Ginsparg-Wilson relation with $R = (a\gamma_5 D)^{2k}$ Ting-Wai Chiu^{a*}^aDepartment of Physics, National Taiwan University, Taipei, Taiwan 106, Republic of China

The Ginsparg-Wilson relation $D\gamma_5 + \gamma_5 D = 2aDR\gamma_5 D$ with $R = (a\gamma_5 D)^{2k}$ is discussed. An explicit realization of D is constructed. It is shown that this sequence of topologically-proper lattice Dirac operators tend to a nonlocal operator in the limit $k \rightarrow \infty$. This suggests that the locality of a lattice Dirac operator is irrelevant to its index.

1. INTRODUCTION

In general, the Ginsparg-Wilson relation [1] for a lattice Dirac operator D can be written as

$$D\gamma_5 f(D) + g(D)\gamma_5 D = 0, \quad (1)$$

where f and g are analytic functions. Then the fermionic action $\mathcal{A}_f = \bar{\psi} D \psi$ is invariant under the generalized chiral transformation

$$\psi \rightarrow \exp[\gamma_5 \theta f(D)] \psi, \quad (2)$$

$$\bar{\psi} \rightarrow \bar{\psi} \exp[\theta g(D)\gamma_5], \quad (3)$$

where θ is a global parameter. In particular, for $f(D) = \mathbf{1} - aRD$ and $g(D) = \mathbf{1} - aDR$, (1) becomes

$$D\gamma_5 + \gamma_5 D = aD(R\gamma_5 + \gamma_5 R)D, \quad (4)$$

where R is any operator. Since $(R\gamma_5 + \gamma_5 R)$ commutes with γ_5 , without loss, we can set R to commute with γ_5 ,

$$R\gamma_5 = \gamma_5 R. \quad (5)$$

Then (4) becomes the usual form of the GW relation

$$D\gamma_5 + \gamma_5 D = 2aDR\gamma_5 D. \quad (6)$$

In the continuum, the massless Dirac operator $D = \gamma_\mu(\partial_\mu + iA_\mu)$ is chirally symmetric and anti-hermitian, i.e.,

$$D\gamma_5 + \gamma_5 D = 0, \quad (7)$$

$$D^\dagger = -D, \quad (8)$$

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which imply that D is γ_5 -hermitian, i.e., $D^\dagger = \gamma_5 D \gamma_5$.

On the lattice, a lattice Dirac operator D cannot satisfy (7) or (8) without losing its other essential properties. However, we can still require D to satisfy the γ_5 -hermiticity,

$$D^\dagger = \gamma_5 D \gamma_5. \quad (9)$$

Multiplying γ_5 on both sides of (6), and using (9), we obtain $D^\dagger + D = 2aD^\dagger R D$, which gives $R^\dagger = R$.

Thus we conclude that, in general, R is a Hermitian operator commuting with γ_5 . From (6), the GW fermion propagator (in the trivial sector) satisfies $D^{-1}\gamma_5 + \gamma_5 D^{-1} = 2aR\gamma_5$, which immediately suggests that R should be as local as possible, i.e., $R \simeq \mathbf{1}$. Otherwise, it may cause additive mass renormalization to D^{-1} , which of course is undesirable. In fact, it has been demonstrated that the most optimal R is $R = r\mathbf{1}$ with some constant r [2].

Nevertheless, theoretically, it is rather interesting to consider R as a function of D , and to see how D behaves with respect to R . It turns out that, in general, the independent variable in the functional form of R must be $(a\gamma_5 D)^2$, if D is normal and γ_5 -hermitian. In particular, for $R(t) = t^n$, $n = 0, 1, \dots$, the sequence of topologically-proper D (27) tend to a nonlocal Dirac operator in the limit $n \rightarrow \infty$. This suggests that the locality of a lattice Dirac is irrelevant to its index.

