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Variations on a theme by a singing wineglass

KUAN-WEN CHEN¹, CHIH-KAI WANG¹, CHIN-LING LU¹ and YIH-YUH CHEN^{2,3,4}

¹ *Hong-wen Senior High School - Taichung, Taiwan, ROC*

² *Physics Department, National Taiwan University - Taipei, Taiwan, ROC*

³ *Institute of Astrophysics, National Taiwan University - Taipei, Taiwan, ROC*

⁴ *National Center of Theoretical Sciences at Taipei - Taipei, Taiwan, ROC*

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Abstract. – A “singing wineglass” is a glass which generates a more or less pure tone when one rubs its rim with a wet finger. The frequency of this tone is lowered when water or wine is poured into the glass. One usually associates this downward frequency shift with an effective mass contributed by the liquid to the vibrating wineglass. We present experimental work and simplified theoretical models to show that this idea requires modifications for a couple of counter-intuitive phenomena.

Undoubtedly, using a wet finger to rub the rim of a wineglass to produce a sound of more or less pure tone must have been a fun and rewarding experience for many party guests. So much so that Benjamin Franklin even modified the idea and invented “armonica” for one’s listening enjoyment [1]. Even scientists could not resist the charm and had also studied the sound generated by such a “singing wineglass” in some length [1–4]. These previous works suggest that the tone one hears consists mainly of the fundamental mode modulated by the rubbing motion of the wet finger [1]. Thus, simply changing the rubbing speed or the magnitude of the applied force does not alter the pitch one hears, though volume modulation can be easily perceived. Despite its popularity, a singing wineglass might exhibit interesting phenomena that go quite contrary to one’s intuition. Nice and easy accounts of this aspect can be found in [1, 2]. Added to the list of counter-intuitive behavior, a singing wineglass enjoys another phenomenon we recently encountered: The unexpected frequency shift associated with the tone when water is present in a glass.

To begin, we first note that adding water to a wineglass readily lowers its frequency. This phenomenon is hardly surprising if one recalls that the added water is forced by the wineglass to participate in the vibration, thus contributing an effective mass to the vibrating glass. If this were the whole story, then we would expect the following scenario to hold: Dipping a solid object into a glass already filled with water to its rim must make the frequency higher, because the amount of water inside the glass is made less by the presence of the foreign object. Wrong! We observed a *lowering* in the tone when a solid column is dipped into the center of the glass while keeping the water level intact. We also measured the frequency of an empty wineglass completely surrounded by water put in a laterally large bucket. This frequency turned out

Fig. 1 – A comparison of frequencies. (a) From left to right, the wineglass is empty, filled outside, filled inside, and filled both in and out to the same water level, respectively. (b) The frequency f_2 decreases to f_4 when a rigid column is dipped into the center while keeping the water level intact.

to be slightly *higher* than that for a glass filled with water inside. Figure 1 summarizes schematically the ordering of the frequencies for various configurations. In table I we also listed the data collected in the table-top experiments we performed, using a piezoelectric microphone for sound sampling and both an oscilloscope and Adobe Audition (a waveform analyzing software) for frequency analysis.

For both cases described above, a larger amount of water clearly contributes less to the effective mass of the vibration. So what is going on here?

This puzzle can be most easily resolved if we forget about the true geometry of the glass and focus on a simplified model problem. The one we start with is a two-dimensional model in which a layer of water of thickness s is in contact with a thin slab of glass of thickness τ_g set into vibration. (See fig. 2.) Setups similar to this, such as a pair of parallel plates sandwiching a layer of different elastic material, appear common in some acoustical applications [5, 6]. In what follows, a subscript l (g) will be used to denote a variable for the liquid (glass). Suppose the displacement of a liquid/glass element from its equilibrium position is denoted by ξ , then the linearized forms of $F = ma$ for the liquid and the glass are

$$\rho_l \frac{\partial^2 \xi_l}{\partial t^2} = -\nabla P, \quad (1)$$

$$\rho_g \tau_g \frac{\partial^2 \xi_g}{\partial t^2} = \hat{F}(\xi_g) + P \hat{n}, \quad (2)$$

where eq. (1) is Euler's equation for fluid motion [7], with P being the pressure deviation of

TABLE I – Wineglass frequencies in hertz for the various water filling configurations depicted in fig. 1. Glasses 1 and 2 are regular ones purchased in a store whereas Glass 3 is a custom-made right circular cylindrical glass filled at different water levels for comparison.

