ELSEVIER

Signal Processing 80 (2000) 1435–1444

**SIGNAL
PROCESSING**

www.elsevier.nl/locate/sigpro

Minimax design of 2-D linear-phase FIR filters with continuous and powers-of-two coefficients[☆]

Ju-Hong Lee^{a,*}, Shih-Jen Yang^b, Ding-Chiang Tang^a

^a*Department of Electrical Engineering, National Taiwan University, Rm 517, 2nd Building, Taipei 106, Taiwan*

^b*Department of Electronic Engineering and Technology, National Taiwan Institute of Technology, Taipei 106, Taiwan*

Received 8 March 1999; received in revised form 6 January 2000

Abstract

This paper deals with the minimax design problem of two-dimensional (2-D) linear-phase FIR digital filters with continuous and powers-of-two (POT) coefficients. First, the minimax continuous-coefficient design problem is formulated as a linear programming problem with inequality constraints. We present a method based on a variant of Karmarkar's algorithm to solve the resulting design problem. During each iteration, the main computing cost required for finding the filter coefficients is to solve a set of linear equations. Based on the obtained continuous coefficients, an efficient method is presented for designing a minimax 2-D FIR filter with POT coefficients in the spatial domain. Simulation results are provided for illustration and comparison. © 2000 Elsevier Science B.V. All rights reserved.

Zusammenfassung

Dieser Artikel behandelt das minimax Entwurfsproblem zweidimensionaler (2-D), linearphasiger FIR Digitalfilter mit kontinuierlichen Koeffizienten und solchen, die Potenzen von Zwei sind (POT, powers-of-two). Zuerst wird das minimax Entwurfsproblem mit kontinuierlichen Koeffizienten als ein lineares Programmierungsproblem mit Ungleichheits-Nebenbedingungen formuliert. Wir präsentieren eine Methode zur Lösung des sich ergebenden Entwurfsproblems basierend auf einer Variante des Algorithmus nach Karmarkar. Während jedem Iterationsschritt besteht der hauptsächliche Rechenaufwand in der Lösung eines Systems linearer Gleichungen. Basierend auf den kontinuierlichen Koeffizienten wird eine effiziente Methode zum Entwurf eines räumlichen 2-D minimax FIR Filters mit POT Koeffizienten präsentiert. Simulationsergebnisse zur Illustration und zum Vergleich werden vorgestellt. © 2000 Elsevier Science B.V. All rights reserved.

Résumé

Cet article traite du problème de conception minimax de filtre numériques FIR à phase linéaire bi-dimensionnels (2-D) ayant des coefficients continus et de type puissance de deux (POT). Tout d'abord, le problème de conception minimax pour des coefficients continus est formulé comme problème de programmation linéaire avec des contraintes d'inégalité. Nous présentons une méthode basée sur une variante de l'algorithme de Karmarkar pour résoudre le problème de conception résultant. A chaque itération, le coût de calcul principal pour les coefficients du filtre est dû à la résolution

[☆]This work was supported by the National Science Council under Grant NSC88-2218-E002-031.

* Corresponding author. Fax: 886-2-367-1909.

d'un ensemble d'équations linéaires. Sur la base des coefficients continus obtenus, une méthode efficace est proposée pour la conception d'un filtre FIR 2-D minimax ayant des coefficients POT dans le domaine spatial. Des résultats de simulation sont fournis à titre d'illustration et de comparaison. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: FIR filters; Algorithms; Optimization

1. Introduction

Two-dimensional (2-D) digital filters have been widely used for processing aerial and satellite photos, geophysical and seismological data, geological and medical images. 2-D finite impulse response (FIR) linear-phase filters are often preferred for the above applications because they possess several preferable characteristics, e.g., guaranteed stability and free from phase distortion. However, the number of independent filter coefficients required for optimally designing a 2-D FIR filter is generally large for practical applications. This leads to a research area of developing an efficient technique for designing 2-D FIR linear-phase filters optimal in the minimax sense.

Many techniques have been presented for the minimax design of 2-D linear-phase FIR filters [2,4–7]. Several exchange-type design techniques were developed based on the Remez exchange algorithm in [4,6]. However, the 2-D basis functions do not satisfy the Haar condition and the alternation theorem could not be applied to the 2-D case. Hence, using these techniques to optimally design 2-D linear-phase FIR filters suffers from two main problems, namely heavy computation load and convergence problems [7]. Based on a multiple simplex exchange concept, an algorithm consisting of an iterative sequence of linear programs was proposed in [2]. To obtain a minimax design, this algorithm still resorts to a variation of the simplex algorithm. Recently, design techniques using weighted least-squares (WLS) algorithm have attracted much attention [5]. Generally, this technique provides only quasi-minimax design results after convergence. In [1], the authors present the design results with powers-of-two (POT) coefficients for 2-D linear-phase FIR filters by using a simulated annealing algorithm. However, it is well known that using simulated-annealing-based

algorithms to design filters usually requires very high computing cost and memory.