First, recall that for any D which is normal ($DD^\dagger = D^\dagger D$) and γ_5 -hermitian, its eigensystem ($D\phi_s = \lambda_s \phi_s$) has the following properties [3]:

- (i) eigenvalues are either real or come in complex conjugate pairs.
- (ii) the chirality ($\chi_s = \phi_s^\dagger \gamma_5 \phi_s$) of any complex eigenmode is zero.
- (iii) each real eigenmode has a definite chirality.
- (iv) sum of the chirality of all real eigenmodes is zero (chirality sum rule).

Assume that the eigenvalues of D lying on a simple closed contour in the complex plane. Then (i) implies that the real eigenvalues of D can only occur at two different points, say at zero and a^{-1} . Then the chirality sum rule reads

$$\sum_{\lambda_s=0, a^{-1}} \phi_s^\dagger \gamma_5 \phi_s = 0, \quad (10)$$

which immediately gives

$$n_+ - n_- + N_+ - N_- = 0. \quad (11)$$

where $n_\pm$ denotes the number of zero modes of $\pm$ chirality, and $N_\pm$ the number of nonzero real (a^{-1}) eigenmodes of $\pm$ chirality. Then we immediately see that any zero mode must be accompanied by a real (a^{-1}) eigenmode with opposite chirality, and the index of D is

$$\text{index}(D) \equiv n_- - n_+ = -(N_- - N_+). \quad (12)$$

It should be emphasized that the chiral properties (ii), (iii) and the chirality sum rule (11) hold for any normal D satisfying the γ_5 -hermiticity, as shown in Ref. [3]. However, in nontrivial gauge backgrounds, whether D possesses any zero modes or not relies on the topological characteristics [11] of D , which cannot be guaranteed by the conditions such as the locality, free of species doublings, correct continuum behavior, γ_5 -hermiticity and the GW relation.

Therefore, we must require that D is normal and γ_5 -hermitian,

$$D^\dagger D = D D^\dagger, \quad D^\dagger = \gamma_5 D \gamma_5.$$

These two equations immediately give

$$\gamma_5 (\gamma_5 D)^2 = (\gamma_5 D)^2 \gamma_5. \quad (13)$$

Thus we have found an example of $R = (a\gamma_5 D)^2$ which is Hermitian and commutes with γ_5 . In

general, R can be any analytic function of $(a\gamma_5 D)^2$, i.e.,

$$R = f[(a\gamma_5 D)^2], \quad f: \text{analytic function}. \quad (14)$$

In particular, for $f(x) = x^n$, it becomes

$$R = (a\gamma_5 D)^{2k}, \quad k = 0, 1, 2, \dots \quad (15)$$

Substituting (15) into the GW relation (6), we obtain

$$\gamma_5 D + D \gamma_5 = 2aD(a\gamma_5 D)^{2k} \gamma_5 D, \quad (16)$$

which is equivalent to Fujikawa's proposal [4]

$$\gamma_5 (\gamma_5 D) + (\gamma_5 D) \gamma_5 = 2a^{2k+1} (\gamma_5 D)^{2k+2}. \quad (17)$$

2. A CONSTRUCTION OF D

So far, the only viable way to construct a topologically-proper lattice Dirac operator is the Overlap [5]

$$D = \frac{1}{2a} (\mathbf{1} + \gamma_5 \epsilon), \quad \epsilon^2 = \mathbf{1} \quad (18)$$

which satisfies the GW relation (6) with $R = 1$. Since D (18) is normal and γ_5 -hermitian, its eigenmodes satisfy the chiral properties (i)-(iv). There are many different ways to implement the Hermitian ϵ in (18). However, it is required to be able to capture the topology of the gauge background. That means, one-half of the difference of the numbers ($h_\pm$) of positive (+1) and negative (-1) eigenvalues of ϵ is equal to the background topological charge Q ,

$$\sum_x \text{tr}(a\gamma_5 D(x, x)) = \frac{1}{2} \sum_x \text{tr}(\epsilon) = \frac{1}{2} (h_+ - h_-) = Q$$

where tr denotes the trace over the Dirac and color space. Otherwise, the axial anomaly of D cannot agree with the topological charge density in a nontrivial gauge background.

