigma}_j] = i\theta_{ij}. \quad (3.9)$$

In terms of the dual B-field on the Dp-brane worldvolume, $\theta = \frac{B}{1+B^2}$. These are exactly the commutation relations obtained by quantizing open strings on a D-brane. Here we want to emphasize again that nowhere we have resorted to string theory.

Note that our definition of the new function $\Phi(\hat{\sigma})$ through $\Phi(\sigma)$ is schematically

$$\Phi(\hat{\sigma}) = \sum \frac{1}{n!} \partial^n \Phi(\sigma) (\Delta\sigma)^n, \quad (3.10)$$

where $\Delta\sigma = \hat{\sigma} - \sigma$ and is a derivative. On first sight, this definition seems to be different from the usual Weyl ordering for a function of noncommutative variables, which is

$$\Phi(\hat{\sigma}_i) = \int dk \tilde{\Phi}(k) e^{ik^i \hat{\sigma}_i}. \quad (3.11)$$

As is well known, the latter definition obeys the star product. Our definition is instead

$$\Phi(\hat{\sigma}_i) = \int dk \tilde{\Phi}(k) e^{ik^i \sigma_i - \frac{1}{2} k^i \theta_{ij} \partial_j}. \quad (3.12)$$

But since $e^A e^B = e^{A+B}$ for A, B commuting, the two definitions are in fact identical. Therefore,

$$f(\hat{\sigma}) g(\hat{\sigma}) = (f * g)(\hat{\sigma}) \quad (3.13)$$

for any functions f, g of $\tilde{X}_a$'s and $\tilde{A}_i$'s with the star product defined by

$$(f * g)(\sigma) = e^{\frac{i}{2} \theta_{ij} \frac{\partial}{\partial \sigma_i} \frac{\partial}{\partial \sigma_j}} f(\sigma) g(\sigma')|_{\sigma' = \sigma}. \quad (3.14)$$

It is now natural to interpret this as defining the noncommutative algebra of functions over a noncommutative space. It is satisfying to see that the star product in noncommutative gauge theory has a simple origin from matrix model.

³ For convenience, we will consider only $U(1)$ gauge symmetry on the D-brane world volume in the following, and its generalization to the $U(N)$ case is straightforward.

It is convenient to introduce a left (resp. right) translational invariant “vacuum” denoted by $\rangle$ (resp. $\langle$) which is annihilated by all derivatives ∂_i acting from the left (resp. right). So for instance

$$\Phi(\hat{\sigma})\rangle = \Phi(\sigma)\rangle. \quad (3.15)$$

As a notation consistent with Stokes’ theorem, integration on the noncommutative space can be denoted as the “vacuum expectation value” $\langle \cdot \rangle^4$,

$$\langle f(\hat{\sigma}) \rangle = \int d^p \sigma f(\sigma),$$

where $\int d^p \sigma \cdot$ is the ordinary integration on a classical space, and it should agree with the large N limit of the trace tr .

Here a puzzle arises. Consider two fields $\Phi_1(\sigma)$ and $\Phi_2(\sigma)$ originally commuting with each other. After the similarity transformation, $\hat{\Phi}_1$ and $\hat{\Phi}_2$ do not commute, while by naively applying (3.6), one gets $[\hat{\Phi}_1, \hat{\Phi}_2] = U_0^\dagger [\Phi_1, \Phi_2] U_0 = 0$. The reason why this naive procedure is not correct is as follows. When turning the matrices Φ_1, Φ_2 into functions of σ_i , we have killed (infinitely) many degrees of freedom since the two original large N matrices are not commuting in general. For instance one can first compactify the space on T^p by imposing constraints like

$$V_i^\dagger X_j V_i = X_j + 2\pi \delta_{ij} R_j, \quad V_i^\dagger X_a V_i = X_a, \quad (3.16)$$

and then let $R_j \rightarrow \infty$ in the end, if one wishes. The similarity transformation is ill-defined after the constraints are imposed. This is just what happens in the warm-up example in Section 2. Instead, if we perform the similarity transformation first and then impose the constraints for compactification (as we were doing here in this section), we get the noncommutativity on the D-brane.

