A Block Representation for Products of Hyperbolic Householder Transform

WEN-MING YAN
Department of Computer Science and Information Engineering
National Taiwan University
Taipei, Taiwan 10764, R.O.C.

KUO-LIANG CHUNG*
Department of Information Management
National Taiwan Institute of Technology
No. 43, Section 4, Keelung Road, Taipei, Taiwan 10672, R.O.C.
kchung@cs.ntit.edu.tw

(Received and accepted February 1996)

Communicated by D. J. Rose

Abstract—In this paper, a block representation for products of hyperbolic Householder transform, which is rich in matrix-matrix multiplications, is presented. Not only the representation is derived by a rather straightforward way, but it also extends the previous results [1,2] to the complex domain.

Keywords—BLAS 3 operation, Block representation, Complex domain, Hyperbolic Householder transform, QR factorization.

1. INTRODUCTION

The Householder transform [3] is very useful in matrix computations and signal processing [4]. In order to increase the performance of the Householder transform for QR factorization on vector supercomputers like CRAY series, Bischof and Van Loan [5] presented the first block Householder transform in terms of WY representation, which is rich in matrix-matrix multiplications, i.e., BLAS 3 operation [6]. Later, Schreiber and Van Loan [1] proposed a compact WY representation. Puglisi [2] presented an improved algorithm for involving more BLAS 3 operations based on the Woodbury-Morrison formula. We refer the reader to [7,8] for numerical behaviors of the compact representation.

In this paper, a block representation for products of hyperbolic Householder transform, which is rich in matrix-matrix multiplications, is presented. Not only the representation is derived by a rather straightforward way instead of using the Woodbury-Morrison formula, but it also extends the previous results [1,2] to the complex domain.

*Author to whom all correspondence should be addressed.

The authors are indebted to the referees and D. J. Rose for their valuable suggestions and comments, which improved the quality and presentation of this paper. We would also like to thank C. Bischof for providing us some related papers. This research was supported in part by the National Science Council of R.O.C. under Contract NSC85-2121-M011-002.
In Section 2, we first introduce the form for the complex Householder transform by Chung and Yan [9], then propose an alternative form and this form will be used to derive the block representation for the hyperbolic Householder transform.

2. THE COMPLEX HOUSEHOLDER TRANSFORM

In [10], the complex Hermitian transform has been developed. Recently, Venkaial, Krishna, and Paulraj [11] also extended the real Householder transform [3] to the complex domain \( C^n \). They first guessed the transform \( H \) being \( H = I - (1 + (a^*z/z^*a))(zz^*/z^*z) \), where \( a, b \in C^n \) and \( z = a - b \), then it was verified that \( Ha = b \) and \( H \) is unitary. Later, Xia and Suter [12] proved the necessary part of the Householder transform [11]. If \( a^*a \neq a^*b \), they first guessed that \( H = I - (1 + y)(zz^*/z^*z) \), where \( y \) is a complex number, then it was shown that \( y = -(z^*b/z^*a) \).

In the work of Chung and Yan [9], a complex Householder transform, \( H = I - (zz^*/z^*a) \), is given. This transform still satisfies the requirements \( Ha = b \) and \( H \) is unitary. Specifically, the transform is shown by a straightforward derivation although the two forms in [11,12] can be simplified to \( I - (zz^*/z^*a) \). Let \( \Phi \) be a diagonal matrix with diagonal entries +1 and -1.

In the hyperbolic Householder transform [13,14], we want to find a hypernormal matrix \( H \) such that \( Ha = b \) and \( H^*H = \Phi \). According to the derivation in [9], we obtain

\[
H = \bar{H}(a, b) = \Phi - \frac{zz^*}{z^*a}, \quad \text{where } z = \Phi a - b.
\]

Note that the above hyperbolic Householder transform is equal to the complex Householder transform when \( \Phi = I \).

For deriving the block representation of the hyperbolic Householder transform, we use an alternative form, \( H = \Phi \bar{H}(a, \Phi b) = I - \Phi wt w^* \), for hyperbolic Householder transform, where \( w = \Phi a - \Phi b \) and \( t = (1/w^*a) \). This alternative form also satisfies \( Ha = b \) and \( H^*H = \Phi \) (see the Appendix).

3. THE BLOCK HYPERBOLIC HOUSEHOLDER TRANSFORM

As what follows, some notations follow those used in [4]. Suppose \( Q_m = H_1 H_2 \ldots H_m \) is a product of these \( m (< n) \) alternative \( n \times n \) hyperbolic Householder matrices as described in Section 2. Let \( Q_m = I - \Phi Y_m T_m Y_m^* \) and \( H_{m+1} = I - \Phi Y_{m+1} t_{m+1}^{-1} y_{m+1}^* y_{m+1} \), where \( Y_m \) is a \( n \times m \) matrix, \( T_m \) is a \( m \times m \) matrix, \( y_{m+1} = \Phi a_{m+1} - \Phi b_{m+1} \) \((a_{m+1}^*\Phi a_{m+1} = b_{m+1}^*\Phi b_{m+1} \text{ and } a_{m+1} \neq b_{m+1})\), and \( t_{m+1} = y_{m+1}^* a_{m+1} \). The derivation to the block representation of \( Q_m H_{m+1} \) is shown as follows.