In this paper, we deal with the minimax design of 2-D linear-phase FIR filters with continuous and POT coefficients. First, an efficient method is developed for designing continuous-coefficient 2-D linear-phase FIR filters optimal in the minimax sense. The design problem is formulated as a linear programming problem with inequality constraints. A primal affine scaling variant of Karmarkar's algorithm is presented to solve the resulting design problem. During each iteration, the main computing cost is to solve a set of linear equations. Then, we present an efficient method to find the POT coefficients of a minimax 2-D FIR filter by solving an optimization problem in the spatial domain. Simulation results show the effectiveness of the proposed methods.

This paper is organized as follows. Section 2 formulates the minimax design problem of 2-D linear-phase FIR filters with continuous coefficients. In Section 3, we present an efficient method based on a primal-form affine scaling variant of Karmarkar's algorithm to solve the resulting design problem. Section 4 presents an efficient method for the minimax design of 2-D FIR filters with POT coefficients. Several simulation examples are provided in Section 5 for illustration and comparison. Finally, this paper is concluded in Section 6.

2. Formulation of the minimax continuous-coefficient design problem

The frequency response of an $N_1 \times N_2$ 2-D FIR digital filter with impulse response $h(n_1, n_2)$ can be expressed as

$$H(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n_1=N_{11}}^{N_{12}} \sum_{n_2=N_{21}}^{N_{22}} h(n_1, n_2) e^{-j(\omega_1 n_1 + \omega_2 n_2)}, \quad (1)$$

where $N_1 = N_{12} - N_{11} + 1$, $N_2 = N_{22} - N_{21} + 1$. For linear-phase response, $h(n_1, n_2)$ must possess some symmetry about the center of coefficient support region, (N_{1c}, N_{2c}) , such as $h(n_1, n_2) = h(2N_{1c} - n_1, 2N_{2c} - n_2)$. Substituting this constraint into (1), we have the corresponding magnitude response for the 2-D FIR filter as follows:

$$H(\omega_1, \omega_2) = \sum \sum a(n_1, n_2) \cos(\omega_1(n_1 - N_{1c}) + \omega_2(n_2 - N_{2c})), \quad (2)$$

where the coefficients $a(n_1, n_2)$ are related to the impulse response of the 2-D FIR filter. For example, let $N_{1c} = N_{2c} = 0$, $N_1 = N_2 = 2N + 1$, and $h(n_1, n_2)$ possesses quadrantal symmetry, then we have

$$\begin{aligned} H(\omega_1, \omega_2) = & h(0, 0) + \sum_{n=1}^N 2[h(n, 0) \cos(n\omega_1) \\ & + h(0, n) \cos(n\omega_2)] \\ & + \sum_{n_1=1}^N \sum_{n_2=1}^N 4h(n_1, n_2) \cos(n_1\omega_1) \\ & \cos(n_2\omega_2). \end{aligned} \quad (3)$$

Accordingly, $a(n_1, n_2)$ are given by

$$\begin{aligned} a(n_1, n_2) &= \begin{cases} h(0, 0) & \text{for } n_1 = n_2 = 0, \\ 2h(n_1, 0) & \text{for } n_1 = 1, 2, \dots, N, \text{ and } n_2 = 0, \\ 2h(0, n_2) & \text{for } n_2 = 1, 2, \dots, N, \text{ and } n_1 = 0, \\ 4h(n_1, n_2) & \text{for } n_1, n_2 = 1, 2, \dots, N. \end{cases} \end{aligned} \quad (4)$$

In general, expression (3) can be rewritten as

$$H(\omega_1, \omega_2) = \sum_{k=0}^{M-1} a_k \phi_k(\omega_1, \omega_2), \quad (5)$$

where M is the number of independent parameters in the coefficient set $\{a(n_1, n_2)\}$, a_k is the k th independent parameter of $\{a(n_1, n_2)\}$, and $\phi_k(\omega_1, \omega_2)$ is the corresponding basis function.

Let $\Omega = (\omega_1, \omega_2)$ denote a sample point in the 2-D frequency plane, $\mathbf{I}_p$ and $\mathbf{I}_s$ represent the passband and stopband regions of the magnitude response $H(\omega_1, \omega_2)$, respectively. Each $\mathbf{I}_p$ and $\mathbf{I}_s$ may consist of several disjoint compact 2-D frequency

bands. The minimax design problem is thus to find the coefficients of $H(\omega_1, \omega_2)$ so that the maximum value over $\mathbf{I}_p$ and $\mathbf{I}_s$ of the following weighted error function is minimized:

$$E(\Omega) = W(\Omega) |H(\Omega) - D(\Omega)|, \quad (6)$$

where

$$\begin{aligned} D(\Omega) &= \begin{cases} 1 & \text{for } \Omega \in \mathbf{I}_p \\ 0 & \text{for } \Omega \in \mathbf{I}_s \end{cases} \quad \text{and} \\ W(\Omega) &= \begin{cases} 1 & \text{for } \Omega \in \mathbf{I}_p, \\ K & \text{for } \Omega \in \mathbf{I}_s, \end{cases} \end{aligned} \quad (7)$$