In the above we have given discussions in the Hamiltonian formulation. The Hamiltonian in the temporal gauge $A_0 = 0$ after the similarity transformation becomes that of the noncommutative $U(1)$ gauge theory. If we want to recover the field A_0 , for consistency, it must be a noncommutative variable too. Therefore the resulting action for the Dp-brane is obtained from the D0-brane Lagrangian

$$L = \text{tr} \left(\frac{1}{2} (D_0 X^\mu)^2 - \frac{1}{4} [X^\mu, X^\nu]^2 \right) \quad (3.17)$$

by replacing X by $\hat{X}$, usual product by star product and $\text{tr}(\cdot)$ by $\langle \cdot \rangle$. Thus we obtain the well-known NCYM Lagrangian.

4. Relation to Seiberg–Witten map

In the previous section we have only showed the effect of conjugation by U_0 on operators, and NCYM is obtained as the 0th order approximation of the exact matrix

⁴ This notation is very natural in noncommutative geometry for the following reason. In the quantum mechanics for a single particle, a state can be denoted in terms of its wave function as $\psi(x)\rangle$, and momentum acts on the state according to the algebraic rules $[p, x] = -i$ and $p\rangle = 0$. The inner product of two states $\psi_1(x)\rangle$, denoted by $\langle \psi_1^\dagger(x) \psi_2(x) \rangle$, is just the integration of $\psi_1^*(x) \psi_2(x)$.

theory in C_{-ij} background. Now we consider the effect of conjugation by U_1 and examine the new fields $\tilde{A}_i$ and $\tilde{X}^a$.

Before we start, we mention that it is straightforward to repeat the idea of [10] to derive the relation between the noncommutative scalars and the usual scalars. Together with the result for the noncommutative $U(1)$ gauge field, it is

$$\hat{X}_a^{\text{sw}} = X_a^{\text{sw}} - \theta_{kl} A_k^{\text{sw}} \partial_l X_a^{\text{sw}}, \quad (4.1)$$

$$\hat{A}_i^{\text{sw}} = A_i^{\text{sw}} - \frac{1}{2} \theta^{kl} A_k^{\text{sw}} (\partial_l A_i^{\text{sw}} + F_{li}^{\text{sw}}), \quad (4.2)$$

up to first order in θ for the gauge group $U(1)$.

For the action of U_0 , the result (3.8) for a Dp -brane solution is exact to all orders in θ . To the first order in θ , the transformation by U_1 is given by

$$\tilde{X}_a = U_1^\dagger X_a U_1 = X_a - \frac{1}{2} \theta_{kl} A_k \partial_l X_a + \cdots, \quad (4.3)$$

$$\tilde{X}_i = U_1^\dagger X_i U_1 = i \left[\partial_i - \frac{1}{4} \theta_{kl} [(\partial_i A_k), \partial_l]_+ \right] + \left[A_i - \frac{1}{2} \theta_{kl} A_k F_{li} \right] + \cdots. \quad (4.4)$$

Ideally, the effect of U_1 should result in the relations (4.1), (4.2) of Seiberg and Witten which map the commutative fields to the noncommutative ones. To the lowest order in θ , we should identify our fields $\tilde{X}$ and $\tilde{A}$ with Seiberg and Witten's noncommutative $U(1)$ fields $\hat{X}^{\text{sw}}$ and $\hat{A}^{\text{sw}}$, because they are multiplied with one another using the star product. However, (4.3) and (4.4) are not exactly the same as the Seiberg–Witten map, so we cannot just identify our X and A with their commutative $U(1)$ fields X^{sw} and A^{sw} . To the first order in θ , the unwanted piece (the second term in the first $[\cdot]$) in (4.4) can be absorbed in a change of coordinates. Let

$$\sigma_k = \sigma'_k + \frac{1}{2} \theta_{kl} A_l(\sigma'), \quad (4.5)$$

which implies a shift in ∂_i ,

$$\partial'_i = \partial_i - \frac{1}{4} \theta_{kl} [(\partial_i A_k), \partial_l]_+ + \cdots. \quad (4.6)$$

