A CRITICAL ANALYSIS OF IDEAL MAGNETOHYDRODYNAMIC MODELS FOR CRAB-LIKE PULSAR WINDS

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ABSTRACT

We present a critical examination of the ideal MHD model for the stationary Crab pulsar wind, which has a terminal flow Lorentz factor $\gamma_\infty \sim 10^6$ and may have a terminal ratio of Poynting flux to kinetic energy flux as low as $\sigma \sim 10^{-2}$ – 10^{-3} . We first show that transitions to a low- σ configuration cannot occur gradually in regions well beyond the light cylinder where the flow has already become ultrarelativistic. This is because the poloidal field lines do not expand sufficiently beyond the fast critical point to convert the electromagnetic energy into flow kinetic energy. As an alternative, we consider whether the acceleration may proceed abruptly, with the flow rapidly passing through the fast critical point and expanding into a low- σ configuration. Such rapid expansion of field lines, analogous to that in a de Laval nozzle, requires the poloidal fields to be highly compressed upstream of the fast critical point. We categorize the rapid field expansion generally into two prototypes: sheetlike and fountain-like, with the former being a spontaneous transition and the latter requiring external pressure supports. Unfortunately, it is shown that both types of rapid acceleration fail to satisfy either the energy and momentum conservation or the MHD flux-freezing condition. Nevertheless, we have pinned down the only situation where a stationary, ideal-MHD low- σ wind may exist. It requires almost the entire wind acceleration to occur in the immediate neighborhood of the light cylinder. Moreover, it also demands drastic modifications to the conventional picture of the pulsar dipole magnetosphere, in that the outer magnetosphere must be dominated by the toroidal fields and that the pulsar wind is carried by only a small fraction of the magnetospheric field lines emerging from the star.

Subject headings: MHD — pulsars: general — relativity — stars: mass loss

1. INTRODUCTION

The Crab pulsar wind is thought to consist of a mixture of relativistically flowing particles and low-frequency electromagnetic field (Rees & Gunn 1974), which radiates non-thermally after crossing a standing shock located about 0.1 pc from the pulsar. For the shocked flow to match the slow expansion of the outer shell, the shock has to be so strong that the material upstream of the shock must be almost unmagnetized. The ratio of Poynting flux to kinetic energy flux, σ_∞ , upstream of the terminal shock has been estimated to be on the order of 10^{-2} (Kennel & Coroniti 1984; Begelman & Li 1992), or even less (Emmering & Chevalier 1987). It is difficult to understand how such a weakly magnetized wind can result from what is thought to be an electromagnetic acceleration process. It was quickly realized after the discovery of pulsars that a *radially* directed relativistic MHD wind driven primarily by the rotating magnetic field can only attain a value of $\sigma_\infty \gg 1$ (Michel 1969; Goldreich & Julian 1970). Kennel, Fujimura, & Okamoto (1983) therefore considered a hot MHD wind model, in which the gas pressure provides most of the acceleration. Ruderman (1981) and Arons (1983) suggested another model, where the wind consists primarily of ions, accelerated to a high speed by the field-aligned electric field in the neutron star magnetosphere.

Nonideal MHD mechanisms have also been proposed to

account for the low- σ_∞ pulsar wind. Coroniti (1990) proposed an oblique rotator model that relies on magnetic field reconnection near the equatorial plane. Magnetic reconnection serves both to eliminate the magnetic flux and to heat the plasma, thereby accelerating it. However, in order to yield a σ_∞ as low as 10^{-2} – 10^{-3} , the reconnection rate has to be so large that the oblique angle of the rotor exceeds 80° , an angle that is well beyond what most pulsar models suggest. Mestel & Shibata (1994) considered the wind acceleration by the field-aligned electric field outside the light cylinder, and they also suggested the need for dissipation in the wind. Along the line of wind acceleration by the field-aligned electric field, Beskin, Gurevich, & Istomin (1983) proposed that at the separatrix of the magnetosphere between the open field and closed field, the pair plasma can be rapidly accelerated. However, the value σ_∞ has not been estimated in their work. Following the same scenarios, Bogovalov (1989) suggested that such a transition can yield a low- σ wind; however, in this analysis, balance of all components of the energy-momentum flux was not properly accounted for. Studies on this particular issue of pulsar winds have also been directed toward engaging dissipationless, nonideal MHD effects, such as Hall currents in pair plasmas, where the field lines no longer need to be frozen into the bulk plasmas (Melatos & Melrose 1996).

Returning to the ideal MHD framework, it has recently

been proven that all axisymmetric, relativistic winds that are driven electromagnetically must collimate asymptotically along the rotational axis (Chiueh, Li, & Begelman 1991). However, once partially collimated, the wind is found to accelerate rather slowly, with Lorentz factor scaling with the axial distance R as $\ln(R)$. The logarithmic acceleration makes it difficult for the flow to reach $\sigma_\infty \sim 10^{-2}$ – 10^{-3} at a distance of 0.1 pc (Li, Begelman, & Chiueh 1991; Li 1993; Begelman & Li 1994). In response to the difficulties associated with the low- σ_∞ wind, Begelman (1998) has recently examined the alternatives and pointed out that instabilities may destroy the axisymmetry of the postshock flow assumed by the previous works (Kennel & Coroniti 1984; Emmering & Chevalier 1987; Begelman & Li 1994). In such a case, σ_∞ becomes undetermined and could be as large as $O(1)$. Even so, it still remains a possibility that σ_∞ is small.

In this paper, we shall critically examine the relativistic ideal-MHD wind model, for which $\sigma_\infty \ll 1$, in the parameter regime appropriate for the Crab-like high γ_∞ pulsar winds. We shall first illustrate, in § 2, how extremely difficult it is for the pulsar wind to collimate and to reach a value of σ well below unity when the flow is smoothly varying in space. We will subsequently show, in § 3, that it is also impossible for the wind to undergo a rapid transition, on the length scale much less than the distance from the star and located at a distance farther than a couple of light cylinder radii from the star, to the low- σ configuration. In § 4, we conclude that the only possible site for the nearly total conversion of electromagnetic energy into flow kinetic energy is through the rapid acceleration at a site very close to the light cylinder; in addition, the conditions that a low- σ_∞ pulsar wind will have to impose on the pulsar magnetosphere are also addressed. The summary and discussions are given in § 5.