where K is the ratio between the passband and stopband ripples. Taking L_1 samples $\{\Omega_{p1}, \Omega_{p2}, \dots, \Omega_{pL_1}\}$ uniformly distributed in $\mathbf{I}_p$ and L_2 samples $\{\Omega_{s1}, \Omega_{s2}, \dots, \Omega_{sL_2}\}$ uniformly distributed in $\mathbf{I}_s$, we evaluate $E(\Omega)$ at these frequency samples and define the following matrices:

$$\begin{aligned} \mathbf{U}_p &= \begin{bmatrix} \phi_0(\Omega_{p1}) & \phi_1(\Omega_{p1}) & \dots & \phi_{M-1}(\Omega_{p1}) \\ \phi_0(\Omega_{p2}) & \phi_1(\Omega_{p2}) & \dots & \phi_{M-1}(\Omega_{p2}) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_0(\Omega_{pL_1}) & \phi_1(\Omega_{pL_1}) & \dots & \phi_{M-1}(\Omega_{pL_1}) \end{bmatrix}, \\ \mathbf{U}_s &= K \begin{bmatrix} \phi_0(\Omega_{s1}) & \phi_1(\Omega_{s1}) & \dots & \phi_{M-1}(\Omega_{s1}) \\ \phi_0(\Omega_{s2}) & \phi_1(\Omega_{s2}) & \dots & \phi_{M-1}(\Omega_{s2}) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_0(\Omega_{sL_2}) & \phi_1(\Omega_{sL_2}) & \dots & \phi_{M-1}(\Omega_{sL_2}) \end{bmatrix}. \end{aligned} \quad (8)$$

Using (8) and letting $\mathbf{a} = [a_0, a_1, \dots, a_{M-1}]^T$, we can formulate the minimax design problem of 2-D linear-phase FIR filters as follows:

$$\begin{aligned} & \text{minimize} && \delta \\ & \text{subject to} && \mathbf{U}_p \mathbf{a} - \delta \mathbf{1}_{L_1} \leq \mathbf{1}_{L_1}, \\ & && -\mathbf{U}_p \mathbf{a} - \delta \mathbf{1}_{L_1} \leq -\mathbf{1}_{L_1}, \\ & && \mathbf{U}_s \mathbf{a} - \delta \mathbf{1}_{L_2} \leq \mathbf{0}_{L_2}, \\ & && -\mathbf{U}_s \mathbf{a} - \delta \mathbf{1}_{L_2} \leq \mathbf{0}_{L_2}, \end{aligned} \quad (9)$$

where δ denotes the maximum value of (6), $\mathbf{1}_L$ and $\mathbf{0}_L$ are the two $L \times 1$ column vectors whose entries are one and zero, respectively. In matrix form, (9)

can be further expressed as

$$\begin{aligned} & \text{maximize} \quad \mathbf{b}^T \mathbf{w} \\ & \text{subject to} \quad \mathbf{A}^T \mathbf{w} \leq \mathbf{c}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} \mathbf{w} &= [\mathbf{a}^T, \delta]^T, \quad \mathbf{b} = [\mathbf{0}_M^T, -1]^T, \\ \mathbf{c} &= [\mathbf{1}_{L_1}^T, -\mathbf{1}_{L_1}^T, \mathbf{0}_{L_2}^T, \mathbf{0}_{L_2}^T]^T, \\ \mathbf{A} &= \begin{bmatrix} \mathbf{U}_p^T & -\mathbf{U}_p^T & \mathbf{U}_s^T & -\mathbf{U}_s^T \\ -\mathbf{1}_{L_1}^T & -\mathbf{1}_{L_1}^T & -\mathbf{1}_{L_2}^T & -\mathbf{1}_{L_2}^T \end{bmatrix}. \end{aligned} \quad (11)$$

Expression (10) represents the dual form of a conventional linear programming problem. The corresponding standard primal form of (10) is given by

$$\begin{aligned} & \text{minimize} \quad \mathbf{c}^T \mathbf{x} \\ & \text{subject to} \quad \mathbf{A}^T \mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \end{aligned} \quad (12)$$

where $\mathbf{x}$ represents the primal variable for the above linear programming (LP) problems.

Basically, (10) and (12) can be solved optimally by using a standard simplex LP package. However, due to the fact that the constraint matrix $\mathbf{A}$ with size $(M+1) \times 2(L_1 + L_2)$ is quite large, directly utilizing the simplex algorithms like those presented in [4] always requires large storage space and converges very slowly. In the next section, we describe how to use a variant of Karmarkar's algorithm to efficiently solve the design problem.

