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Locality of optimal lattice domain-wall fermions

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Abstract

It is shown that the effective 4D lattice Dirac operator of optimal lattice domain-wall fermions with finite N_s (in the fifth dimension) is exponentially local for sufficiently smooth gauge background.

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Recently, it has been shown that one can construct a lattice domain-wall fermion action such that the effective 4D lattice Dirac operator preserves the chiral symmetry optimally for any given finite N_s in the fifth dimension [1].

Explicitly, the action of optimal lattice domain-wall fermions reads¹

$$\begin{aligned} \mathcal{A} = & \sum_{s,s'=1}^{N_s} \sum_{x,x'} \bar{\psi}(x,s) \\ & \times \left[(\omega_s D_w(x,x') + 1) \delta_{s,s'} \right. \\ & + (\omega_s D_w(x,x') - 1) P_- \delta_{s',s+1} \\ & + (\omega_s D_w(x,x') - 1) P_+ \delta_{s',s-1} \left. \right] \\ & \times \psi(x',s') \end{aligned}$$

$$\begin{aligned} - \frac{m_q}{2m_0} \sum_{x,x'} & \left[\bar{\psi}(x,1) (\omega_1 D_w(x,x') - 1) \right. \\ & \times P_+ \psi(x',N_s) \\ & + \bar{\psi}(x,N_s) (\omega_{N_s} D_w(x,x') - 1) \\ & \left. \times P_- \psi(x',1) \right] \end{aligned} \quad (1)$$

with boundary conditions

$$P_+ \psi(x,0) = - \frac{m_q}{2m_0} P_+ \psi(x,N_s),$$

$$P_- \psi(x,N_s+1) = - \frac{m_q}{2m_0} P_- \psi(x,1),$$

and the quark fields with bare mass m_q can be constructed from the left and right boundary modes

$$q(x) = P_- \psi(x,1) + P_+ \psi(x,N_s),$$

$$\bar{q}(x) = \bar{\psi}(x,1) P_+ + \bar{\psi}(x,N_s) P_-.$$

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¹ In this Letter, we suppress the lattice spacings (a and a_5), as well as the Dirac and color indices, which can be restored easily.

Here D_w is the 4D Wilson–Dirac operator with a negative parameter ($-m_0$),

$$D_w = \sum_{\mu=1}^4 \gamma_\mu t_\mu + W - m_0, \quad m_0 \in (0, 2),$$

$$t_\mu(x, x') = \frac{1}{2} [U_\mu(x) \delta_{x', x+\mu} - U_\mu^\dagger(x') \delta_{x', x-\mu}],$$

$$W(x, x') = \sum_{\mu=1}^4 \frac{1}{2} [2\delta_{x, x'} - U_\mu(x) \delta_{x', x+\mu} - U_\mu^\dagger(x') \delta_{x', x-\mu}],$$

and the weights

$$\omega_s = \frac{1}{\lambda_{\min}} \sqrt{1 - \kappa'^2 \text{sn}^2(v_s; \kappa')}, \quad s = 1, \dots, N_s, \quad (2)$$

where $\text{sn}(v_s; \kappa')$ is the Jacobian elliptic function with argument v_s (3) and modulus $\kappa' = \sqrt{1 - \lambda_{\min}^2 / \lambda_{\max}^2}$, ($\lambda_{\max}^2$ and $\lambda_{\min}^2$ are the maximum and the minimum of the eigenvalues of H_w^2 , $H_w = \gamma_5 D_w$). The weights $\{\omega_s\}$ are obtained from the roots ($u_s = \omega_s^{-2}$, $s = 1, \dots, N_s$) of the equation

$$\delta_Z(u) = \begin{cases} 1 - \sqrt{u} R_Z^{(n,n)}(u) = 0, & N_s = 2n + 1, \\ 1 - \sqrt{u} R_Z^{(n-1,n)}(u) = 0, & N_s = 2n. \end{cases}$$