2. IMPOSSIBILITY OF SMOOTH ACCELERATION TO LOW σ

Stationary, axisymmetric relativistic MHD winds have been investigated by many authors (e.g., Lovelace et al. 1986; Camenzind 1987; Li, Chiueh, & Begelman 1992). They consist of a set of nested flux surfaces (about the symmetry axis), along which the loaded plasma flows. Usually, these surfaces are labeled by the so-called poloidal flux function Ψ , defined from the poloidal field strength B_p by $B_p R = \nabla \Psi \times e_\phi$ in a cylindrical coordinate system (R, ϕ, Z) (where e_ϕ is the azimuthal unit vector and the subscript p represents the poloidal component). Along each flux surface, there exist four conserved quantities: the mass flux per unit magnetic flux

$$k(\Psi) = \rho V_p / B_p \quad (1)$$

(where ρ is the proper density), the field line rotation rate $\Omega(\Psi)$, the specific energy

$$\mu(\Psi) = \gamma c^2 - \frac{B_\phi R \Omega}{(4\pi k)}, \quad (2)$$

and the specific angular momentum

$$l(\Psi) = R[V_\phi - B_\phi / (4\pi k c)]. \quad (3)$$

The four velocities and the flow Lorentz factor have been denoted by (V_p, V_ϕ) and γ , respectively; a cold flow has been assumed and the gravity ignored in the above expressions.

These conserved quantities, together with the flux-freezing condition

$$V = k B / \rho + \gamma \Omega R e_\phi, \quad (4)$$

yield the following expressions for γ , B_ϕ , and V_ϕ in terms of the conserved quantities and an auxiliary dimensionless quantity $\tau = 4\pi k^2 c^2 / (\rho \Omega^2 R^2)$, which measures the relative importance of the kinetic energy flux to the Poynting flux. At each dimensionless radius $x = R/R_L$ (with the light-cylinder radius defined as $R_L \equiv c/\Omega$), we have

$$\gamma = \frac{\alpha c^2 - \mu \tau x^2}{c^2(1 - (1 + \tau)x^2)} = \frac{\mu}{c^2} \frac{\tau}{1 + \tau} \left[1 + \frac{1}{1 + \tau} \left(1 - \frac{\alpha}{\gamma} \right) \frac{1}{x^2} + \dots \right], \quad (5)$$

$$B_\phi = -\frac{4\pi k l}{R} \frac{1 - (\mu/l\Omega)x^2}{1 - (1 + \tau)x^2} = -\frac{4\pi k c^2}{R\Omega} \frac{\mu/c^2}{1 + \tau} \left[1 + \left(\frac{\alpha}{\mu/c^2} - \frac{\tau}{1 + \tau} \right) \frac{1}{x^2} + \dots \right], \quad (6)$$

$$V_\phi = \frac{x^2[l\Omega\tau - \alpha c^2]}{R\Omega[1 - (1 + \tau)x^2]} = \frac{l}{R} \frac{\tau}{1 + \tau} \left\{ 1 - \frac{1}{1 + \tau} \left[\frac{\alpha}{\gamma} + \left(\frac{\alpha}{\gamma(1 + \tau)} - 1 \right) \frac{1}{x^2} \right] + \dots \right\}, \quad (7)$$

where the parameter $\alpha \equiv (\mu - l\Omega)/c^2$ measures the initial loading of kinetic energy per rest mass energy at the star, and is on the order of unity if the plasma is injected into the wind nonrelativistically or transrelativistically. Thus, for an electromagnetically driven pulsar wind, α is much less than the total energy per rest mass energy μ/c^2 (by a factor $\sim 10^6$ for the Crab). For winds that have reached ultrarelativistic speeds ($\gamma \gg 1$) well outside the light cylinder ($x \gg 1$), it suffices to keep the leading order terms in x^{-2} and γ^{-1} to obtain the approximate expressions (below eqs. [5], [6], and [7]) from equations (5), (6), and (7), respectively. Through equations (5) and (7) and the flux-freezing condition $V_p = k B_p / \rho$, the identity $\gamma^2 = 1 + (V_p/c)^2 + (V_\phi/c)^2$ can be cast into the following algebraic “wind” equation which relates τ (and thus all other flow quantities) to the poloidal field strength B_p along a flux surface:

$$\left\{ \frac{\mu - l\Omega - (\mu/l\Omega)x^2}{c^2[1 - (1 + \tau)x^2]} \right\}^2 = 1 + \left(\frac{\tau B_p R^2}{4\pi k R_L^2 c} \right)^2 + \left(\frac{l x^2}{R c} \right)^2 \left[\frac{1 + \tau - (\mu/l\Omega)}{1 - (1 + \tau)x^2} \right]^2. \quad (8)$$

Equation (8) has been analyzed in depth by Begelman & Li (1994); to the lowest order in x^{-2} and γ^{-1} , it is reduced to

$$\gamma = \mu - B_p R^2 / (4\pi k R_L^2 c), \quad (9)$$

which ties the flow kinetic energy γ to the poloidal field geometry (through B_p). That is, an increase in flow kinetic energy γ is always accompanied by a decrease in the Poynting flux which is proportional to $B_p R^2$. In other words, the divergence of a flux tube at a rate faster than radial expansion will always lead to an increase of kinetic energy. As

shown in Begelman & Li (1994), the fraction of magnetic energy converted into flow kinetic energy always remains very small up to the fast magnetosonic point, where $V_p^2 = V_f^2 = [B_p^2(1 - x^2) + B_\phi^2]/4\pi\rho$. (The subscript f stands for the fast magnetosonic point.) It follows that

$$\mu \cong \frac{(B_p R^2)_f}{4\pi k R_L^2 c}. \quad (10)$$

The kinetic energy can become comparable to or even larger than the magnetic energy only outside the fast point, where the ratio of kinetic to magnetic energy flux is given approximately by $(B_p R^2)_f/B_p R^2 - 1$. Therefore, the only way to attain a low value of σ_∞ is for the flux tubes to diverge significantly outside the fast point, thus substantially lowering $B_p R^2$.