An explicit realization of ϵ in (18) is the Neuberger-Dirac operator [6] with

$$\epsilon = \frac{H_w}{\sqrt{H_w^2}} \quad (19)$$

where

$$H_w = \gamma_5 (\gamma_\mu t_\mu + W - m_0 a^{-1}), \quad 0 < m_0 < 2 \quad (20)$$

$$t_\mu(x, y) = \frac{1}{2a} [U_\mu(x)\delta_{x+\hat{\mu},y} - U_\mu^\dagger(y)\delta_{x-\hat{\mu},y}] \quad (21)$$

$$W(x, y) = \frac{1}{2a} \sum_\mu [2\delta_{x,y} - U_\mu(x)\delta_{x+\hat{\mu},y} - U_\mu^\dagger(y)\delta_{x-\hat{\mu},y}]$$

To generalize this construction for $k > 0$, we multiply both sides of the GW relation (6) by R and redefine $D' = RD$, then we have

$$D'\gamma_5 + \gamma_5 D' = 2D'\gamma_5 D' . \quad (22)$$

It can be shown that D' is γ_5 -hermitian. Now (22) is in the same form of the GW relation with $R = 1$. Thus, one can construct D' in the same way as the Overlap

$$D' = RD = \frac{1}{2a}(\mathbf{I} + \gamma_5 \epsilon), \quad \epsilon^2 = \mathbf{I}, \quad (23)$$

provided that a proper realization of ϵ can be obtained. Using (5), we obtain

$$D' = RD = a^{-1}\gamma_5(a\gamma_5 D)^{2k+1}, \quad (24)$$

which immediately yields

$$D = a^{-1}\gamma_5(a\gamma_5 D')^{1/(2k+1)}, \quad (25)$$

where the $(2k+1)$ -th real root of the Hermitian operator $(a\gamma_5 D')$ is assumed. Then (25) suggests that if the ϵ in (23) is expressed in terms of a Hermitian operator H , i.e., $\epsilon = \frac{H}{\sqrt{H^2}}$, then H is required to behave as $(\gamma_\mu D_\mu)^{2k+1}$ in the continuum limit such that D can become $\gamma_\mu D_\mu$ after taking the $(2k+1)$ -th root in (25). Thus, H must contain the term $(\gamma_\mu t_\mu)^{2k+1}$, where $\gamma_\mu t_\mu$ is the naive lattice fermion operator defined in (21). Then another term must be added in order to remove the species doublers in $(\gamma_\mu t_\mu)^{2k+1}$. So, we add the Wilson term to the $(2k+1)$ -th power, i.e., W^{2k+1} . Finally, a negative mass term $-\frac{1}{2}a^{-(2k+1)}$ is inserted such that ϵ is able to detect the topological charge Q of the gauge background, i.e.,

$$\frac{1}{2} \sum_x \text{tr} \left(\frac{H}{\sqrt{H^2}} \right) = Q. \quad (26)$$

Putting all these terms together, we have

$$H = \gamma_5 \left[(-1)^k (\gamma_\mu t_\mu)^{2k+1} + W^{2k+1} - \frac{a^{-(2k+1)}}{2} \right]$$

which, at $k = 0$, reduces to the H_w in Eq. (20). Then (25) can be rewritten as

$$D = a^{-1} \left(\frac{1}{2} \right)^{\frac{1}{(2k+1)}} \gamma_5 \left(\gamma_5 + \frac{H}{\sqrt{H^2}} \right)^{\frac{1}{(2k+1)}} \quad (27)$$

which agrees with Fujikawa's construction [4]. Note that (27) can also be written in the form $D = D_c(\mathbf{I} + aRD_c)^{-1}$, where D_c is dependent on the order k . The general properties of D have been derived in [7]. In particular, the index of D is independent of the order k , and the eigenvalues of D fall on a closed contour with their real parts bounded between zero and a^{-1} . In Fig. 1, the eigenvalues of D are plotted for $k = 0, 1, 2$ respectively, with the same background gauge field.

