Catastrophe theory from a pedestrian’s point of view

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Catastrophe theory was originally developed as a sort of all-encompassing theory to describe certain singular behavior of a system. Though losing its momentum and competition to the “new” chaos theory, catastrophe theory nevertheless provides us with a unified description on why very different systems might share similar features. We give a brief, intuitive introduction to the basic ideas of catastrophe theory.

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Speaking of catastrophe theory, one cannot help forming a vivid image of a bunch of poor dinosaurs running about for their lives when a huge meteor hit the “wrong” earth at the “wrong” time. Such was the case when my third-grader son saw me preparing the slides for this lecture. Obviously someone has visited too many a museum of natural sciences. Nope, the catastrophe theory we will be talking about does not cause any environmental damages. Rather, it is a mathematical theory put forth by R. Thom in the 1960’s and, after the publication of his famous book “Structural Stability and Morphogenesis” [1] and through the vigorous crusades of Zeeman[2], became quite a fashionable term in the 1970’s. Even though the theory is quite general in a sense, it did not thrive to the extent as its proponents would like it to, perhaps because the “new” chaos theory turned out to contain a richer set of dynamical behavior as well as to be able to deal with things in a more quantitative manner. Though catastrophe theory seems to have waned quite a bit, it never loses its charm in providing us with a somewhat universal description of phenomena arising in both natural and social sciences.

So, what is catastrophe theory? Instead of giving a definition, let us start with a famous example: The caustic (brightly focused) pattern observed in a coffee mug (Fig. 1). If we look at the simplified two-dimensional model of Fig. 2, which shows parallel rays horizontally impinging upon a circular mirror (= wall of the mug) of unit radius from the left, we immediately identify the source of this caustic pattern: It is there because the rays reflected off the mirror obviously depend on the “impact parameter” \(u\). Thus, we may conveniently think of the points on each ray as being described by some function \(f(x, y, u) = 0\). Because a given point \((x, y)\) on the envelope corresponds to the bunching of many nearby rays with slightly different \(u\), and because all these rays obey \(f(x, y, u) = 0\) (for the same \(x\) and \(y\) but not necessarily the same \(u\)), it must be true that
\( \partial f(x, y, u) / \partial u = 0 \). In principle, we may use \( f = 0 \) and \( \partial f / \partial u = 0 \) to eliminate \( u \) and obtain an equation \( g(x, y) = 0 \). This equation then describes the envelope, or the caustic in our context. Instead of using \( u \) as the parameter for the family of incident rays, we may just as well use the angle \( \theta \) of Fig. 2 for the job, and this turns out to be a more convenient choice. It is quite straightforward to show that the points \((x, y)\) on a given reflected ray satisfy

\[
-x \sin 2\theta - y \cos 2\theta - \sin \theta = 0.
\]

Differentiating this equation with respect to \( \theta \) and setting it to zero, we may solve for the caustic. The answer, in parametric form using \( \theta \) as a parameter, is:

\[
\begin{align*}
x &= \frac{3}{2} \cos \theta - \cos^3 \theta, \\
y &= \sin^2 \theta.
\end{align*}
\]

This equation shows that, as \( \theta \) becomes very small, \( y \) goes like \( \theta^3 \) whereas \( x \) deviates from \( 1/2 \) like \( \theta^2 \):

\[
\begin{align*}
x - \frac{1}{2} &\approx \frac{3}{4} \theta^2, \\
y &\approx \theta^3.
\end{align*}
\]
FIG. 2: A simplified two-dimensional model illustrating how the caustic pattern in a mug is generated. Horizontally impinging rays coming from the left strike the circular mirror and get reflected to form an envelope. Each of the reflected ray can be described by some function \( f(x, y, u) = 0 \).

This is why the caustic develops a cusp near \((x, y) = (1/2, 0)\), which corresponds to \(\theta = 0\).