The rate at which a flux tube diverges is governed by the cross-field force balance equation

$$\begin{aligned} \kappa R = e_c \cdot \{ \nabla [B_\phi^2 + B_p^2(1 - x^2)] \\ + 2(B_\phi^2 - B_p^2 x^2 - 4\pi\rho V_\phi^2) \nabla \ln(R) \} R \\ \times \{ 2B_p^2 [1 - (1 + \tau)x^2] \}^{-1}, \end{aligned} \quad (11)$$

where κ is the poloidal field line curvature and e_c is the cross-field unit vector pointing toward the axis. From equations (5)–(7) and the “wind” equation (8), it is straightforward to express the “effective” pressure and the toroidal velocity as

$$\begin{aligned} P_{\text{eff}} &\equiv B_\phi^2 + B_p^2(1 - x^2) \\ &= \frac{B_\phi^2}{\gamma^2(1 - x^{-2})} [1 + O(\gamma^{-1}) + O(x^{-4})] \end{aligned} \quad (12)$$

and

$$4\pi\rho V_\phi^2 = \tau B_p^2 [1 + O(\gamma^{-1}) + O(x^{-2})]. \quad (13)$$

If we assume that the variation of magnetic field in the cross-field direction occurs on a scale of order R , the spatial derivative can be estimated to be $1/R$. One thus obtains

$$|\kappa R| \lesssim \frac{1}{1 + \tau} \left(\frac{1}{\gamma^2} + \frac{\beta}{x^2} \right) + \frac{1}{x^2}, \quad (14)$$

from the cross-field equation (11) where β is a constant of order unity. Clearly, $\kappa R \ll 1$ if both γ and x are much larger than unity. One may further estimate the flux surface collimation rate by noting that the poloidal field line curvature $\kappa = d\theta/dS$, where θ is the angle between the tangent of the poloidal field line and the equator and S is the length along the field line taken to be on the order of R . The change of θ along any field line over a distance ΔR becomes $\Delta\theta \sim (\ln \Delta x)/[\gamma^2(1 + \tau)]$ in the region $x \gg \gamma(1 + \tau)^{1/2} \gg 1$. On the other hand, in the region $1 \ll x \ll \gamma(1 + \tau)^{1/2}$, one has $\Delta\theta \sim |\Delta x/x^3|$. The former is a generalization of the logarithmic collimation found in the recent works of Chiueh, Li, & Begelman (1991), Li, Chiueh, & Begelman (1992), Eichler (1993), Begelman & Li (1994), and Tomimatsu (1994). The latter collimation rate may remain valid over many decades of the local wind radius since γ can be extremely large (up to $\gamma \sim 10^6$ in the Crab pulsar wind).

In either case, the curvature κ of the poloidal field lines is always much smaller than $1/R$, which implies that the quantity $B_p R^2$ (which is exactly constant for a purely radial wind) changes only by a very small factor (on the order of the change in angle $\Delta\theta \ll 1$ estimated above) along the flux

surfaces. From equation (9), it follows that the change in the flow kinetic energy will be negligible compared with the total energy. Therefore, we conclude that the poloidal field lines in a smoothly varying ultrarelativistic MHD flow must be nearly straight far beyond the light cylinder radius, and in this region efficient conversion of Poynting flux to kinetic energy flux is impossible. The fundamental reason for this behavior is that the magnetic pressure force in the cross-field direction is canceled almost exactly by a cross-field electric force in the region where both γ and x (eq. [5]) are large. The resulting “effective” pressure gradient ∇P_{eff} is insufficient to bend the field lines in any significant way so that a diverging channel configuration, needed for efficient acceleration, can never be set up.

It should be stressed that the above analysis does not exclude a possible smooth acceleration from a high σ to a low σ within a couple of light cylinder radii from the star. However, we regard this situation as nearly impossible because there is not a sufficient solid angle into which the magnetic flux may expand, unless the flux tubes are severely compressed before sudden expansion. Nevertheless, a smooth transition to σ of order unity is relatively easy to attain near the light cylinder since $B_p R^2$ needs only to decrease by a factor of 2 after the flow passes the fast point.

3. DIFFICULTIES ASSOCIATED WITH RAPID TRANSITION FROM HIGH- σ TO LOW- σ

We now examine the possibility of expanding the pulsar wind into a low- σ configuration in a more extreme situation in which the wind changes its configuration along the field lines on a length scale much smaller than the local radius R . That is, the assumption employed in the last section $|\nabla| \sim R^{-1}$ for arriving at equation (14) will no longer be valid. Again, we shall confine the discussions to regions at least a couple of light-cylinder radii away from the star, in order to facilitate the much simpler approximate expressions given in the approximations below equations (5), (6), and (7), respectively, for the following analyses.

To begin, we assume that the pulsar spindown luminosity is distributed fairly evenly among the energy-carrying open field lines emerging from the neutron star, and hence our discussions will focus on the situations where the poloidal field expands, along the field lines, by a factor of σ_∞^{-1} within a short length scale $\Delta(\ll R)$. Such an abrupt expansion requires a large poloidal field curvature $\kappa \sim \Delta^{-1}$. In general, rapid expansion can be categorized into two classes of geometrically distinct configurations that have the desired large curvature. One class has a sheetlike transition (Figs. 1a, 1b) and the other a fountain-like transition (Figs. 1c, 1d).