3. IN THE LIMIT $k \rightarrow \infty$

One of the salient features of the sequence of topologically-proper Dirac operators (27) is that the amount of chiral symmetry breaking (i.e., r.h.s. of (6)) decreases as the order k increases. However, at finite lattice spacing, the chiral symmetry breaking of D cannot be zero even in the limit $k \rightarrow \infty$, since D satisfies the GW relation (6). In the limit $k \rightarrow \infty$, the only possibility for D to break the chiral symmetry is that $D(p)$ becomes a piecewise continuous function in the Brillouin zone, with discontinuities somewhere at $|p| \simeq a^{-1}$, and $\gamma_5 D(p) + D(p)\gamma_5 = 0$ for $|p| < a^{-1}$. Since such a $D(p)$ is non-analytic at infinite number of p , the corresponding $D(x, y)$ must be nonlocal in the position space.

If $D(x, y)$ is nonlocal in the free fermion limit (i.e., $U_\mu(x) \rightarrow 1$), then it must be nonlocal in any smooth gauge background. So, it suffices to examine the non-analyticity of $D(p)$ in the free fermion limit.

In the free fermion limit, the GW Dirac operator (27) in momentum space can be written as

$$D(p) = D_0(p) + i\gamma_\mu D_\mu(p), \quad (28)$$

where

$$D_0(p) = a^{-1} \left[\frac{1}{2} \left(1 + \frac{u(p)}{N(p)} \right) \right]^{\frac{k+1}{2k+1}},$$

$$D_\mu(p) = D_0(p) \sqrt{\frac{1}{a^2 t^2(p)} \left(\frac{N(p) - u(p)}{N(p) + u(p)} \right)} \sin(p_\mu a),$$

Figure 1. The eigenvalues of D in a nontrivial gauge background for $k = 0, 1, 2$ respectively.

$$u(p) = a^{-(2k+1)} \left(\left[\sum_{\mu} (1 - \cos(p_{\mu}a)) \right]^{2k+1} - \frac{1}{2} \right)$$

$$t^2(p) = a^{-2} \sum_{\mu} \sin^2(p_{\mu}a),$$

$$N(p) = \sqrt{[t^2(p)]^{2k+1} + [u(p)]^2}.$$

At the zeroth order ($k = 0$), $D(p)$ is analytic in the Brillouin zone. However, in the limit $k \rightarrow \infty$, $D(p)$ tends to a nonanalytic function with discontinuities at each point on two concentric hypersurfaces inside the Brillouin zone [8]. In four dimensions, the inscribed hypersurface is specified by the equation

$$\sum_{\mu=1}^4 s_{\mu}^2 = 1,$$

while the circumscribing one by

$$6 + \sum_{\mu=1}^4 c_{\mu}(c_{\mu} - 4) + \sum_{\mu=1}^3 \sum_{\nu>\mu}^4 c_{\mu}c_{\nu} = 0,$$

where $c_{\mu} = \cos(p_{\mu}a)$ and $s_{\mu} = \sin(p_{\mu}a)$. This implies that $D(x, y)$ is nonlocal in the limit $k \rightarrow \infty$.

On the other hand, $D(x, y)$ is local at the zeroth order ($k = 0$), for gauge backgrounds which are sufficiently smooth at the scale of the lattice spacing [9,10]. Since the index of D (27) is independent of the order k , this suggests that the locality of a lattice Dirac operator is irrelevant to its index. In other words, if a GW Dirac operator which is γ_5 -hermitian, doubler-free, and has correct continuum behavior in the free fermion limit, *but* always has zero index [11], then the cause may *not* be due to its nonlocality.

To summarize, we have provided a nontrivial example of D (27), which is *topologically-proper* but tends to a *nonlocal* operator in the limit $k \rightarrow \infty$. Unlike the nonlocal and chirally-symmetric Dirac operator

$$D_c = a^{-1} \frac{\sqrt{H_w^2} + \gamma_5 H_w}{\sqrt{H_w^2} - \gamma_5 H_w} \quad (29)$$

which has poles (i.e., the nonzero real eigenmodes) for nontrivial background gauge fields (as a consequence of the chirality sum rule [3]), the present example (27) breaks the chiral symmetry according to the GW relation, thus its nonzero real (a^{-1}) eigenmodes are well-defined for any order k .

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