	Glass 1	Glass 2	Glass 3	Glass 3 (lower water level)
f_1	1020	1280	1030	1030
f_2	930	928	831	917
f_3	918	857	829	915
f_4	836	692	714	814
thickness (mm)	1.72	1.86	2.4	2.4

Fig. 2 – A simplified two-dimensional model for the wineglass problem: A layer of liquid of thickness s is sandwiched between a thin slab of glass and a solid wall. Nothing varies in the z -direction. (a) 3D view, (b) top view.

the liquid from its equilibrium configuration, $\hat{n}$ is the outward normal vector of the fluid at an appropriate interface, ρ denotes the mass density, and $\hat{F}$ is the restoring force per unit area whose detailed form is not important for our purpose [8]. The pressure P is to be solved using the incompressibility condition

$$\nabla \cdot \vec{\xi}_l = 0. \quad (3)$$

(Water can be treated as incompressible in our model because the sound speed in water, about 1500 m/s, is much greater than the flexural wave speed on the glass rim, which is estimated to be at most 200 m/s in our experiments.)

These variables also satisfy the following boundary conditions:

$$\vec{\xi}_l \cdot \hat{n} = \vec{\xi}_g \cdot \hat{n} \quad (\text{at the liquid-glass interface}), \quad (4)$$

$$\vec{\xi}_l \cdot \hat{n} = 0 \quad (\text{at the solid wall } y = s), \quad (5)$$

which simply say that the liquid cannot penetrate either the glass or the solid wall. (But notice that there *can* and in general will be slipping motion between the liquid and the wall/glass.)

Assuming a sinusoidal time dependence, we can verify that the displacement vector $\vec{\xi}_g$ for the flat glass at hand can be approximated as

$$\vec{\xi}_g = (0, A \sin kx \sin \omega t) \quad (6)$$

for some wave number k and small amplitude A . The reason there is no displacement in the x -direction is that glass is inextensible for our case [2], and this form guarantees that the glass wall has a constant length in the linear approximation when undergoing flexural wave motion. To solve for the fluid displacement $\vec{\xi}_l$, we can take the divergence of eq. (1) and make use of eq. (3) to yield $\nabla^2 P = 0$. With the assumption that the liquid-glass interface is located at $y = 0$ and the solid wall is at $y = s$, we can quickly derive the following solution using

separation of variables:

$$\vec{\xi}_l = \left(-A \cos kx \frac{\cosh k(s-y)}{\sinh ks} \sin \omega t, A \sin kx \frac{\sinh k(s-y)}{\sinh ks} \sin \omega t \right), \quad (7)$$

$$P = -\rho_l A \frac{\omega^2}{k} \sin kx \frac{\cosh k(s-y)}{\sinh ks} \sin \omega t. \quad (8)$$

At the liquid-glass interface $y = 0$ we therefore have

$$P\hat{n} = \rho_l \frac{\omega^2}{k} \frac{\cosh ks}{\sinh ks} \vec{\xi}_g, \quad (9)$$

which reduces eq. (2) to

$$-\omega^2 \rho_g \tau_g \left(1 + \frac{\rho_l}{\rho_g \tau_g k} \coth ks \right) \vec{\xi}_g = \hat{F}(\vec{\xi}_g). \quad (10)$$

On the other hand, the original frequency ω_0 for the case when the liquid is absent obviously satisfies

$$-\omega_0^2 \rho_g \tau_g \vec{\xi}_{g,0} = \hat{F}(\vec{\xi}_{g,0}). \quad (11)$$

Hence, a comparison of these two equations tells us that

$$\omega^2 \left(1 + \frac{\rho_l}{\rho_g \tau_g k} \coth ks \right) = \omega_0^2. \quad (12)$$

Because $\coth ks$ is a monotonically decreasing function of s , the inevitable conclusion is that the frequency is *lower* when s is smaller, that is, when there is less water.

Having arrived at this result, we see why the idea of an effective mass contributed by the fluid may be *formally* justified. This is all embedded in eq. (10), where influence of liquid explicitly enters the scene. Tracing back, we know it arises as a direct consequence of eq. (9). Thus, a clear understanding of what this equation means will help solve the puzzle.