Common to both classes is that such a sudden fanout configuration requires a sufficiently large solid angle into which the field lines can expand. If, before the transition, all field lines have already occupied a substantial fraction of the 4π solid angle, then further expansion of field lines by a large factor σ_∞^{-1} will be impossible. Hence, the poloidal field lines need to be bunched within a small solid angle before the sudden expansion takes place. The field-line bunching also takes advantage of the de Laval nozzle effect, through which the flow can be accelerated to pass through the magnetosonic point.

The major difference between the two types of transition lies in the following features. For the sheetlike transition, individual poloidal field lines fan away from the rest of the

FIG. 1.—Four typical sudden transitions from a large- σ flow. (a), (b) sheet-like transitions; (c), (d) fountain-like transitions.

field lines sparsely and the distance between fanout locations of neighboring field lines is large compared with the original distance between these lines. Thus, there exists a one-dimensional transition layer, across which the wind configuration changes sharply on the length scale of Δ , but along which the configuration varies smoothly on the length scale of R . By contrast, the fountain-like transition requires most field lines to fan out at the same location within a region of size Δ . The sheetlike transition may occur spontaneously, whereas the fountain-like transition must be pressure-supported by an external medium as will be further elucidated later. Note that we have two small parameters, σ_∞ and Δ/R , which are generally related; the smaller Δ/R is, i.e., the tighter the field bunching, the smaller the resultant σ_∞ is expected to be.

Figures 2 and 3 set the stage for the discussions to follow. They illustrate various qualitative aspects of the solutions to equations (5), (6), and (7). Shown in Figure 2 are the poloidal magnetic “flux” $B_p R^2$ within a prescribed flux tube as functions of the normalized cylindrical radial distance x for the two kinds of bunched field line configurations. Shown in Figure 3 are the corresponding $|B_\phi R R_L|/B_p R^2$ and four-velocities of the flow speed. Solid lines are for the field configuration where field lines are abruptly bunched and then suddenly expand; both sheetlike (Fig. 1a) and fountain-like (Fig. 1c) transitions may assume this kind of field configuration. On the other hand, the dashed lines refer to the cases where the bunching of field lines occur gradually over many light-cylinder radii before sudden field line expansion; again, both sheetlike (Fig. 1b) and fountain-like (Fig. 1d) transitions may assume this configuration.

The flow speed in the abrupt bunching/expansion case (solid lines in Figs. 2 and 3) remains nonrelativistic all the

way up to the most compressed location, beyond which the field lines suddenly fan out and acceleration rapidly takes place. The location of the transition can in principle be arbitrarily far from the Alfvén (or light-cylinder) radius. Before the rapid acceleration, the field lines are tightly wound in the toroidal direction, as illustrated by the large ratio of $|B_\phi R_L|/B_p R$, an indication of the storage of energy in the form of toroidal magnetic fields. Using equations (5) and (6) for γ and B_ϕ , as well as the definition of γ in terms of the flow velocity, we find that in the nonrelativistic regime the flow velocity obeys

$$V_p \approx c \frac{B_p R}{B_\phi R_L}, \quad V_\phi \approx \frac{R V_p^2}{2 R_L c}. \quad (16)$$

The first expression indeed shows that $|B_p R|/B_\phi R_L$ must be small for a nonrelativistic flow. In this regime, expansion of poloidal field B_p leads to deceleration of the flow, analogous to the subsonic flow in a de Laval nozzle. The fast magnetosonic point is located immediately downstream of the most compressed region. Immediately upstream of the fast point, where the flow begins to become relativistic, we note that the value of $B_\phi R R_L$ decreases to track that of $B_p R^2$, as is revealed by equation (12) in the large γ limit. Rapid expansion of field lines after the fast point gives rise to major acceleration of the wind as a result of decompression of the toroidal field. We also note that reducing $B_p R^2$ by merely a factor of 2 after the fast point makes the flow acquire a σ value of about unity, bringing the flow energy and electromagnetic energy into equipartition. Any further reduction of $B_p R^2$ will result in a low value of σ_∞ . Since the compression and expansion both occur within a short distance, the flow at a distance Δ before the fast point is still nonrelativistic.

FIG. 2.—log-linear plots of the quantities $B_p R^2$ (in units of $4\pi k c R_L^2$) as a function of the cylindrical radius (in unit of R_L) for the two types of bunching/expansion field configurations discussed in the text. Solid line refers to the configurations indicated in Figs. 1a and 1c, and dashed lines to those indicated in Figs. 1b and 1d).

For the gradual bunching cases followed by rapid expansion (Figs. 2 and 3 *dashed lines*), it is necessary that the transition layer extends to regions far from the light cylinder. Upstream of the transition, the nonrelativistic region is

relatively small and the relativistic region extends over a distance of many light cylinder radii. In contrast to the previous case, the flow upstream of the transition has already become ultrarelativistic, and the transition is from

FIG. 3.—log-linear plots of the ratio $|B_\phi R R_L|/B_p R^2$ and the dimensionless poloidal four-speed V_p/c for the corresponding poloidal field configurations shown in Fig. 2. Solid and dashed lines refer to the same types of bunching/expansion configurations as those in Fig. 2. Points labeled by F mark the fast critical points.

a $\sigma \gg 1$ ultrarelativistic flow to a $\sigma \ll 1$ ultrarelativistic flow.

3.1. Sheetlike Transition

We shall again focus on the cross-field force balance (eq. [11]) near the transition. Because the poloidal electric force (terms with $B_p^2 x^2$ in eq. [11]) dominates over the poloidal magnetic force outside the light cylinder, the sign of the magnetic field line curvature, determined by the sign of the denominator in equation (11) obeys what the electric curvature force dictates. Its direction is opposite to that of the magnetic curvature force but the same as the flow centrifugal force (the term with $B_p^2 \tau x^2$ in the denominator of eq. [11]). Figures 1a and 1b show two prototype sheetlike transitions. At these transitions, the flows are accelerated rapidly across the layer, as signified by the sudden expansion of the poloidal fields. Referring to equation (12), we find that the drastic increase in the Lorentz factor γ to a downstream value approximately equal to $\mu \sim 10^6$ warrants a sudden decrease of P_{eff} in equation (12). Such a pressure drop creates the curvature with the desired sign, which may be sufficiently large to turn the poloidal field line to a direction almost normal to the layer, thus creating a sudden fanout configuration. In turn, the backreaction of the accelerated flow may compress the upstream field to keep the tightly bunched upstream field lines intact.