The argument v_s in (2) is

$$v_s = (-1)^{s-1} M \text{sn}^{-1} \left(\sqrt{\frac{1 + 3\lambda}{(1 + \lambda)^3}}; \sqrt{1 - \lambda^2} \right) + \left[\frac{s}{2} \right] \frac{2K'}{N_s}, \quad (3)$$

where

$$\lambda = \prod_{l=1}^{N_s} \frac{\Theta^2(\frac{2lK'}{N_s}; \kappa')}{\Theta^2(\frac{(2l-1)K'}{N_s}; \kappa')}, \quad (4)$$

$$M = \prod_{l=1}^{\lfloor \frac{N_s}{2} \rfloor} \frac{\text{sn}^2(\frac{(2l-1)K'}{N_s}; \kappa')}{\text{sn}^2(\frac{2lK'}{N_s}; \kappa')}, \quad (5)$$

K' is the complete elliptic integral of the first kind with modulus κ' , and Θ is the elliptic theta function. Note that $\lambda_{\max}^{-1} \leq \omega_s \leq \lambda_{\min}^{-1}$, since $\text{sn}^2(\cdot; \cdot) \leq 1$.

After adding pseudofermions (Pauli–Villars fields) with fixed mass $m_q = 2m_0$ to the action (1), one can derive the effective 4D lattice Dirac operator for the

internal quark loops

$$D(m_q) = r(D_c + m_q)(1 + rD_c)^{-1}, \quad r = \frac{1}{2m_0}, \quad (6)$$

where

$$rD_c = \frac{1 + \gamma_5 \mathcal{S}_{\text{opt}}}{1 - \gamma_5 \mathcal{S}_{\text{opt}}},$$

$$\mathcal{S}_{\text{opt}} = \begin{cases} H_w R_Z^{(n,n)}(H_w^2), & N_s = 2n + 1, \\ H_w R_Z^{(n-1,n)}(H_w^2), & N_s = 2n, \end{cases} \quad (7)$$

and $R_Z(H_w^2)$ is the Zolotarev optimal rational approximation for the inverse square root of H_w^2 ,

$$R_Z^{(n,n)}(H_w^2) = \frac{d_0}{\lambda_{\min}} \prod_{l=1}^n \frac{1 + h_w^2/c_{2l}}{1 + h_w^2/c_{2l-1}} \quad (8)$$

and

$$R_Z^{(n-1,n)}(H_w^2) = \frac{d'_0}{\lambda_{\min}} \frac{\prod_{l=1}^{n-1} (1 + h_w^2/c'_{2l})}{\prod_{l=1}^n (1 + h_w^2/c'_{2l-1})} \quad (9)$$

in which $h_w^2 = H_w^2 / \lambda_{\min}^2$, and the coefficients d_0, d'_0, c_l and c'_l are expressed in terms of elliptic functions [2] depending on N_s and $\lambda_{\max}^2 / \lambda_{\min}^2$.

In the limit $N_s \rightarrow \infty$, $\mathcal{S}_{\text{opt}} \rightarrow H_w (H_w^2)^{-1/2}$, thus rD_c (7) becomes chirally symmetric. Therefore, in the massless limit ($m_q = 0$) and $N_s \rightarrow \infty$, D (6) is exactly equal to the overlap Dirac operator [3,4], and satisfies the Ginsparg–Wilson relation [5]

$$D\gamma_5 + \gamma_5 D = 2D\gamma_5 D.$$

This implies that the effective 4D Dirac operator of optimal DWF is exponentially local for sufficiently smooth gauge background, and is topologically-proper (i.e., with the correct index and axial anomaly), similar to the case of overlap Dirac operator. At finite N_s , the effective 4D lattice Dirac operator is exactly equal to the overlap Dirac operator with $(H_w)^{-1/2}$ approximated by Zolotarev optimal rational polynomial.

The valence quark propagator coupling to physical hadrons is

$$(D_c + m_q)^{-1} = r(1 - rm_q)^{-1} [D^{-1}(m_q) - 1],$$

where $D^{-1}(m_q)$ can be computed via the 5-dimensional lattice Dirac operator of optimal DWF.