We now look at the same problem in a way that admits generalizations. First of all, we note that a ray actually is a dynamical thing, because it propagates in space as time passes. Thus, we should write the position vector \(\vec{x}\) of a particular ray as some function \(\vec{h}\) of the time \(t\) and the parameter \(u\). In abstract notations, we thus have

\[
\vec{x} = \vec{h}(u, t).
\]

Quite generally, \(\vec{h}\) will not always be an invertible function, because light rays may cross or even form caustics. But what if we “lift” the trajectories up to a higher-dimensional space? Wouldn’t this somehow remove the multi-valuedness of \(\vec{h}\)? Let’s see.

Defining

\[
\vec{X} \equiv (\vec{x}, u) = (\vec{h}(u, t), u),
\]

then we have a two-dimensional surface in the three-dimensional space of \((\vec{x}, u)\). Next, we investigate if this surface is regular or not. To do this, we need to check if there is any combination in the small variations of \(u\) and \(t\) such that we get no net change in \(\vec{X}\). Thus, we need to consider

\[
d\vec{X} = (d\vec{h}, du) = \left(\frac{\partial \vec{h}}{\partial u} du + \frac{\partial \vec{h}}{\partial t} dt, du\right).
\]

Now, if there is any combination in \(du\) and \(dt\) such that \(d\vec{X} = 0\), then the above expression implies that it must be true that \(du = 0\). This then implies \(\partial \vec{h}/\partial t = 0\). But \(\partial \vec{h}/\partial t\) actually
is the velocity of the light ray in the physical space, which under normal conditions should not vanish. Hence, we conclude that \( \vec{X} \) as parametrized by \( u \) and \( t \) (Eqn.3) is quite regular: locally the surface will not “wrinkle” or do anything weird. In fact, for the problem at hand, this implies that the surface normal \( (\partial \vec{X} / \partial u, \partial \vec{X} / \partial t) \) is always defined.

As a concrete example of the previous idea, we may go back to the coffee mug and “lift” the caustic, as the model in Fig. 3 shows. The pictures shown here are snapshots of a short movie I produced. Basically, what I did was use sticks to represent the rays reflected off the circumference of the mug, lift them up in the direction perpendicular to all the sticks (Fig. 3-(a)), then rotate the model to show you how they look in the three-dimensional space (Figs. 3-(b) through 3-(e)). In Fig. 3-(f) I have inserted a surface conforming to the sticks to guide you with the visual; and in later frames I rotated the whole thing to a different orientation, then dropped the sticks altogether, leaving only the surface (Fig. 3-(g)). Finally, I inserted a board to cut through the model at different locations (Fig. 3-(h) and Fig. 3-(i)) to show you the various cross-sections of the surface. The last picture, Fig. 3-(j), is identical to Fig. 3-(a): we are looking straight down at the “lifted” model, except that it is showing the surface without the sticks.

But why did we bother going through all this trouble just to “lift” the rays into forming a surface? Well, we know that a surface with a well-defined normal has no “kinks”
or “wrinkles” whatsoever and hence can be viewed as locally dividing the three-dimensional space into an “upper half” and a “lower half.” As such, we may devise a function $f(\vec{x}, u)$ of the point $(\vec{x}, u)$ so that it is identically zero on the surface and its gradient, $\nabla f$, coincides with the surface normal. Then, the lifted surface is characterized by $f = 0$, whereas the caustic corresponds to $\partial f / \partial u = 0$, because we can have rays with slightly different $u$’s all meeting at the same point $\vec{x}$. To summarize: The surface we are looking at is consisted of points satisfying $f = 0$; and the caustic corresponds to $\partial f / \partial u = 0$.