Let us now examine the above physical picture in quantitative detail. We shall regard the transition layer as one-dimensional, and examine the appropriate Rankine-Hugoniot relations for such an ideal MHD transition. Let the direction normal to the layer be the $\hat{e}_1$ direction, the azimuthal direction $\hat{e}_3$, and the direction perpendicular to the above $\hat{e}_2$. The energy-momentum flux appropriate for such a transition can be written as

$$F\gamma c + \frac{E_2 B_3}{4\pi} = T^{01}, \quad (17)$$

$$FV_1 + \frac{B_3^2 + B_2^2 + E_2^2 - B_1^2 - E_1^2}{8\pi} = T^{11}, \quad (18)$$

$$FV_2 - \frac{B_1 B_2 + E_1 E_2}{4\pi} = T^{21}, \quad (19)$$

$$FV_3 - \frac{B_1 B_3}{4\pi} = T^{31}, \quad (20)$$

where $F = \rho V_1$, the mass flux. Conservation of the energy momentum flux demands that $T^{\beta 1}$ be continuous across the layer.

Since the fast point, at which the flow kinetic energy is much smaller than the total energy, is located within the transition layer, we thus seek upstream and downstream solutions that are electromagnetically and kinetically dominated, respectively. We can immediately write down a constraint on the upstream condition if such a transition is to occur. Since the downstream flow is kinetic-dominated and since any cold flow must satisfy $F^2 \gamma'^2 c^2 = F^2 (c^2 + V_1'^2 + V_2'^2 + V_3'^2)$, where the primes denote the downstream quantities, we therefore find that the downstream $T^{\beta 1}$ must obey the equation

$$(T^{01})^2 \approx F^2 c^2 + (T^{11})^2 + (T^{21})^2 + (T^{31})^2. \quad (21)$$

From conservation of all components of the energy-momentum flux, equation (21) must also be satisfied by the

upstream $T^{\beta 1}$. One may hence substitute the appropriate upstream values of $T^{\beta 1}$ into equation (21) to obtain

$$(B_1 B_3 x)^2 \approx (4\pi F c)^2 + \frac{1}{4}[B_3^2 + (B_2^2 - B_1^2)(1 - x^2)]^2 + [B_1 B_2 (1 - x^2)]^2 + (B_1 B_3)^2, \quad (22)$$

To extract the relevant terms, we have also subtracted $E_2^2 - B_1^2$ from T^{11} in equation (22) (the second term on the right side of eq. [22]) since both E_2 and B_1 are not changed across the transition layer as a result of $\mathbf{V} \times \mathbf{E} = 0$ and $\mathbf{V} \cdot \mathbf{B} = 0$. Furthermore, in order for a comparison with Figures (2) and (3) to be made, we have, in equation (22), identified $B_p^2 = B_1^2 + B_2^2 \cong B_2^2$, $B_\phi = B_3$, and $R^2 B_p^2 / R_L^2 = E_1^2 + E_2^2 \cong E_1^2$.

For the case of sudden field bunching/expansion (Fig. 1a and the *solid lines* in Figs. 2 and 3), we find that, due to the large B_3^2 in a nonrelativistic upstream flow, equation (22) can never be satisfied. This is due primarily to the fact that $B_\phi^2 \gg B_p^2 x^2 \approx B_2^2 x^2 \geq B_2^2$ for a nonrelativistic upstream flow followed from equation (16). The component $T^{11} (\approx P_{\text{eff}})$ therefore becomes too large for equation (22) to be satisfied. This finding excludes the transition shown in Figure 1a.

By contrast, equation (22) can be satisfied when the magnetic flux assumes the gradual bunching configuration (Fig. 1b and the *dashed lines* in Figs. 2 and 3), where the upstream $\gamma \gg 1$ and $R \gg R_L$. This can be shown as follows. Ignoring the small rest mass term $F^2 c^2$, we may cast equation (22) into the form:

$$[B_3^2 - B_2^2(x^2 - 1)]\{B_1^2(x^2 - 1) - \frac{1}{2}[B_3^2 - (B_2^2 + B_1^2)(x^2 - 1)]\} \approx \frac{1}{2}[B_1^2(1 - x^2)]^2. \quad (23)$$

Furthermore, from equation (12), we find that

$$T^{11} + B_1^2(x^2 - 1)/4\pi \approx [B_3^2 - (B_2^2 + B_1^2)(x^2 - 1)]/8\pi \approx (B_3^2/8\pi\gamma^2), \quad (24)$$

a small quantity when γ is large. This quantity can be inserted into equation (23) to solve for B_1 , and it follows that B_1 must satisfy

$$B_1^2 x^2 \approx (\sqrt{5} - 1) \frac{B_3^2}{2\gamma^2}, \quad (25)$$

in order for the upstream flow to satisfy equation (21).

However, equation (21) is only one of the four energy-momentum conservation laws; other components need also be examined. We shall now show that simultaneous conservation of T^{21} and T^{11} reveals a difficulty for this seemingly plausible transition. Since the upstream $T^{11} \approx B_\phi^2/8\pi\gamma^2 - B_1^2 x^2/4\pi \sim O(B_1^2 x^2/4\pi)$ (cf. eqs. [24] and [25]), we find that $T^{11} \ll T^{21} \approx B_1 B_2 x^2/4\pi$ due to the fact that $B_1 \ll B_2$ upstream of the transition. However, downstream of the transition, if the flux-freezing condition for ideal MHD flows is to hold, the poloidal flow must be along the field line, which should have the "1" component greater than, or at least comparable to, the "2" component in order for the downstream magnetic fields to exhibit a fanout configuration. This requirement demands that $T^{11} \geq T^{21}$ downstream of the transition, in contradiction to the relation $T^{11} \ll T^{21}$ upstream of the transition. Consequently, the transition shown in Figure 1b is also ruled out.