3. Minimax continuous-coefficient design method

According to the formulations shown by (10) and (12), we can solve the design problem by using either the dual-form or the primal-form LP algorithms. Among the algorithms for solving linear programming problems, Karmarkar's algorithm introduced in [9] has become the most notable one. It has been shown in [3] that the primal-form affine scaling variant of Karmarkar's algorithm generally produces better results than the dual-form affine scaling variant of Karmarkar's algorithm. Moreover, the convergence of the primal-form affine scaling algorithm has been proved in [3, pp. 152–155]. Therefore, we adopt the primal-form affine scaling variant of Karmarkar's algorithm

developed in [8] for solving (12) to find the filter coefficients optimal in the minimax sense.

Assume that an initial solution $\mathbf{x}$ which satisfies the constraints is given. Then, $\mathbf{x}$ is mapped into a vector $\mathbf{y}^0$ with all elements equal to one as follows:

$$\mathbf{y} = \mathbf{D}_x^{-1} \mathbf{x}, \quad (13)$$

where $\mathbf{D}_x$ denotes a diagonal matrix containing the components of $\mathbf{x}$. Based on the mapping, we create

$$\tilde{\mathbf{A}} = \mathbf{A} \mathbf{D}_x \quad \text{and} \quad \tilde{\mathbf{c}} = \mathbf{D}_x \mathbf{c}. \quad (14)$$

To satisfy the equality constraints, we project $\tilde{\mathbf{c}}$ onto the null space of $\tilde{\mathbf{A}}$ to obtain

$$\tilde{\mathbf{c}}_p = \mathbf{P} \tilde{\mathbf{c}}, \quad (15)$$

where

$$\mathbf{P} = \mathbf{I} - \tilde{\mathbf{A}}^T (\tilde{\mathbf{A}} \tilde{\mathbf{A}}^T)^{-1} \tilde{\mathbf{A}} \quad (16)$$

denotes the projection operator. Next, we move from the initial $\mathbf{y}$ to $\mathbf{y}^1$ in the direction $-\tilde{\mathbf{c}}_p$ to reduce the transformed objective function in the maximum rate according to

$$\mathbf{y}^1 = \mathbf{1} - \alpha \frac{\tilde{\mathbf{c}}_p}{\max_i (\mathbf{e}_i^T \tilde{\mathbf{c}}_p)}, \quad (17)$$

where the required step size $\alpha \in (0,1)$ is chosen so that $\mathbf{y}^1 > \mathbf{0}$, $\mathbf{1}$ represents a vector with appropriate size and all elements equal to one, and $\mathbf{e}_i$ is a vector with appropriate size and the i th element equal to one and the others equal to zero. After obtaining $\mathbf{y}^1$, we then find a new feasible solution $\mathbf{x}^1$ for (12) by performing the inverse mapping.

Consider the projection operator given by (16). Substituting (14) into (16) yields

$$\mathbf{P} = \mathbf{I} - \mathbf{D}_x \mathbf{A}^T (\mathbf{A} \mathbf{D}_x^2 \mathbf{A}^T)^{-1} \mathbf{A} \mathbf{D}_x. \quad (18)$$

Accordingly, we have from (14), (15), and (18) that

$$\tilde{\mathbf{c}}_p = \mathbf{D}_x [\mathbf{I} - \mathbf{A}^T (\mathbf{A} \mathbf{D}_x^2 \mathbf{A}^T)^{-1} \mathbf{A} \mathbf{D}_x^2] \mathbf{c}. \quad (19)$$

Next, we define the following two vectors:

$$\mathbf{w} = (\mathbf{A} \mathbf{D}_x^2 \mathbf{A}^T)^{-1} \mathbf{A} \mathbf{D}_x^2 \mathbf{c} \quad \text{and} \quad \mathbf{r} = \mathbf{c} - \mathbf{A}^T \mathbf{w}, \quad (20)$$

where $\mathbf{w}$ represents the dual variable associated with the primal variable $\mathbf{x}$. Consequently, the primal variable $\mathbf{x}^{k+1}$ at the $(k+1)$ th iteration is found by the following equation:

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \frac{\alpha}{\gamma} \mathbf{d}_x, \quad (21)$$

where

$$\begin{aligned} \mathbf{d}_x &= \mathbf{D}_x^2 \mathbf{r} \quad \text{and} \quad \gamma = \max_i (\mathbf{e}_i^T \tilde{\mathbf{c}}_p) \\ &= \max_i (\mathbf{e}_i^T \mathbf{D}_x \mathbf{r}) = \max_i \left(\frac{\mathbf{e}_i^T \mathbf{D}_x^2 \mathbf{r}}{x_i} \right) \end{aligned} \quad (22)$$

with x_i being the i th entry of $\mathbf{x}$.

Based on the above presentation, we summarize the proposed method by giving a design procedure step-by-step as follows:

Algorithm 1

Step 1: Initialization. Choose an initial guess $\mathbf{x}^0$ which satisfies the equality constraints $\mathbf{A}^T \mathbf{x}^0 = \mathbf{b}$ of (12). We can simply choose $\mathbf{x}^0 = [\mathbf{x}_1^T \ \mathbf{x}_2^T]^T$, where $\mathbf{x}_1 = \mathbf{x}_2 = (1/2J)\mathbf{1}$, $2J$ is equal to the number of sample points in frequency. Set the iteration number $k = 0$.