This operator ∂' is chosen such that it is anti-Hermitian and satisfies $[\partial'_i, \sigma'_j] = \delta_{ij}$ and $[\partial'_i, \partial'_j] = 0$ (up to first order in θ)⁵. At the same time,

$$A_i(\sigma) = A_i(\sigma' + \frac{1}{2} \theta A) = A_i(\sigma') + \frac{1}{2} \theta_{kl} (\partial_k A_i) A_l + \cdots,$$

and similarly for X_a , giving exactly the extra pieces we were missing from the Seiberg–Witten map in (4.3) and (4.4). Finally, we obtain

$$\hat{X}_a = \tilde{X}_a(\hat{\sigma}'), \quad \text{with } \tilde{X}_a(\sigma') = X_a(\sigma') - \theta_{kl} A_k \partial_l X_a + \cdots, \quad (4.7)$$

$$\hat{A}_i = \tilde{A}_i(\hat{\sigma}'), \quad \text{with } \tilde{A}_i(\sigma') = A_i(\sigma') - \frac{1}{2} \theta_{kl} A_k (\partial_l A_i + F_{li}) + \cdots, \quad (4.8)$$

where $\hat{\sigma}' = \sigma' + \frac{i}{2} \theta \partial'$. These equations are exactly of the same form as the Seiberg–Witten map, thus $\tilde{X}$, $\tilde{A}$ can be identified with the noncommutative $U(1)$ fields $\hat{X}^{\text{sw}}$, $\hat{A}^{\text{sw}}$ in the Seiberg–Witten map and X , A with the commutative $U(1)$ fields. To be exact, the separation of U into U_0 and U_1 should be adjusted order by order in θ in order to reproduce the Seiberg–Witten map.

⁵ Compared with the usual definition of derivatives after a change of coordinates $\sigma \rightarrow \sigma'$, (4.6) differs by the additional term $\frac{1}{4} \theta_{kl} \partial_i \partial_l \partial_k A_l$. This term can be accounted for by the change of integration measure due to the Jacobian $\frac{\partial \sigma}{\partial \sigma'}$.

It is obviously more complicated to implement this kind of derivation to higher orders in θ , and one expects to meet ambiguities if all we need is a map between the noncommutative and commutative variables which preserves gauge transformations [15]. On the other hand, it is also easy to see that in principle the results at the first order in θ can be extended to all orders by solving a particular differential equation with respect to θ . From $\hat{X} = U^\dagger X U$ and $U = \exp(-\theta_{ij} J^{ij})$, where $J_{ij} = \frac{1}{4} \text{tr} [[X^i, X_\mu], X^j]_+ P_\mu$, one derives

$$\delta \hat{X} = \delta \theta_{ij} [J^{ij}, \hat{X}], \quad (4.9)$$

for a variation $\delta \theta_{ij} \propto \theta_{ij}$. This is just the first order term in θ in the expansion of $\hat{X}$ in terms of X , with all X 's replaced by $\hat{X}$. Note that the derivation is valid only if θ_{ij} is varied by scaling. For a different path of variation of θ , the result is in general different [15]. Note also that (4.9) is only analogous to the differential equation in Ref. [10]; they are different by a change of variables.

Note that we will not be able to perform the change of coordinates (4.5) for the case of $U(N)$ because that requires the coordinates σ to turn into a matrix. The best one can do is to take the $U(1)$ part of A for this change of coordinates. On the other hand, one should wonder why should one use a single set of coordinates for all the N D-branes, while the background values of $B + F$ can be different on each D-brane. For the cases in which the expectation values of $B + F$ change significantly from brane to brane, one expects that with only a single set of coordinates satisfying $[\sigma_i, \sigma_j] = i\theta_{ij}$, NCYM will not be able to give a good description of the system for any choice of θ . It is just because our formulation gives the exact result, and the most general background of $B + F$ can be any $U(N)$ matrix, that we are led to a situation where it seems natural to introduce a matrix of coordinates. A different consideration that seems to lead to the same conclusion is to start from a suitable theory of Matrix open string and consider a sector of the theory which describes N coincident D-branes. Analogous to the situation in Ref. [6–9], it seems natural that the D-brane worldvolume will emerge as $U(N)$ matrices with noncommuting matrix elements. We leave this possibility for further studies.