In this Letter, we show that the effective 4D lattice Dirac operator (6) is exponentially local for sufficiently smooth gauge configurations. Since Eq. (6) can be rewritten as

$$D(m_q) = rm_q + (1 - rm_q)D, \quad (10)$$

where

$$D = rD_c(1 + rD_c)^{-1} = \frac{1}{2}(1 + \gamma_5 \mathcal{S}_{\text{opt}}), \quad (11)$$

thus it suffices to assert the exponential locality of D (11).

First, we rewrite (8) and (9) in partial fractions

$$R_Z^{(n,n)}(H_w^2) = (1 + h_w^2/c_{2n}) \sum_{l=1}^n \frac{b_l}{1 + h_w^2/c_{2l-1}}, \quad (12)$$

$$R_Z^{(n-1,n)}(H_w^2) = \sum_{l=1}^n \frac{b_l}{1 + h_w^2/c_{2l-1}}, \quad (13)$$

where

$$c_l = \frac{\text{sn}^2\left(\frac{lK'}{N_s}; \kappa'\right)}{1 - \text{sn}^2\left(\frac{lK'}{N_s}; \kappa'\right)}, \quad (14)$$

$$b_l = \frac{1}{\lambda_{\min}} \frac{2\lambda}{1 + \lambda} \frac{1}{M} \frac{\prod_{i=1}^{n-1} (1 - c_{2l-1}/c_{2i})}{\prod_{i=1, i \neq l}^n (1 - c_{2l-1}/c_{2i-1})}, \quad (15)$$

λ and M are defined in (4) and (5), respectively.

Therefore, it suffices to examine the locality of each term in (12) and (13), by expanding each fraction in terms of orthogonal polynomials, similar to the approach used in [6]. Here I expand each fraction in term of Chebycheff polynomials of the first kind

$$\begin{aligned} & \frac{1}{1 + h_w^2/c_{2l-1}} \\ &= \frac{2c_{2l-1}}{(b-1) \sinh \theta_l} \left(1 + 2 \sum_{k=1}^{\infty} e^{-k\theta_l} T_k(z) \right), \end{aligned} \quad (16)$$

where

$$b = \frac{\lambda_{\max}^2}{\lambda_{\min}^2}, \quad (17)$$

$$\theta_l = \cosh^{-1} \left(\frac{b+1+2c_{2l-1}}{b-1} \right), \quad (18)$$

$$z = \frac{b+1-2h_w^2}{b-1}. \quad (19)$$

The expansion (16) is valid provided that $\theta_l \neq 0$ (i.e., $\lambda_{\min}^2 > 0$), which is ensured by any gauge configuration satisfying [7]

$$\forall_{\text{plaquette}} \|1 - U(p)\| < \epsilon < \frac{1 - |1 - m_0|^2}{6(2 + \sqrt{2})} \quad (20)$$

since

$$\begin{aligned} & \left(\sqrt{1 - 6(2 + \sqrt{2})\epsilon} - |1 - m_0| \right)^2 \\ & \leq \lambda_{\min}^2 < \lambda_{\max}^2 \leq (|4 - m_0| + 4)^2. \end{aligned} \quad (21)$$

Because h_w only has nearest neighbor coupling, it follows that $h_w^2(x, y) = 0$ for all separations bigger than two lattice spacings, i.e., $\sum_{\mu} |x_{\mu} - y_{\mu}| > 2a$. Thus, for the Chebycheff polynomial $T_k(z)$ of degree k , $[T_k(z)](x, y) = 0$ for $\sum_{\mu} |x_{\mu} - y_{\mu}| > 2ka$. Then, (16) gives

$$\begin{aligned} & \left(\frac{1}{1 + h_w^2/c_{2l-1}} \right)(x, y) \\ &= \frac{2c_{2l-1}}{(b-1) \sinh \theta_l} \sum_{k=m}^{\infty} e^{-k\theta_l} [T_k(z)](x, y), \end{aligned} \quad (22)$$

where $m = \sum_{\mu} |x_{\mu} - y_{\mu}|/2a \geq 1$. Using the triangle inequality and the property $\|T_k(z)\| \leq 1$, we turn Eq. (22) into the inequality

