A cost-optimal parallel tridiagonal system solver

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Abstract. We first show how to transform the solution of an $n \times n$ tridiagonal system into suffix computations of continued fractions. Then a parallel substitution scheme is introduced to compute the suffix values. The derived parallel algorithm allows the tridiagonal system to be solved in $O(\log n)$ time on an unshuffle network with $\Theta(n/\log n)$ processors. It is cost-optimal in the sense that processor number times execution time is minimized. Our solver is conceptually simple and easy for implementation.

Keywords. Continued fraction, Suffix values, Recurrence equations, Substitution scheme, Tridiagonal system, Unshuffle.

1 Introduction

Consider a system of linear equations $Ax = b$, where the $n \times n$ coefficient matrix A is tridiagonal, that is, only elements directly on, directly above, and directly below the diagonal are nonzero and all the other entries are zero. The solution of a tridiagonal system lies in the heart of many important problems, such as the spline interpolation by cubic polynomials and the solutions of partial differential equations. It is in fact very easy to derive a sequential method for solving the tridiagonal systems in $O(n)$ time [7].

Based on the LU decomposition of matrix A , Stone [17,18] first proposed a recursive-doubling method to solve the system in $O(\log n)$ time using $\Theta(n)$ processors in parallel. Sameh and Kuck [16] showed that by the use of QR factorization the system can be solved in $O(\log n)$ time with $\Theta(n)$ processors too. The odd-even reduction method, which was due to Hockney [10], also provides parallel algorithms with the same cost [9,13]. Wang [20] presented a divide-and-conquer method which has the same cost and is suitable for both parallel and vector computers. Recently, the odd-even reduction method is applied to solve the tridiagonal system on the perfect shuffle network [11]. The time needed is $O(\log n)$ and the number of processors used is $n/(1 + \alpha)$, where $\alpha = (\text{the time to communicate one floating-point number})/(\text{the time to perform one floating-point arithmetic operation})$.

In this paper, we first show that the solution of a tridiagonal system can be converted into continued fraction based computations which correspond to the Gaussian elimination and the back-substitution applied on the system. Then, a parallel substitution scheme is introduced to compute the suffix values of the continued fractions. The derived parallel algorithm allows the tridiagonal system to be solved in $O(\log n)$ time on a network in which $\Theta(n/\log n)$ processors communicate through an unshuffle routing mechanism. Our solver greatly reduces the number of processors used previously and is cost-optimal in the sense that processor number times execution time is minimized.

2 Transformation to continued fractions

Suppose we have the following tridiagonal system:

$$\begin{aligned} d_1 x_1 + e_1 x_2 &= b_1, \\ c_i x_{i-1} + d_i x_i + e_i x_{i+1} &= b_i \quad \text{for } i = 2, 3, \dots, n-1, \\ c_n x_{n-1} + d_n x_n &= b_n. \end{aligned}$$

Assuming $d_1 \neq 0$, we can eliminate x_1 from the second equation to get the new equation

$$d'_2 x_2 + e_2 x_3 = b'_2,$$

where $d'_2 = d_2 - c_2 e_1 / d_1$ and $b'_2 = b_2 - c_2 b_1 / d_1$. Next, we use the above equation to eliminate x_2 from the third equation (assuming $d'_2 \neq 0$) and get

$$d'_3 x_3 + e_3 x_4 = b'_3,$$

where $d'_3 = d_3 - c_3 e_2 / d'_2$ and $b'_3 = b_3 - c_3 b'_2 / d'_2$. In general, at stage $(k-1)$, $2 \leq k \leq n$, we eliminate x_{k-1} from the k th equation (assuming $d'_{k-1} \neq 0$) and get

$$d'_k x_k + e_k x_{k+1} = b'_k,$$

with

$$d'_k = d_k - c_k e_{k-1} / d'_{k-1}, \quad (2.1)$$

and

$$b'_k = b_k - c_k b'_{k-1} / d'_{k-1}. \quad (2.2)$$

This procedure is the well-known Gaussian elimination.

During the back-substitution for obtaining x_i , $i = n, n-1, \dots, 1$, we first have (assuming $d'_n \neq 0$)

$$x_n = \frac{b'_n}{d'_n}$$

and then

$$x_k = \frac{b'_k - e_k x_{k+1}}{d'_k} \quad (2.3)$$

for $k = n-1, n-2, \dots, 1$.

Up to here, the solution of the tridiagonal system has been transformed into three recurrence equations, namely, (2.1), (2.2) and (2.3), for d'_k , b'_k and x_k respectively. In the remainder of this section, these recurrences will be further transformed into suffix computations of continued fractions.

Notation. The following form represents a finite continued fraction (CF):

$$l_1 + \frac{m_2}{l_2 + \frac{m_3}{\dots \frac{m_n}{l_{n-1} + \frac{m_n}{l_n}}}}$$

Such a representation for CF is typographically cumbersome and can be replaced by a more compact form:

$$l_1 + \frac{m_2}{l_2 +} \frac{m_3}{l_3 +} \dots \frac{m_n}{l_n}.$$

Let us define a second-order linear recurrence:

$$\begin{aligned} f_0 &= 1, & f_1 &= d_1, \\ f_k &= d_k f_{k-1} - c_k e_{k-1} f_{k-2} \quad \text{for } k \geq 2. \end{aligned} \quad (2.4)$$

It is easily seen that $d'_k = f_k/f_{k-1}$ for $k \geq 1$. The computation of d'_k breaks down when $f_{k-1} = 0$. Nevertheless, this rarely happens in practice although there is no theoretical guarantee for it [1].