Let \( Q_{m+1} = Q_m H_{m+1} \), then we have

\[
Q_{m+1} = (I - \Phi Y_m T_m Y_m^*) (I - \Phi y_{m+1} t_{m+1}^{-1} y_{m+1}^*) = I - \Phi E,
\]

where \( E = Y_m T_m Y_m^* + y_{m+1} t_{m+1}^{-1} y_{m+1}^* - Y_m T_m Y_m^* \Phi y_{m+1} t_{m+1}^{-1} y_{m+1}^* \). Since the leftmost side of each term of \( E \) is \( Y_m \) or \( y_{m+1} \); the rightmost side of each term of \( E \) is \( Y_m^* \) or \( y_{m+1}^* \), we let

\[
E = Y_{m+1} T_{m+1} Y_{m+1}^*.
\]

where

\[
Y_{m+1} = (Y_m \quad y_{m+1}) \quad \text{and} \quad T_{m+1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

Hence, we have

\[
Y_{m+1} T_{m+1} Y_{m+1}^* = Y_m A Y_m^* + Y_m B y_{m+1}^* + y_{m+1} C Y_m^* + y_{m+1} D y_{m+1}^* = Y_m T_m Y_m^* + y_{m+1} t_{m+1}^{-1} y_{m+1}^* - Y_m T_m Y_m^* \Phi y_{m+1} t_{m+1}^{-1} y_{m+1}^*.
\]
Block Representation

where \( A = T_m, B = -T_mY_m^*\Phi t_{m+1}^{-1}Y_{m+1}, C = 0 \), and \( D = t_{m+1}^{-1} \). It follows that

\[
T_{m+1} = \begin{pmatrix} T_m & -T_mY_m^*\Phi t_{m+1}^{-1}Y_{m+1} \\ 0 & t_{m+1}^{-1} \end{pmatrix}.
\]  

(3)

By (1)-(3), we have

\[
Q_{m+1} = I - \Phi Y_{m+1}T_{m+1}Y_{m+1}^*.
\]  

(4)

Equation (4) extends the previous result [1] to the complex domain.

By induction, we have \( Q_k = I - \Phi Y_kT_kY_k^* \) for \( k = 1, 2, \ldots \), where

\[
Y_k = (y_1 \ y_2 \ \cdots \ y_k),
\]

\[
T_k = \begin{pmatrix} T_{k-1} & -T_{k-1}Y_{k-1}^*\Phi t_k^{-1}Y_k \\ 0 & t_k^{-1} \end{pmatrix},
\]

and \( T_1 = t_1^{-1} \). By (3), we also have

\[
T_{m+1}^{-1} = \begin{pmatrix} T_{m+1}^{-1} & -Y_m^*\Phi t_{m+1}Y_{m+1} \\ 0 & t_{m+1}^{-1} \end{pmatrix}.
\]

Let \( S_k = T_k^{-1} \) for \( k = 1, 2, \ldots \), then it follows that \( Q_k = I - \Phi Y_kS_k^{-1}Y_k^* \), where

\[
S_k = T_k^{-1} = \begin{pmatrix} S_{k-1} & Y_{k-1}^*\Phi y_k \\ 0 & t_k \end{pmatrix},
\]  

(5)

Now we consider \( s_{k,ij} \), the \( ij \) entry of \( S_k \), by (5), it is given by

\[
s_{k,ij} = s_{j,ij} = y_i^*\Phi y_j, \quad 1 < i < j \leq k,
\]

\[
s_{k,ij} = s_{i,ij} = 0, \quad 1 < j < i \leq k,
\]

\[
s_{k,ii} = s_{i,ii} = t_i.
\]

That is, we have

\[
S_k = \text{diag}(t_1, t_2, \ldots, t_k) + A_k,
\]

where \( A_k = [a_{ij}] \) and \( a_{ij} = y_i^*\Phi y_j \) for \( 1 < i < j \leq k; a_{ij} = 0 \) otherwise. The above block representation, \( Q_k = I - \Phi Y_kS_k^{-1}Y_k^* \), is the same as the one [2] when \( \Phi = I \). On the other hand, our block representation also extends the previous result [2] to the complex domain.

APPENDIX

From

\[
w^*a + a^*w = (\Phi(a - b))^*a + a^*(\Phi(a - b)) = a^*\Phi a - b^*\Phi a + a^*\Phi a - a^*\Phi b,
\]

\[
w^*\Phi w = (\Phi(a - b))^*(\Phi(a - b)) = (a - b)^*\Phi(a - b)
\]

\[
= a^*\Phi a - a^*\Phi b - b^*\Phi a + b^*\Phi b
\]

and

\[
w^*a + a^*w - w^*\Phi w = a^*\Phi a - b^*\Phi b = 0,
\]

it yields to

\[
H^*\Phi H = \left(I - \Phi \frac{ww^*}{w^*a}\right)^*\Phi \left(I - \Phi \frac{ww^*}{w^*a}\right) = \Phi - \left(\frac{1}{w^*w} + \frac{1}{w^*a} - \frac{w^*\Phi w}{(w^*a)(a^*w)}\right)ww^*
\]

\[
= \Phi - \frac{w^*a + a^*w - w^*\Phi w}{(w^*a)(a^*w)}ww^* = \Phi.
\]

Finally, \( Ha = b \) can be verified as follows:

\[
Ha = \left(I - \Phi \frac{ww^*}{w^*a}\right)a = a - \Phi w = a - (a - b) = b.
\]
REFERENCES