As a more complicated but realistic example, we may consider a two-dimensional $\vec{u} \equiv (u, v)$. This is realized in, say, the formation of rainbows when the rays emitted by the sun undergo a series of reflection and refraction when they hit a raindrop (Fig. 4). Here, the light source is not a line (as modeled above for the coffee mug) but rather an area, thus the two-dimensional variable $\vec{u}$. Also, $\vec{x}$ is a three-dimensional vector in the present case. If we “lift” the “surface” as we did before by introducing $\vec{X} \equiv (\vec{x}, \vec{u}) = (\vec{h}(\vec{u}, t), \vec{u})$, then we will have a three-dimensional “surface” embedded in a five-dimensional space. Again, the “surface” can be characterized by $f = 0$ for some function $f(\vec{x}, \vec{u})$, and the caustic is determined by the condition that the directional derivative of $f$ along some direction in the $\vec{u}$-space vanishes. When we do find a caustic satisfying this condition, it will be more convenient for us to locally change the variables from $(u, v)$ to some $(u', v')$ such that on the caustic we have $\partial f / \partial u' = 0$ and $\partial f / \partial v' \neq 0$, the latter condition being generically possible. When this latter condition is indeed satisfied, however, we see that the variable $v'$ is inessential for the investigation of the local behavior of caustics and thus can be dropped. Then in a sense we are back to the situation with the coffee mug in which we only need to deal with one “relevant” or “essential” variable $u'$. The moral: Even if a system might possess a lot of variables, a judicious choice of variables often kills most of them, leaving only a very small number of the relevant variables. But if things do not go that well, and we happen to bump up with $\partial f / \partial u' = 0$ and $\partial f / \partial v' = 0$, then a degeneracy has occurred. This, however, can still be satisfactorily dealt with in catastrophe theory, as will be seen later.

If the conditions $f = 0$ and $\partial f / \partial u = 0$ vaguely remind you of something resembling
the requirement for the static equilibrium configuration of a mechanical system, then congratulations! You are right on! Because the stable static configuration of a mechanical system necessarily corresponds to a state with minimal potential energy, we may think of $f$ as the force or the torque acting on the object, so that equilibrium configurations correspond to $f = 0$, while $\partial f/\partial u = 0$ then represents the coexistence of two admissible equilibrium configurations. When this happens, typically the object is on the verge of selecting one (stable) configuration over another (unstable) configuration so that one might observe an “exchange of stability.” As an illustration, we may consider how a toy block floats on the surface of water: Its orientation depends crucially on the relative density of the block with respect to water, and also on the aspect ratio of the block. For instance, Fig. 5 shows how a two-dimensional square block floats in water when its specific weight is 0.15, 0.23, 0.27, and 0.90, respectively. Just how the block switches from one configuration (upright) to another (tilted) as a function of the specific weight is not our concern here. Suffices to say that the analysis can be facilitated with the use of catastrophe theory (for related discussions, please see [3]).

Let’s get back to the coffee mug. A point $(x_0, y_0)$ sitting right on the caustic corresponds to a point $(x_0, y_0, u_0)$ on the lifted surface which satisfies $f = 0$ and $\partial f/\partial u = 0$. Therefore, slightly away from this point the function $f$ generally behaves like

$$f = -a(\Delta u)^2 + b\Delta x + c\Delta y + \ldots,$$

where $\Delta$ refers to a small change (for instance, $\Delta u \equiv u - u_0$), and $a$, $b$, and $c$ are some numbers. To investigate what the lifted surface looks like near $(x_0, y_0, u_0)$, we set $f = 0$. Then, after setting $(b\Delta x + c\Delta y)/a \equiv \eta$ and, we may write the equation of the surface as

$$(\Delta u)^2 \approx \eta.$$

This is shown in Fig. 6(a), in which an “inessential” coordinate $\xi$ is also drawn to remind ourselves that the surface does not change in any qualitative way along other directions. Glancing at this picture, I guess you won’t be surprised to find that people call it a “fold.”
FIG. 6: (a) The geometry of the lifted surface generally takes up the shape of a “fold” near a caustic. Along the direction of the so-called inessential variable ($\xi$ in the figure) the shape of the surface does not change in any qualitative way. (b) Identifying the “folds” (dashed curves) on the lifted surface for the coffee mug caustics.

In Fig. 6(b), we see clearly how the “folds” (dashed curves) on the lifted surface correspond to the coffee mug caustics.

All this is fine. But one cannot help noticing that there is certain point on the coffee mug caustics of Fig. 1 which appears to be the coalescence of two fold-type caustics discussed above. Clearly, things get more degenerate near this point. How do we deal with this situation?