One feature of this equation is that it predicts a positive pressure when the glass bulges out. This is certainly in line with our intuition, because the pressure is then always counter-acting the restoring force of the deformed glass, thus contributing a positive effective mass. The presence of the factor $\coth ks$ does deserve further discussion, though, since it becomes extremely large for small s , making the pressure increase without bound. The way we understand this feature is as follows. When a fluid element driven by the vibrating glass rushes towards the solid wall at the other end, it simply has to stop in view of the impermeability condition (eq. (5)). But a small stopping distance s implies a large velocity gradient in the y -direction, which in turn means there must be a compensating flow in the x -direction because the fluid is incompressible. This lateral fluid motion has a large velocity, which can only be driven by a large pressure gradient. Hence, the pressure necessarily rises when s is small. This is the physical mechanism behind eq. (9). Of course, one can also directly verify this argument by looking at eq. (7), paying special attention to the relative orders of magnitude of the two components of the displacement vector $\vec{\xi}_l$ at $y = 0$ for the case when $s \approx 0$. In this sense, however, we see that the concept of an added effective mass from the liquid should not be taken too literally, because the physically existing small amount of water simply cannot account for the large effective mass.

One can generalize the above model to make it slightly more realistic by considering a layer of fluid sandwiched between two concentric right circular cylinders of radius r_1 and

r_2 , respectively. We will assume that the outer one having a radius r_2 is made of glass and the inner one of rigid material. If we adopt a cylindrical coordinate system and write $\vec{\xi} = \xi_\theta \hat{e}_\theta + \xi_r \hat{e}_r$, then it is straightforward to show that the area enclosed by the glass exhibiting flexural motion remains a constant in the first-order approximation of the displacement. In other words, our setup of the problem is self-consistent, and we do not have to work with the fully three-dimensional problem at this stage. Resorting to separation of variables, we find the solution to be

$$\xi_r = A \left(r^{m-1} - \frac{r_1^{2m}}{r^{m+1}} \right) \cos m\theta \sin \omega t, \quad (13)$$

$$\xi_\theta = -A \left(r^{m-1} + \frac{r_1^{2m}}{r^{m+1}} \right) \sin m\theta \sin \omega t, \quad (14)$$

$$P = \rho_l \frac{\omega^2}{m} A \left(r^m + \frac{r_1^{2m}}{r^m} \right) \cos m\theta \sin \omega t \quad (15)$$

for some integer $m > 1$ and some number A . (Previous work shows that one has $m = 2$ as the lowest mode; see [2, 4].) Before going on, we should remark that, for the case when $r_1 = 0$, ref. [4] contains one mistake: In order to satisfy the incompressibility condition, the radial and the tangential components of the velocity of the *bulk fluid* must have the same magnitudes, as opposed to the 2 : 1 ratio claimed in sect. VI of this reference. However, the fact that the glass in our problem only sustains flexural wave does imply that this ratio is 2 : 1 for the *glass* on the rim. (See ref. [2] and p. 579 of ref. [9].) This then implies that actually there *is* a slipping motion between the fluid and the glass, as was already mentioned before.

The results above allow one to quickly derive

$$P = \rho_l \frac{\omega^2}{(m/r_2)} \left(\frac{1 + (r_1/r_2)^{2m}}{1 - (r_1/r_2)^{2m}} \right) \xi_r \quad (16)$$

at the liquid-glass interface. This equation clearly is the cylindrical counterpart of eq. (9), with the obvious identification of $k = m/r_2$. And once again, we know this implies that the liquid-filled cylindrical glass vibrates at a frequency higher than that for one having a solid placed at its center.

This calculation also implies that the frequency satisfies

$$\omega^2 \left(1 + \frac{\rho_l}{\rho_g \tau_g k} \left(\frac{1 + (r_1/r_2)^{2m}}{1 - (r_1/r_2)^{2m}} \right) \right) = \omega_0^2, \quad (17)$$

which closely resembles eq. (12). For the case when $r_1 = 0$ we then have

$$\omega^2 \left(1 + \frac{\rho_l}{\rho_g \tau_g k} \right) = \omega_0^2. \quad (18)$$

Of course, we can just as well solve the exterior problem by placing the fluid outside the cylindrical glass. Interestingly, the algebra shows that the formula works out exactly like eq. (18). Thus, for a very thin glass, the interior and the exterior problems turn out to produce the same tone! But for thicker glass wall, our argument above does suggest a *higher* frequency for the exterior problem, which has been experimentally verified in table I. (The fact that f_2 and f_3 differ by only 2 Hz for Glass 3, whose geometry conforms much better to that investigated in our model, suggests that the thin-glass approximation is a valid one, and that 2.4 mm can be considered as thin enough for this purpose.)