The above discussions rule out the sheetlike transitions.

3.2. Fountain-like Transition

For the fountain-like transitions, the poloidal field lines fan out at the same location. Shown in Figures 1c and 1d are the two prototypes of such transitions. The fountain-like transitions must be pressure-supported by some external media. By squeezing the flow, the external medium (*shaded region*) permits the conversion of internal energy into flow kinetic energy in the subsonic flow. Such an external support is common for the laboratory nozzle flows, and it offers a possibility for the fountain-like transition to low- σ configurations.

One may envision the external medium to be a vacuum toroidal magnetic field generated by a net current within the flow, a toroidal magnetized plasma detached from the rotation drive (Fig. 1c), or a cold and dense gravitating gas which is also detached from the rotation drive (Fig. 1d). The sudden expansion of the wind at the transition imposes stringent conditions on the choice of the external medium. Compare the regions of the boundaries labeled “i” and “ii”. At “i,” the wind has not yet expanded. Having a lower value of γ , the effective pressure P_{eff} in the upstream wind is much larger than that at location “ii” (see the cross-field force balance equation, eqs. [11] and [12]). Within a distance of order Δ , the length scale over which the transition takes place, the external medium must manage to match such a pressure drop, which can amount to a factor of several orders of magnitude in accordance with the ratio of the downstream to upstream Lorentz factors.

Consider the case where the external pressure is provided by a vacuum toroidal magnetic field, appropriate for a self-confined jet as shown in Figure 1c (Appl & Camenzind 1993). The vacuum field decreases as $1/R$ outside the flow, not sufficiently rapidly to match the wind pressure drop when the flow abruptly expands; therefore it cannot be a viable candidate.

If the external medium consists of a toroidal region of magnetized plasma, the situation can be improved. The electric current within the plasma can be such that both the poloidal and toroidal magnetic fields are screened out by the electric current within a distance of order Δ (see Fig. 1c). The tension force of the magnetic layer not only can confine the plasma itself but also serves to pinch the flow, creating a nozzle opening. Similarly, if the external medium consists of a cold and dense gravitating gas, the gravitational scale height can be so small that the gas may effectively lie within a narrow layer (see Fig. 1d), where the gas pressure drops substantially over a distance Δ in order for it to match to that of the rapidly expanding flow.

To scrutinize whether the above external media may work, let us consider the energy-momentum budget across the narrow strip of a rectangular box enclosed by boundaries 1 and 2 in Figures 1c and 1d. If the boundaries 1 are sufficiently long such that the boundaries 2 are in the vacuum, we can ignore the energy-momentum flux contributed from boundaries 2 and again treat the transition as a one-dimensional problem, similar to the sheet transition with, however, the following modifications. A static, external magnetized plasma carries no energy flux (T_{ex}^{01}) or angular momentum flux (T_{ex}^{31}) but does carry poloidal momentum flux (T_{ex}^{11} and T_{ex}^{21}), which can be injected through the upstream boundary into the transition layer. Modifications to equations (18) and (19) are to add T_{ex}^{11} and T_{ex}^{21} to the wind counterparts, T_w^{11} and T_w^{21} , in the upstream

boundary occupied by the confining plasma. (We have used the same convention that the direction normal to the transition layer is the $\hat{e}_1$ direction.) The magnitude of the external stress tensor $T_{\text{ex}}^{\alpha\beta}$ can be estimated by taking an appropriate surface average on either side of the transition layer. Note that the external medium must be in static equilibrium, and hence T_{ex}^{11} , T_{ex}^{21} , and T_{ex}^{22} must be of the same size; their values should be about the effective cross-field pressure of the upstream flow, $T_{\text{ex}}^{22} \approx P_{\text{eff}}$, in order to for the external plasma to be in contact equilibrium with the upstream flow.

In contrast to the layer transition, we now have $B_1 \approx B_p \gg B_2$ and $E_2 \gg E_1$ in the upstream region. From equation (10), the upstream quantity T_w^{11} has a magnitude about $(B_\phi^2 + B_p^2[x^2 - 1])/8\pi$, (much) larger than its $P_{\text{eff}} \sim (B_\phi^2 - B_p^2[x^2 - 1])/8\pi$ for a (relativistic) nonrelativistic upstream flow. Hence, $T_{\text{ex}}^{11} (\sim T_{\text{ex}}^{21} \sim P_{\text{eff}}) \ll T_w^{11}$.

Again, the same difficulties that we faced for the layer transition recur. In the case of nonrelativistic upstream flow, the upstream energy flux T^{01} is too small to balance the upstream T^{11} ; therefore, equation (21) can never be satisfied due to the inequality $B_\phi \gg B_p$ for nonrelativistic flows. Since the static confining plasma carries no T_{ex}^{01} and too little T_{ex}^{11} in comparison with T_w^{11} , the addition of the confining plasma barely has any effect in helping equation (21) to be satisfied. This consideration rules out the transitions such as Figure 1c.

On the other hand, equation (21) can be satisfied when the upstream flow is ultrarelativistic (Fig. 1d) since T_w^{01} and T_w^{11} are about the same size and are able to balance each other. Other components of the upstream momentum flux are smaller than the upstream T^{11} by a factor of about x^{-2} or about the upstream $(B_2/B_1)^2$, and therefore they contribute little to equation (21). Identical to the considerations for the corresponding case of the sheetlike transition (Fig. 1b), one finds that equation (21) is satisfied.