Step 2: At the k th iteration, compute $\mathbf{w} = [\mathbf{a}^T \ \delta]^T = (\mathbf{A} \mathbf{D}_x^2 \mathbf{A}^T)^{-1} \mathbf{A} \mathbf{D}_x^2 \mathbf{c}$ according to the following process:

(2.1): Construct two diagonal matrices $\mathbf{D}_i = \text{diag}(\mathbf{x}_i)$, $i = 1, 2$.

(2.2): Compute $\mathbf{f}_1 = \mathbf{D}_2 \mathbf{x}_2 - \mathbf{D}_1 \mathbf{x}_1$, $\mathbf{M}_1 = \mathbf{D}_1^2 + \mathbf{D}_2^2$, $\mathbf{c}_1 = -(\mathbf{1}^T \mathbf{M}_1 \mathbf{1})^{-1}$, $\mathbf{c}_2 = \mathbf{f}_1^T \mathbf{d}$, $\mathbf{v}_1 = \mathbf{U}^T \mathbf{f}_1$, $\mathbf{v}_2 = \mathbf{U}^T \mathbf{M}_1 \mathbf{d}$, where

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_p \\ \mathbf{U}_s \end{bmatrix}.$$

(2.3): Compute $\mathbf{a} = (\mathbf{U}^T \mathbf{M}_1 \mathbf{U} + \mathbf{c}_1 \mathbf{v}_1 \mathbf{v}_1^T)^{-1} (\mathbf{v}_2 + \mathbf{c}_1 \mathbf{c}_2 \mathbf{v}_1)$ and $\delta = \mathbf{c}_1 (\mathbf{v}_1^T \mathbf{a} - \mathbf{c}_2)$.

Step 3: Compute $\delta / \max\{|\mathbf{U} \mathbf{a} - \mathbf{d}|\}$ and stop the design procedure if $\delta / \max\{|\mathbf{U} \mathbf{a} - \mathbf{d}|\} \geq \eta$, where η is a preset small positive number. Otherwise, go to Step 4.

Step 4: Compute $\mathbf{d}_x = \mathbf{D}_x^2 \mathbf{r} = \mathbf{D}_x^2 (\mathbf{c} - \mathbf{A}^T \mathbf{w})$ according to the following process:

(4.1): Compute $\mathbf{d}_1 = \mathbf{D}_1^2 (\mathbf{d} + \delta \mathbf{1} - \mathbf{U} \mathbf{a})$, $\mathbf{d}_2 = \mathbf{D}_2^2 (-\mathbf{d} + \delta \mathbf{1} + \mathbf{U} \mathbf{a})$.

(4.2): Construct $\mathbf{d}_x = [\mathbf{d}_1^T \ \mathbf{d}_2^T]^T$.

Step 5: Compute step size α/γ from (22) and update the primal variable according to $\mathbf{x}^{k+1} = \mathbf{x}^k - (\alpha/\gamma) \mathbf{d}_x$. Then, set $k = k + 1$ and go to Step 2.

4. Powers-of-two coefficient design method

After obtaining the optimal continuous coefficients by Algorithm 1, we present an efficient

method for finding the POT coefficients which make the corresponding 2-D linear-phase FIR filter have minimax frequency response. The considered design problem is formulated as the following discrete optimization problem:

$$\text{minimize} \quad \|W(\Omega)[D(\Omega) - GH(\Omega)]\|_\infty$$

$$\text{subject to} \quad a_n \in \left\{ s : s = \sum_{k=1}^{B_n} c_k 2^{-p_k}, c_k \in \{0, 1\}, \right.$$

$$\left. p_k \in \{1, 2, \dots, I\} \right\},$$

$$\text{for } n = 0, 1, 2, \dots, M-1, \quad \sum_{n=0}^{M-1} B_n \leq N_T, \quad (23)$$

where $\|x\|_\infty$ denotes the L_∞ norm of x , G the scaling factor for coefficient discretization, B_n the number of POT terms used for representing a_n , I the coefficient wordlength excluding sign bit, and N_T is the upper bound of the total number of POT terms used for representing all of a_i , $i = 0, 1, \dots, M-1$.

However, (23) reveals that directly solving (23) requires to perform an optimization in the frequency domain and, hence, leads to a time-consuming task. Therefore, we resort to a suboptimal but efficient method which solves this problem in the time domain. Assume that the optimal continuous-coefficient vector obtained by Algorithm 1 is $\mathbf{a}_c$. It is appropriate to efficiently solve the problem of (23) by finding an optimal POT coefficient vector $\mathbf{a}_p$ which minimizes the peak error $\|G_o \mathbf{a}_p - \mathbf{a}_c\|_\infty$ subject to the same constraints given in (23), where G_o is the optimal scaling factor which produces the smallest possible peak approximation error. Moreover, the constraints of (23) can be expressed in a matrix form as follows:

$$\begin{aligned} \hat{\mathbf{C}} = & \left\{ \mathbf{C} : \mathbf{C} = [C_{ij}]_{1 \leq i \leq M, 1 \leq j \leq 2I}, C_{ij} \in \{0, 1\} \text{ and} \right. \\ & \left. \sum_{i=1}^M \sum_{j=1}^{2I} C_{ij} \leq N_T \right\}. \end{aligned} \quad (24)$$