It would also be interesting to relate our results to recent discussions on possible relations between noncommutative variables and commutative variables [16–18].

5. Other components of the C field

In this section we consider the effects of other components of C . Let us consider C_{ijk} first. In this case the unitary operator U is simply a function of X^i , so the X 's are not modified by conjugation by U , but their conjugate momenta are changed. It is easy to see that if the directions labeled by i, j, k are compactified, C_{ijk} will change the spectrum as

$$P_i \rightarrow P_i - \frac{i}{2} C_{ijk} [X_j, X_k]. \quad (5.1)$$

It appears that without compactification, the above similarity transformation does not change physics, this is easy to see by applying the transformation to the Hilbert space,

rather than to operators. An alternative argument is that C_{ijk} are not moduli in 11 dimensions. With compactification, the story can change. If one dimension is compactified first, the matrix string results. Let this dimension be x^1 , then $C_{1ij} = B_{ij}$. It is well known in string theory that on a torus, B_{ij} becomes a genuine moduli, this implies that to get nontrivial physics we need to compactify two more dimensions. We conclude that on T^3 parametrized by (x^i, x^j, x^k) , C_{ijk} does have physical effects. This is compatible with Eq. (5.1). Upon compactification, the second term is proportional to $C_{ijk} F_{jk}$. The zero modes of F_{ij} will shift the canonical momentum P_i . If this shift is not quantized (a vector on the momentum lattice), then there is a net physical effect. This effect was first discussed in Ref. [19] based on physical arguments.

The case of $C_{+ij} \neq 0$ is a little more interesting. Since in the light-cone gauge $X^+ = P^+ \tau = (N/R)\tau$, we have

$$U = \exp\left(-\frac{1}{2R} C_{+ij} N \text{tr}[X^i, X^j] \tau\right). \quad (5.2)$$

It is nontrivial only when $\text{tr}[X^i, X^j] \neq 0$. However,

$$iN \text{tr}[X^i, X^j]$$

is simply the membrane charge and is a conserved quantity. The operator (5.2) simply shifts the energy by a quantity proportional to the membrane charges.

If $C_{+-i} \neq 0$, both X and P will be modified by the similarity transformation by a time-dependent piece. If we compactify X_i , its momentum is quantized and the physical effect of C_{+-i} is manifested in the momentum spectrum, in a way similar to the effect of a Wilson line. But here the shift in spectrum is time-dependent. It may be interesting to study in more details these cases.

The linear coupling of C field in matrix theory was also obtained in Refs. [20,21], it may be interesting to make connection of our formulation with their approaches.

6. Discussions

We expect that our construction will shed light on the problem of working out the complete Seiberg–Witten map. It seems that serious progress in examining the AdS/CFT correspondence in a B field background can be made only after the Seiberg–Witten map is well understood. In general, one needs to find a general way to construct local and gauge invariant operators. We leave this problem to future investigation.

The method of studying effects of C field in matrix theory can be readily generalized to the IIB matrix model [22]. There one starts with the Schild action S_0 for a fundamental string. To include the B field, one adds a term to the Schild action,

$$S = S_0 + \int B_{ij} \{X^i, X^j\} d^2\sigma. \quad (6.1)$$

Upon discretizing the world sheet, the second term becomes

$$iB_{ij} \text{tr}[X^i, X^j]. \quad (6.2)$$

To investigate the physical effects of this new term, one may focus attention on a D-brane solution. It was already pointed out in Ref. [23] that a D-string and in general a D-brane solution naturally introduces the NCSYM as the effective world-volume theory. It would be interesting to study this issue further by combining the term (6.2) with the ordinary IIB matrix action.

Coming back to matrix theory, it is rather surprising to us that introducing a constant C background is implemented by a similarity transformation, although this is imposed on us by the Chern–Simons term in the membrane action. One may ask the deeper question in light of our observation: how does a constant background emerges from a would-be background independent formulation? In the past, adding background inevitably introduces new degrees of freedom, this certainly is not in line with the idea of a background independent formalism. A background should emerge as a collective solution of the existent degrees of freedom. Our work seems to be a step further in this direction. We still put in the background “by hand”, but by merely reshuffling the old degrees of freedom through a similarity transformation, instead of introducing new ones.

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