However, the frozen-in condition gives the same problem as it did before. We have $T_{\text{ex}}^{11} \sim T_{\text{ex}}^{21} \sim T_w^{21} \ll T_w^{11}$, and thus the static confining plasma adds too little to the upstream momentum flux in the $\hat{e}_1$ direction, only increasing the T^{21} by a small amount of the same order as the upstream T_w^{21} . It follows that the supporting wall can hardly help in substantially increasing the upstream ratio of T^{21}/T^{11} . Much like the sheetlike transitions, conservation of the momentum flux, with $T^{21} \ll T^{11}$, demands the downstream kinetically dominated flow to be almost parallel to the $\hat{e}_1$ direction. This is in contradiction to the desired fanout configuration requiring $T^{21} \gtrsim T^{11}$. It also rules out the transition such as Figure 1d.

The primary reason that the supporting wall fails to provide an efficient nozzle effect is that the energy-momentum stress tensor for a sub-Alfvénic ultrarelativistic MHD flow is extremely anisotropic, with the dominant component of momentum flux along the flow; this is in great contrast to that of a nonrelativistic MHD flow. As a result, the supporting wall can only exert very little pressure to confine the flow, thereby adding little to the stress balance. It follows that the nozzle effect is not necessarily an efficient means for converting electromagnetic energy to flow kinetic energy in an already ultrarelativistic flow. Moreover, the fact that the wind is dominated by magnetic pressure as opposed to thermal pressure further exacerbates the wind acceleration. For a thermal wind, the internal energy can be completely depleted in favor of the flow

kinetic energy. The asymptotic internal energy density decreases with distances as $P \sim \rho^\Gamma \sim r^{-2\Gamma}$, which can be much smaller than the asymptotic flow energy density, $\rho\gamma^2 \sim r^{-2}$, for any reasonable equation of state with an adiabatic index $\Gamma > 1$. However, for a magnetically dominated wind, the asymptotic electromagnetic energy density is dominated by $(B_\phi^2 + E_p^2)/2$, which decreases with distance as r^{-2} , the same as the scaling by which the kinetic energy density decreases. Consequently, complete conversion of electromagnetic energy into flow energy is indeed difficult for a magnetically driven wind at distances farther than a couple of light cylinder radii from the neutron star.

4. CONDITION ON SUDDEN WIND ACCELERATION NEAR THE LIGHT CYLINDER

We are therefore left with only one possibility: that stationary MHD wind acceleration, from a high- σ configuration to a low- σ configuration, takes place no farther than a fraction of R_L outside the light cylinder. Such a rapid acceleration cannot be located well inside the light cylinder, either, because it has to occur outside the Alfvén critical point, which in turn is located at the cylindrical radius $R_L(\Omega/\mu)^{1/2}$. In the limit of a relativistic wind driven primarily by the magnetic field of a rotating neutron star, one finds that $\Omega/\mu \rightarrow 1$ from below, and hence the Alfvén critical point is located immediately inside the light cylinder. The two bounds narrow the region of interest to be very close to the light cylinder. This conclusion also indicates that virtually all the current closure in the pulsar wind should occur nearly within the magnetosphere, if indeed the wind acceleration is through the ideal MHD mechanism.

Can an ideal MHD wind be rapidly accelerated to a low- σ configuration near the light cylinder? We have no rigorous answer to this question since the answer must depend on the global electrodynamics of the entire pulsar magnetosphere, in the inner part of which the plasmas are believed to be nonneutral and the ideal MHD breaks down (Michel 1991; Mestel & Shibata 1994). This is a subject beyond the scope of the present work. However, even within the ideal MHD framework, there exist conditions to be satisfied if the answer is affirmative. In particular, we find that only a small fraction, $\sigma_\infty^{1/2}$, of all field lines emerging from the dipole magnetosphere can become open and loaded with the pulsar wind. Moreover, we find that the pulsar magnetosphere must be greatly distorted from the dipole configuration; specifically, the field lines must be tightly wound up, i.e., dominated by the toroidal field, in order to store the neutron-star rotational energy. We come to these conclusions with the following considerations.

The luminosity of the terminal Poynting flux L_{EM} in the wind is $\sigma_\infty L_{\text{spin}}$, where L_{spin} is the spindown luminosity. Once the wind reaches the asymptotic state, we have $L_{\text{EM}} \sim cB_p^2 R^4/R_L^2$ and the poloidal field lines are almost radial so that $B_p^2 R^4 \approx \Psi^2 = \text{constant}$, where Ψ is the total poloidal magnetic flux in the wind. Furthermore, we have found that the only possible site for rapid acceleration is near the light cylinder; by interpolating the radial fields from the infinity back to the acceleration site, it follows that the field strength $B_p \sim (\sigma_\infty L_{\text{spin}} R_L^2/R^4 c)^{1/2} \sim (\sigma_\infty L_{\text{spin}}/R_L^2 c)^{1/2}$ near the light cylinder where $R \sim R_L$. On the other hand, the dipole radiation of a vacuum magnetic field asserts that near the light cylinder $B_{p,\text{dipole}} \sim B_\phi \sim E_p$, and $L_{\text{EM,dipole}} (= cE_p B_\phi R_L^2) \sim$

L_{spin} . Thus the strength of the dipole poloidal magnetic field at the light cylinder is $B_{p,\text{dipole}} \sim (L_{\text{spin}}/cR_L^2)^{1/2}$. For the observed Crab luminosity, this estimate is consistent with the dipole neutron star magnetosphere with a 10^{12} G surface field. Comparing the estimated B_p and $B_{p,\text{dipole}}$, we find that $B_p \sim (\sigma_\infty)^{1/2} B_{p,\text{dipole}}$ immediately after the rapid wind acceleration near the light cylinder. In other words, if the postacceleration pulsar wind also occupies a nearly 4π solid angle as the dipole radiation does, it follows that only a small fraction ($\sigma_\infty^{1/2}$) of the dipole field lines participate in the pulsar wind.