Let $\mathbf{P}$ be a $2I \times 1$ vector given by $\mathbf{P} = [p(1) \ p(2) \ \dots \ p(2I)]^T = [-2^{-1} \ -2^{-2} \ \dots \ -2^{-I} \ 2^{-1} \ 2^{-2} \ \dots \ 2^{-I}]^T$. Then, $\mathbf{C}\mathbf{P}$ is a feasible solution for (23) if $\mathbf{C} \in \hat{\mathbf{C}}$. As a result, the corresponding optimization problem in the time domain can be

expressed as follows:

$$\text{Minimize } \|G_o \mathbf{C} \mathbf{P} - \mathbf{a}_c\|_\infty. \quad (25)$$

For a given scaling factor G , we note that the discrete parameter vector which minimizes $\|G\mathbf{a} - \mathbf{a}_c\|_\infty$ is the rounded version of $\mathbf{a}_c/G$. Consider the determination of the optimal scaling factor G_o . We present a time-saving manner as follows. Let S be a set of dense discrete scaling factors uniformly distributed in $[G_m, 2]$, where G_m is a suitable positive real value selected to ensure that the L_∞ norm of $\mathbf{a}_c/G_o$ is less than one. For each $G \in S$, we find the corresponding rounded parameter vector $\mathbf{a}_G = \mathbf{a}_c/G$. Then, we compute the corresponding $\mathbf{C}_G \in \hat{\mathbf{C}}$ such that

$$\|G\mathbf{C}_G \mathbf{P} - \mathbf{a}_c\|_\infty = \min_{\mathbf{C} \in \hat{\mathbf{C}}} \|\mathbf{G} \mathbf{C} \mathbf{P} - \mathbf{a}_c\|_\infty. \quad (26)$$

The cost function associated with the $\mathbf{C}_G$ is defined as

$$J(G) = \max(|\mathcal{W}(\Omega)[\mathbf{D}(\Omega) - \mathbf{U}\mathbf{a}_G]|). \quad (27)$$

The optimal scaling factor G_o is the value of G which minimizes $J(G)$, i.e.,

$$J(G_o) = \min_{G \in S} J(G). \quad (28)$$

Finally, we find the matrix $\mathbf{C}_{G_o}$ and the optimal POT coefficient vector $\mathbf{C}_{G_o} \mathbf{P}$ corresponding to G_o .

The following design procedure summarizes the proposed design method:

Algorithm 2

Step 1: Obtain the continuous-coefficient vector $\mathbf{a}_c$ using Algorithm 1. Choose an appropriate increment ΔG for adjusting G . Set the initial value $J_0 = 1$ for $J(G)$ and $G = G_m = \langle Q[\max(|\mathbf{a}_c|)2^B]2^{-B}/(1 - 2^{-B})\Delta G \rangle \Delta G$, where the positive integer B is chosen so that each POT coefficient of $\mathbf{a}_G$ is less than one, $Q[x]$ denotes rounding x to the nearest integer, and $\langle x \rangle$ is the largest integer not greater than x .

Step 2: Set $\mathbf{C}_G$ to a zero matrix and compute $\mathbf{a}_G = Q[\mathbf{a}_c 2^B/G]2^{-B}$.

Step 3: If the largest entry of $\mathbf{a}_G$ is less than 2^{-I} , we stop the design process. Otherwise, find the largest entry $E(n)$ of $\mathbf{a}_G$ and k such that $|E(n) - p(k)| = \min_{1 \leq i \leq 2I} |E(n) - (1 - C_{ni})p(k)|$.

Step 4: Compute $J(G)$ from (27). If $J(G) < J_0$, then set $J_0 = J(G)$, $G_o = G$, and $\mathbf{a}_p = \mathbf{a}_G$.

Step 5: Set $G = G + \Delta G$. If $G \leq 2$, then go to Step 2. Otherwise, go to Step 6.

Step 6: Take the corresponding $\mathbf{a}_p$ and G_o as the resulting POT coefficient vector and the optimal gain factor, respectively. Then, stop the design process.

5. Simulation examples

In this section, we present several simulation results of designing 2-D linear phase FIR filters. These designs are performed on a personal computer with Pentium CPU using MATLAB programming language. For all designs, the ratio K between the error ripples in passband and stopband is set to one and the 2-D filter with quadrantal symmetry shown by (3) is used for comparison. Under the same design specifications, the continuous-coefficient design results of using the design techniques provided by Harris and Mersereau [2], Burnside and Parks [4], and Hsieh et al. [5] are also listed in the tables for comparison. In the case of the designs with POT coefficients, we compare the results of the proposed method with those of [1].