$$\begin{aligned} & \left\| \left(\frac{1}{1 + h_w^2/c_{2l-1}} \right)(x, y) \right\| \\ & \leq \frac{2c_{2l-1}}{(b-1) \sinh \theta_l} \frac{e^{-m\theta_l}}{(1 - e^{-\theta_l})}, \end{aligned}$$

and it follows that

$$\begin{aligned} & \left\| \left(\sum_{l=1}^n \frac{b_l}{1 + h_w^2/c_{2l-1}} \right)(x, y) \right\| \\ & \leq \frac{2}{b-1} \sum_{l=1}^n \frac{b_l c_{2l-1}}{\sinh \theta_l} \frac{1}{(1 - e^{-\theta_l})} \\ & \quad \times \exp \left(-\frac{\theta_l}{2a} \sum_{\mu} |x_{\mu} - y_{\mu}| \right) \\ & < \frac{2}{b-1} \left(\sum_{l=1}^n \frac{b_l c_{2l-1}}{\sinh \theta_l} \frac{1}{(1 - e^{-\theta_l})} \right) \\ & \quad \times \exp \left(-\frac{\theta_1}{2a} \sum_{\mu} |x_{\mu} - y_{\mu}| \right), \end{aligned} \quad (23)$$

where the last inequality is due to the property ($\theta_1 < \theta_2 < \dots < \theta_n$) from (18), since ($0 < c_1 < c_3 < \dots < c_{2n-1}$) from (14). From (11)–(13), and (23), it follows that D (11) is exponentially-local, and so is $D(m_q)$ (10).

Obviously, the above analysis also ensures that the overlap Dirac operator with $(H_w^2)^{-1/2}$ approximated by Zolotarev optimal rational polynomial is exponentially-local. Note that, in practice, the least upper bound of (23) is much smaller than that on the r.h.s. of (23). This implies that $D(x, y)$ is still exponentially local even for rough gauge configurations bounded by an ϵ much larger than its least upper bound in (20).

Further, it is straightforward to assert the differentiability of D (11) with respect to the gauge field, as well as the exponential locality of D in the presence of an isolated near-zero modes of H_w^2 , thus the details can be omitted.

Finally, it is instructive to observe that the decay constant θ_1 in (23) is a monotonic decreasing function of N_s . In the limit $N_s \rightarrow \infty$, $c_1 \rightarrow 0$, and θ_1 becomes the minimum,

$$\theta = \cosh^{-1} \left(\frac{b+1}{b-1} \right) = \cosh^{-1} \left(\frac{\lambda_{\max}^2 + \lambda_{\min}^2}{\lambda_{\max}^2 - \lambda_{\min}^2} \right).$$

In other words, at any finite N_s , $D(x, y)$ is more localized than itself at $N_s = \infty$. In the limit $\omega_s = 1$, the effective 4D lattice Dirac operator is equivalent to the overlap Dirac operator with polar approximation, and c_1 is replaced with $d_1 = \tan^2(\pi/2N_s)$, thus its decay constant (θ_1) is always smaller (i.e., less localized) than that of the optimal DWF, for any finite N_s .

For the conventional DWF with open boundary conditions, it has been estimated [8] that the effective 4D lattice Dirac operator at $N_s = \infty$ behaves as

$$\|D_{\text{eff}}(x, y)\| \lesssim \left(\sum_{\mu} |x_{\mu} - y_{\mu}| \right) \exp \left(-\frac{\tilde{\theta}}{2a} \sum_{\mu} |x_{\mu} - y_{\mu}| \right).$$

However, for finite N_s , the decay constant of $D_{\text{eff}}(x, y)$ has not been obtained. Note that the factor $(\sum_{\mu} |x_{\mu} - y_{\mu}|)$ in $D_{\text{eff}}(x, y)$ makes it decay slower than purely exponential decay. Nevertheless, this is only an artifact due to the expansion [8] in terms of Chebycheff polynomials of the second kind. If one expands the fraction in terms of Chebycheff polynomials of the first kind as (16), then one obtains a purely exponential bound,

$$\|D_{\text{eff}}(x, y)\| \lesssim \exp \left(-\frac{\tilde{\theta}}{2a} \sum_{\mu} |x_{\mu} - y_{\mu}| \right).$$

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