Actually, all we have to do is go ahead with an even higher-order Taylor expansion! Firstly, we know that $f = 0$, $\partial f / \partial u = 0$ and $\partial^2 f / \partial u^2 = 0$ at this degenerate point. Then, slightly away from this point we may expand $f$ into

$$f \approx -a(\Delta u)^3 + (b_1 \Delta x + c_1 \Delta y)(\Delta u)^2 + (b_2 \Delta x + c_2 \Delta y)\Delta u + (b_3 \Delta x + c_3 \Delta y),$$

where $a \neq 0$ and $b_j$’s and $c_j$’s are some numbers. Again, identifying $(b_3 \Delta x + c_3 \Delta y)/a \equiv \eta$ and $(b_2 \Delta x + c_2 \Delta y)/a \equiv \xi$, we see that

$$f \approx a \left( -(\Delta u)^3 + (b_4 \xi + c_4 \eta)(\Delta u)^2 + \xi \Delta u + \eta \right)$$

for some constants $b_4$ and $c_4$. Clearly, the $(\Delta u)^2$ term is much smaller than the two terms following it and hence can be safely ignored. Thus, the function $f$ may be effectively treated as

$$f \propto -(\Delta u)^3 + \xi \Delta u + \eta,$$

and the lifted surface $f = 0$ has a local behavior described by

$$-(\Delta u)^3 + \xi \Delta u + \eta = 0.$$
which is shown in Fig. 7. We see clearly that the origin \((\xi, \eta, \Delta u) = (0, 0, 0)\) is the merging point of two folds (Folds 1 and 2), as expected. This surface, being one of the most frequently encountered in catastrophe theory, is called a “cusp.” Attempts trying to apply catastrophe theory to give a qualitative account of the singular behavior one observes in different fields usually are based on this “cusp catastrophe.” Because of this, it really deserves a closer inspection. But before embarking on this, we should quickly point out its relevance to our coffee mug caustics: Looking down at the surface from the positive \(\Delta u\) axis, we see that the caustics correspond to

\[
-(\Delta u)^3 + \xi \Delta u + \eta = 0,
\]

\[
\frac{\partial}{\partial u} \left( -(\Delta u)^3 + \xi \Delta u + \eta \right) = 0,
\]

which can be solved to yield

\[
\xi = 3(\Delta u)^2, \tag{7}
\]

\[
\eta = -2(\Delta u)^3. \tag{8}
\]

Comparing Eqns.7 and 8 with Eqns.1 and 2 for the coffee mug caustics, we see that the prediction of this general theory agrees with the solution for the actual problem.

At this point you may be wondering why we would use catastrophe theory to re-examine the coffee mug caustics for which the exact solution is readily available. The reason is simple: Catastrophe theory goes beyond what the exact treatment tells us! For instance, without doing the much harder exact analysis for the more general case when the incident rays are somewhat tilted or when the wall of the mug deviates slightly from a perfect circular shape, there really is no telling if the cusp caustics would still be there if the aforementioned changes were made. In other words, the approach of solving for the exact solution cannot tell if the observed phenomenon is generic in nature. And yet catastrophe theory as we presented above shows that the answer is positive, because what
we did was a series of reductions and simplifications of the most general perturbations of the special case. To summarize: *In one scoop, catastrophe theory considers all the possible small configurations deviating from the given degenerate case.* It is in this respect that catastrophe theory is truly powerful and useful.

Now, we examine the geometry of the cusp catastrophe. Two salient features of the cusp catastrophe are the phenomena of hysteresis and discontinuous jump as we vary the “control parameters” $\xi$ and $\eta$. To motivate, we recall that for a mechanical system the function $f$ often represents a sort of restoring force or restoring torque. Using this analogy, the sign of $f$ as a function of the variable $\Delta u$ is shown in Fig. 8 in virtue of Eqn.5. Since a positive $f$ drives $\Delta u$ upward while a negative $f$ does the opposite, we see that the various parts of the S-shaped cross-section of the surface acquires different stability. In particular, the dotted segment sandwiched between the top and the bottom branches of S is unstable.