Fig. 1. The dense grid for the design of 2-D circularly symmetric low-pass FIR digital filter.

The dense grid of frequency in the first quadrant is $(L + 1) \times (L + 1)$ uniformly distributed over I_p and I_s for all designs as shown in Fig. 1. Moreover, we take the grid points at the corresponding edge frequencies into account for evaluating the filter's performance at bandedges. The sampling density is given by $L/2N$ for each design case, where $L + 1$ is the number of grid points taken along each frequency direction and $N = (N_1 - 1)/2 = (N_2 - 1)/2$ as shown in (3). The parameters α and η required for continuous-coefficient designs are set to 0.99 and 0.99999, respectively. For POT coefficient designs, the coefficient wordlength I and the increment ΔG for adjusting G are set to 11 and 2^{-15} , respectively.

Example 1. Here, we consider the circularly symmetric low-pass 2-D filter design. Several continuous-coefficient designs with different specifications by using Algorithm 1, and the techniques of [2,4,5] are presented for illustration and comparison. Table 1 lists the design results in terms of the peak error ripple for several different specifications. We observe from Table 1 that the proposed design method provides the best design results except in the 9×9 case with $\omega_p = 0.35\pi$ and $\omega_s = 0.65\pi$, where the proposed method performs slightly worse than [5].

Example 2. In this example, we perform the POT coefficient designs for a circularly symmetric

Table 1

A comparison between the algorithms for the design of 2-D circularly symmetric FIR filters

Passband frequency	Stopband frequency	Sampling density	Filter size	Algorithm	Peak ripple magnitude
0.45 π	0.55 π	15	5 $\times$ 5	Harris	0.3673
				Hsieh	0.3693
				Algorithm 1	0.3651
			7 $\times$ 7	Harris	0.2473
				Hsieh	0.2472
				Algorithm 1	0.2456
			9 $\times$ 9	Harris	0.2330
				Hsieh	0.2375
				Algorithm 1	0.2324
			5 $\times$ 5	Harris	0.2670
				Burnside	0.2664
				Hsieh	0.2661
0.4 π	0.6 π	20	7 $\times$ 7	Algorithm 1	0.2654
				Harris	0.1272
				Burnside	0.1263
			9 $\times$ 9	Hsieh	0.1282
				Algorithm 1	0.1248
				Harris	0.1141
			5 $\times$ 5	Burnside	0.1141
				Hsieh	0.1179
				Algorithm 1	0.1139
			7 $\times$ 7	Harris	0.1905
				Hsieh	0.1956
				Algorithm 1	0.1878
0.35 π	0.65 π	15	7 $\times$ 7	Harris	0.0656
				Hsieh	0.0657
				Algorithm 1	0.0639
			9 $\times$ 9	Harris	0.0557
				Hsieh	0.0550
				Algorithm 1	0.0553

Table 2
A comparison between the algorithms for the design of 2-D circularly symmetric FIR filters with powers-of-two coefficients

Passband frequency	Stopband frequency	Filter size	Algorithm	N_T	Peak ripple magnitude
0.4π	0.6π	5×5	Banzato	10	0.26816
			Algorithm 2	10	0.26790
0.4π	0.6π	9×9	Banzato	22	0.11615
			Algorithm 2	22	0.12072
0.45π	0.55π	7×7	Banzato	19	0.27514
			Algorithm 2	19	0.25044
0.425π	0.575π	7×7	Banzato	19	0.19532
			Algorithm 2	19	0.18126
0.35π	0.65π	7×7	Banzato	19	0.07969
			Algorithm 2	19	0.06637

low-pass 2-D filter with some different specifications using Algorithm 2. Table 2 shows the design results in terms of peak error ripple. The results designed by using the simulated annealing (SA) algorithm of [1] are also listed for comparison. We note from this table that the proposed method is more effective than the SA algorithm of [1] except in the 9×9 case where the proposed method performs slightly worse than [1]. The impulse responses of the 7×7 filters with continuous and POT coefficients designed by using the proposed methods are listed in Table 3, while the corresponding magnitude responses are shown in Figs. 2 and 3, respectively.

Example 3. Here, a quadrantly symmetric deinterlacing filter with size 3×9 is designed. The desired diamond-shaped frequency response is given by

$$D(\omega_1, \omega_2) = \begin{cases} 1 & \text{for } \omega_1 + \omega_2 \leq \omega_p, \\ 0 & \text{for } \omega_1 + \omega_2 \geq \omega_s \end{cases} \quad (29)$$

with impulse response constraints given by $h(0,0) = 0.5$ and $h(n_1, n_2) = 0$ for (n_1, n_2) both even or both odd as described in [1]. The filters with $\omega_p = 0.8\pi$ and $\omega_s = 1.2\pi$ designed by using Algorithms 1 and 2 have impulse responses listed in Table 4, while the resulting magnitude responses are plotted in Figs. 4 and 5, respectively. The resulting peak ripple is 0.32725634 (−9.70223854 dB)