Fig. 3

Fig. 4

Fig. 3 – Water level dependence of the wineglass frequency. The slope 3.1 is close to the predicted value of $n + 1$, with $n = 2$.

Fig. 4 – Liquid density dependence of the wineglass frequency. The liquids used (in increasing density) are: alcohol, linseed oil, water, vinegar, and brine, respectively.

For a right circular cylindrical glass of height H filled with liquid up to a height h , one will have to solve the full three-dimensional problem, because now there is an extra restoring force coming from the bending of the glass wall from its upright position. However, since we are dealing with a linear system here, we know this force must be proportional to the pressure of the fluid in an averaged sense. Thus, under certain approximations, we may absorb this force into the pressure term so as to reduce the problem into an equivalent two-dimensional model. Specifically, defining $\langle P \rangle \equiv \int_0^h P dz/h$, $\langle \xi_{lr} \rangle \equiv \int_0^h \xi_{lr} dz/h = \int_0^h \xi_{gr} dz/h$, and $\langle \xi_{gr} \rangle \equiv \int_0^H \xi_{gr} dz/H$, and in view of eqs. (1) and (2), we now have

$$-\rho_l \omega^2 \langle \xi_{lr} \rangle = -\hat{e}_r \cdot \nabla \langle P \rangle, \quad (19)$$

$$-\rho_g \tau_g \omega^2 \langle \xi_{gr} \rangle = \langle \hat{e}_r \cdot \hat{F}(\vec{\xi}_g) \rangle + \gamma \frac{h}{H} \langle P \rangle \hat{n}, \quad (20)$$

$$\langle \xi_{lr} \rangle = \frac{\int_0^h \xi_{gr} dz}{\int_0^H \xi_{gr} dz} \frac{H}{h} \langle \xi_{gr} \rangle, \quad (21)$$

where γ is some numerical factor. Approximating the radial displacement of the glass by that of a vertical bar clamped at its base and free at the top end, one has [10]

$$\xi_{gr} \propto (\cosh \alpha + \cos \alpha) (\cosh \alpha z/H - \cos \alpha z/H) - (\sinh \alpha - \sin \alpha) (\sinh \alpha z/H - \sin \alpha z/H),$$

where $\alpha \approx 1.8751$. Due to cancellations in the next higher-order terms in the Taylor expansion of the hyperbolic and sinusoidal functions, the above is well approximated by

$$\xi_{gr} \propto (\cosh \alpha + \cos \alpha) (\alpha z/H)^2 - (\sinh \alpha - \sin \alpha) \frac{(\alpha z/H)^3}{3}.$$

For $h/H \lesssim 0.5$, we may retain only the lowest-order term and take $\xi_{gr} \propto z^n$ for $n = 2$; and for higher water level it is found that $n = 1.5$ is a good approximation [4]. At any rate, eqs. (16) and (21) suggest

$$\langle P \rangle \approx \gamma \rho_l \frac{\omega^2}{(m/r_2)} \left(\frac{1 + (r_1/r_2)^{2m}}{1 - (r_1/r_2)^{2m}} \right) \left(\frac{h}{H} \right)^n \langle \xi_{gr} \rangle,$$

which immediately yields

$$\omega^2 \left(1 + \gamma \frac{\rho_l}{\rho_g \tau_g k} \left(\frac{1 + (r_1/r_2)^{2m}}{1 - (r_1/r_2)^{2m}} \right) \left(\frac{h}{H} \right)^{n+1} \right) = \omega_0^2, \quad (22)$$

in analogy with our previous two-dimensional analysis. With the previously mentioned custom-made Pyrex glass having $H = 11.74$ cm, $r_2 = 4.02$ cm, $\tau_g = 0.24$ cm and $\rho_g = 2.23$ gm/cm³, we see in fig. 3 that the agreement is quite good if we take $n = 2$. This supplements the result of [4] when the water level is not high. In addition, we have also verified the ρ_l -dependence using various liquids. This is shown in fig. 4.

Concluding, our experiments show that the finite-size effect of the liquid on the tone of the singing wineglass appears counter-intuitive if one attempts to explain it in terms of the “effective fluid mass” driven by the glass. But using simplified models, we were able to derive quantitative relations and support them via experiments. In particular, we show that a more coherent way of viewing the whole situation is to place the emphasis on how the fluid pressure varies as the system geometry is changed; and an intuitive picture explaining it is also proposed.

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