What this implies is that if we fix the control parameter $\eta$ and continuously increase $\xi$ from Point A to Point B on the surface, then a further increase in $\xi$ will instantly force the system into jumping from Point B to Point C, as Fig. 9 shows. Indeed, it is because of this possibility of having a drastic change in the system configuration when one continuously varies the control parameters of a system that the theory derives its name.

Once we are at Point C, we may again decrease $\xi$, but only to find that the system now follows the path $C \rightarrow D$ without retracing the path it came from. And, having reached D, a further decrease in $\xi$ will force the system into jumping straight down from D to the
FIG. 9: For fixed $\eta$, a continuous change in $\xi$ will make the system go from Point $A$ to Point $B$ continuously. But upon further increase in $\xi$, the system will jump abruptly from Point $B$ to Point $C$. Decreasing $\xi$ from Point $C$ does not make the system retrace where it came from. Instead, it goes from $C$ to $D$, and a further decrease in $\xi$ will make the system to jump straight down to the lower branch of the S-shaped cross-section.

lower branch of the S-shaped cross-section. This behavior is analogous to the hysteresis loop one encounters in magnetic systems. Because physicists are familiar with this type of behavior in the study of the phase transitions of matter, it probably comes as no surprise that catastrophe theory also finds its way in thermal physics, particularly in the Ginzburg-Landau theory of phase transitions. However, since we now know that this type of mean field theory usually does not correctly reproduce the critical behavior near the transition region, we probably should not put that much emphasis on it.

Because catastrophe theory shows that a simple but generic system can exhibit both continuous and abrupt changes as one “tunes” the system parameters $\xi$ and $\eta$, people have been trying to apply it to a host of vastly different disciplines. A famous, albeit somewhat controversial, example is Zeeman’s account of the aggressive behavior in the dog. (For a simple description, please see [5].) However interesting the idea may seem, I am a bit reserved in this type of applications, because catastrophe theory does give certain quantitative predictions, such as how the control parameters must scale with each other near a degenerate point, as Eqns.7 and 8 exemplify, whereas one usually does not even have a hold of how to define the control parameters for this type of systems.

Lest you think that cusp catastrophe is all there is to catastrophe theory, we must quickly add that higher order catastrophes are certainly possible. For instance, patterning after the same kind of reductions we performed to arrive at Eqn.5, we easily obtain

$$f \propto (\Delta u)^4 + \xi(\Delta u)^2 + \eta(\Delta u) + \zeta$$
FIG. 10: Caustics of a higher order nature: the swallowtail.

FIG. 11: (a) Caustic pattern caused by a full moon shining through a window. (b) The surface of the window has many bumps each acting like a small magnifying glass.

for the case which is one order higher in degeneracy than cusp catastrophe. This time, the caustics in the space of the control parameters \((\xi, \eta, \zeta)\) form a surface, in contrast to our two-dimensional coffee cup model, which has curves as the caustics. Fig. 10 shows this surface, which is aptly termed “swallowtail” for an obvious reason, but (interestingly) by the blind mathematician B. Morin[6]. Going one order even higher, and we have something called “butterfly,” though the geometry must always be projected down to our mere three-dimensional space, which makes it rather difficult to visualize.

As said before, we do not have to restrict ourselves to only a scalar variable \(u\), as most optical phenomena typically involve at least two variables \(u\) and \(v\) because (two-dimensional) area lights are usually the source of illumination. Catastrophe theory also gives a rather clear classification here. Instead of adding more headache to my audience by
copying extra formulas, let me just show you a photo I took on the night of the Full-Moon Festival of 2006, when I was seriously still struggling for what to show for higher order catastrophes for this talk. Having worked all day on the PowerPoint for this lecture, I was truly amazed by the caustic demonstration right there on my bathroom window when I took a break. Here in Fig. 11(a) we are looking at the full moon through the window whose surface is consisted of small rectangular bumps each acting like a magnifying glass (Fig. 11(b)). This caustic pattern can be analyzed. But, with a beautiful bright full moon outside waiting to be marveled at and savored, who cares?

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