Table 3
The impulse responses of the designed 7×7 continuous and powers-of-two coefficient filters with $\omega_p = 0.35\pi$ and $\omega_s = 0.65\pi$ for Example 2

Continuous		Powers-of-two
G_o		23279×2^{-15}
$h(0,0)$	2.029391E−001	$2^{-2} + 2^{-5} + 2^{-8}$
$h(1,0)$	1.393165E−001	$-2^{-4} + 2^{-2} + 2^{-7} + 2^{-10}$
$h(1,1)$	9.537099E−002	$2^{-3} + 2^{-7} + 2^{-10}$
$h(2,0)$	3.311791E−002	$2^{-5} + 2^{-6}$
$h(2,1)$	1.070574E−002	2^{-6}
$h(2,2)$	−2.231470E−002	-2^{-5}
$h(3,0)$	−2.249081E−002	-2^{-5}
$h(3,1)$	−1.930922E−002	$-2^{-5} + 2^{-8}$
$h(3,2)$	−1.107001E−002	-2^{-6}
$h(3,3)$	−7.866187E−004	-2^{-10}

Table 4
Impulse responses of the filters designed for Example 3

Continuous		Powers-of-two
G_o		28728×2^{-15}
$h(0,0)$	5.000000E−001	$2^{-1} + 2^{-1} + 2^{-7}$
$h(1,0)$	2.229294E−001	2^{-2}
$h(0,1)$	2.117561E−001	$-2^{-7} + 2^{-2}$
$h(1,1)$	0	0
$h(0,2)$	0	0
$h(1,2)$	−2.199933E−002	$-2^{-5} + 2^{-7}$
$h(0,3)$	2.487472E−002	2^{-5}
$h(1,3)$	0	0
$h(0,4)$	0	0
$h(1,4)$	−2.513395E−002	-2^{-5}

Fig. 2. Perspective plot of the designed continuous-coefficient 7×7 filter with $\omega_p = 0.35\pi$ and $\omega_s = 0.65\pi$.

Fig. 3. Perspective plot of the designed powers-of-two coefficient 7×7 filter with $\omega_p = 0.35\pi$ and $\omega_s = 0.65\pi$.

for the designed continuous-coefficient filter, it becomes 0.329382 (-9.646001 dB) for the designed filter with POT coefficients. I and ΔG for this case are set to 11 and 2^{-15} , respectively.

6. Conclusion

This paper has presented methods for the minimax design of two-dimensional (2-D) linear-phase

Fig. 4. Perspective plot of the designed 3×9 continuous-coefficient filter in Example 3.

Fig. 5. Perspective plot of the designed 3×9 powers-of-two coefficient filter in Example 3.

FIR filters having continuous and powers-of-two coefficients. First, we formulate the design problem with continuous coefficients under the minimax error criterion. The resulting linear programming problem with inequality constraints can be solved by a primal-form affine scaling variant of Karmarkar's algorithm. Next, an efficient method has been presented to proceed with the design of minimax 2-D filters with powers-of-two coefficients. The effectiveness of each of the proposed methods has been demonstrated by several design examples.

References

- [1] L. Banzato, N. Benvenuto, G.M. Cortelazzo, A design technique for two-dimensional multiplierless FIR filters for video applications, *IEEE Trans. Circuit Systems Video Technol.* 2 (September 1992) 273–284.
- [2] D. Burnside, T.W. Parks, Optimal minimax two-dimensional FIR design using a multiple simplex exchange, *Proceedings of IEEE International Symposium on Circuits and Systems*, Vol. 2, 1994, pp. 593–596.
- [3] S.-C. Fang, S. Puthenpura, *Linear Optimization and Extensions: Theory and Algorithms*, Prentice-Hall, Englewood Cliffs, NJ, 1993.
- [4] D.B. Harris, R.M. Mersereau, A comparison of algorithms for minimax design of two-dimensional linear phase FIR digital filters, *IEEE Trans. Acoust. Speech Signal Process.* ASSP-25 (December 1977) 492–500.
- [5] C.-H. Hsieh, C.-M. Kuo, Y.-D. Jou, Y.-L. Han, Design of two-dimensional FIR digital filters by a two-dimensional WLS technique, *IEEE Trans. Circuits Systems II* 44 (May 1997) 348–356.
- [6] Y. Kamp, J.P. Thiran, Chebyshev approximation for two-dimensional nonrecursive digital filters, *IEEE Trans. Circuits Systems CAS-22* (March 1975) 208–218.
- [7] J.S. Lim, *Two-Dimensional Signal and Image Processing*, Prentice-Hall, Englewood Cliffs, NJ, 1990.
- [8] S.A. Ruzinsky, E.T. Olsen, L_1 and L_∞ minimization via a variant of Karmarkar's algorithm, *IEEE Trans. Acoust. Speech Signal Process.* ASSP-37 (February 1989) 245–253.
- [9] R.J. Vanderbei, M.S. Meketon, B.A. Freedman, A modification of Karmarkar's linear programming algorithm, *Algorithmica* 1 (1986) 395–407.