Since the poloidal field declines at least by a factor σ_∞^{-1} during the rapid transition, the poloidal field strength prior to the major expansion near the light cylinder must have a value $B_p \sim \sigma_\infty^{-1/2} B_{p,\text{dipole}}$, which is more than 1 order of magnitude larger than the estimated dipole fields if $\sigma_\infty \sim 10^{-2}$ – 10^{-3} . Hence the pressure balance with the closed-field magnetosphere cannot be maintained if the magnetosphere retains more or less a dipole configuration. This difficulty can be circumvented only if the neutron star magnetosphere drastically deviates from the dipole configuration near the light cylinder. If so, not only the poloidal field strength increases, but the toroidal field strength can also be amplified by an even larger factor.

Since the rapid transition occurs near the light cylinder, it belongs to the category of sudden bunching/expansion cases in the ideal MHD limit, for which the upstream flow must be nonrelativistic. From the discussions preceding equation (16), we have indeed found that $B_\phi/B_p \sim c/V_p \gg 1$ prior to the major acceleration. As a self-consistency check, we find that the Poynting flux is approximately $\Delta\omega V_p B_\phi B_p/4\pi \sim \Delta\omega c B_p^2/4\pi \sim (\Delta\omega/4\pi\sigma_\infty) c B_{p,\text{dipole}}^2$, with $\Delta\omega$ being the solid angle subtended by the active poloidal flux tube prior to the field-line expansion. After the active flux tube expands by a factor approximately $4\pi/\Delta\omega \sim \sigma_\infty^{-1}$, we recover that the luminosity associated with the actual Poynting flux is roughly the same as the spindown luminosity L_{spin} estimated from the dipole-radiation formula.

In sum, we have derived certain conditions to be satisfied if the Crab pulsar wind is to convert nearly its entire electromagnetic energy into kinetic energy near the light cylinder, through ideal MHD processes. First, only a small fraction $\sigma_\infty^{1/2}$ of the magnetic flux emerging from the neutron star carries the pulsar wind. Second, the neutron-star magnetosphere must deviate from the dipole configuration by a large factor; in particular, the toroidal field must dominate greatly over the poloidal field. The first requirement is consistent with the model of Beshkin et al. (1983), in which the pulsar wind plasmas are only loaded onto the small amount of open-field flux near the boundary of the closed-field magnetosphere. The second requirement indicates that the wind-loaded magnetic field prior to the major acceleration must have a force-free configuration since the kinetic energy is negligible and almost all the neutron-star rotational energy is stored in the toroidal magnetic field. In fact, in order to maintain pressure balance, the closed-field magnetosphere must also have an equally large toroidal magnetic field. In this regard, before the abrupt wind acceleration takes place, there can be no substantial difference between the closed-field and open-field regions. This is also consistent with the physical picture of Beskin et al. (1983), which asserts the sudden breakup of the closed field lines into the wind-loaded open field lines at the separatrix of the magnetosphere.

5. SUMMARY AND DISCUSSIONS

Mechanisms for producing the low- σ , relativistic pulsar winds in the stationary ideal MHD limit are explored. Effective conversion of electromagnetic energy into kinetic energy in such winds demands the flow-carrying flux tubes to diverge more rapidly than radially. However, the degree of flux-tube divergence is found to be severely constrained by the cross-field force balance and the condition of flux-freezing in the ideal MHD limit. Specifically, we have rigorously shown that it is impossible to arrange for any situation beyond a couple of light cylinder radii from the star where the electromagnetic energy can be largely converted into kinetic energy. The reason is that the magnetic pressure needed for pushing open the flux tube to yield flow acceleration is canceled to a large degree by the induced electric field once the flow becomes ultrarelativistic. Moreover, since the effective pressure within the ultrarelativistic flow is low, even adding an external support still cannot shape the flow to have the desired fanout configuration. With the proof given in §§ 2 and 3, we have narrowed down the only possible site for the major pulsar-wind acceleration to be located in the immediate neighborhood of the light cylinder.

In addition, we find that the acceleration of an ideal MHD pulsar wind to low- σ near the light cylinder requires serious modifications to the conventional picture of the pulsar magnetosphere. Contrary to the standard view, the outer magnetosphere would have to be dominated by the toroidal field and the wind plasma loaded onto only a small fraction of the magnetic flux emerging from the neutron star. Although our conclusions are qualitatively consistent with the existing model advocated by Beskin et al. (1983), their work does not address whether the wind-loaded flux tube can actually expand rapidly to assume a low- σ_∞ configuration. Nevertheless, such a model has the appropriate

ingredients to be a plausible contender for solving the low- σ_∞ problem and should be examined in more detail, especially in the region outside the light cylinder.

Our considerations apply only if the wind is stationary, obeys ideal MHD and attains low σ_∞ . The argument that the Crab pulsar wind *must* have a very low σ_∞ is based on the assumption of axisymmetry in the postshock nebular field. Begelman (1998) has challenged this argument, on the ground that axisymmetry will be destroyed by instabilities. MHD acceleration would have no difficulty in producing a wind with $\sigma_\infty \sim O(1)$, which may be consistent with the data according to Begelman (1998). Alternatively, the low- σ_∞ pulsar wind may be time-dependent (containing large-amplitude waves) (Lou 1993a, 1993b) and/or even engaged in dissipation (such as magnetic reconnection or shocks) (Coroniti 1990; Mestel & Shibata 1994). These physical processes are excluded from the framework of our analysis, and hence our proof does not apply to them.

Recent high-resolution X-ray observations of the Crab Nebula show unprecedentedly clear images of the X-ray torus and bipolar jets (Hester et al. 1995). Detailed modeling of the pulsar wind is therefore required to address how the open field lines should partition themselves into those along the equator and those along the rotational axis. Since our results indicate that the acceleration site should be near the light cylinder, this new observation points to the need to consider both the open field lines and closed magnetosphere together for understanding how a rapidly rotating neutron star loses its angular momentum and transports its rotational energy through two separate channels.

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