By (2.4), $d'_n = f_n/f_{n-1} = (d_n f_{n-1} - c_n e_{n-1} f_{n-2})/f_{n-1} = d_n + (-c_n e_{n-1})/(f_{n-1}/f_{n-2})$. So d'_n can be expressed as the following CF:

$$d'_n = d_n + \frac{-c_n e_{n-1}}{d_{n-1} +} \frac{-c_{n-1} e_{n-2}}{d_{n-2} +} \cdots \frac{-c_3 e_2}{d_2 +} \frac{-c_2 e_1}{d_1}.$$

We may assume that n is a power of 2; otherwise, pad d'_n with a special CF

$$\frac{0}{1+} \frac{0}{1+} \cdots \frac{0}{1}$$

to enlarge n to be a power of 2 without altering the computation results.

The k th suffix value of d'_n is d'_k , i.e.

$$d'_k = d_k + \frac{-c_k e_{k-1}}{d_{k-1} +} \frac{-c_{k-1} e_{k-2}}{d_{k-2} +} \cdots \frac{-c_3 e_2}{d_2 +} \frac{-c_2 e_1}{d_1}.$$

Obviously, d'_k can be sequentially converted into its rational form in $O(k)$ time. Our goal is to speed up the computation of obtaining the rational form of d'_k for all $k = 1, 2, \dots, n$.

3 The parallel substitution scheme

Let the rational form $(r_1 + r_2 d'_1)/(r_3 + r_4 d'_1)$ be represented by $(r_1, r_2, r_3, r_4, d'_1)$. Here d'_i can be considered as a pseudo variable for substitution. The CF of, say, d'_8 can be decomposed into 8 sub-CFs:

$$\begin{aligned} d'_8 &= d_8 + \frac{-c_8 e_7}{d'_7} = (-c_8 e_7, d_8, 0, 1, d'_7), \\ d'_7 &= d_7 + \frac{-c_7 e_6}{d'_6} = (-c_7 e_6, d_7, 0, 1, d'_6), \\ &\vdots \\ d'_3 &= d_3 + \frac{-c_3 e_2}{d'_2} = (-c_3 e_2, d_3, 0, 1, d'_2), \\ d'_2 &= d_2 + \frac{-c_2 e_1}{d'_1} = (-c_2 e_1, d_2, 0, 1, d'_1), \\ d'_1 &= \frac{d_1}{1 + d'_0} = (d_1, 0, 1, 1, d'_0). \end{aligned}$$

The pseudo-variable d'_0 will be replaced by zero after all the substitutions are completed.

Since substituting $d'_i = (s_1 + s_2 d'_j)/(s_3 + s_4 d'_j)$ into $(r_1 + r_2 d'_i)/(r_3 + r_4 d'_i)$ gives $((r_1 s_3 + r_2 s_1) + (r_1 s_4 + r_2 s_2) d'_j)/((r_3 s_3 + r_4 s_1) + (r_3 s_4 + r_4 s_2) d'_j)$, the substitution operation which uses 8 multiplications and 4 additions can be defined as:

$$\begin{aligned} &(r_1, r_2, r_3, r_4, d'_i) \circ (s_1, s_2, s_3, s_4, d'_j) \\ &= (r_1 s_3 + r_2 s_1, r_1 s_4 + r_2 s_2, r_3 s_3 + r_4 s_1, r_3 s_4 + r_4 s_2, d'_j). \end{aligned}$$

Lemma 3.1. *The substitution operation $\circ$ is associative.*

Proof. For any $(r_1, r_2, r_3, r_4, d'_i)$, $(s_1, s_2, s_3, s_4, d'_j)$ and $(t_1, t_2, t_3, t_4, d'_k)$, we have

$$\begin{aligned} & \left[(r_1, r_2, r_3, r_4, d'_i) \circ (s_1, s_2, s_3, s_4, d'_j) \right] \circ (t_1, t_2, t_3, t_4, d'_k) \\ &= (r_1 s_3 + r_2 s_1, r_1 s_4 + r_2 s_2, r_3 s_3 + r_4 s_1, r_3 s_4 + r_4 s_2, d'_j) \circ (t_1, t_2, t_3, t_4, d'_k) \\ &= (r_1 s_3 t_3 + r_2 s_1 t_3 + r_1 s_4 t_1 + r_2 s_2 t_1, r_1 s_3 t_4 + r_2 s_1 t_4 + r_1 s_4 t_2 + r_2 s_2 t_2, \\ & \quad r_3 s_3 t_3 + r_4 s_1 t_3 + r_3 s_4 t_1 + r_4 s_2 t_1, r_3 s_3 t_4 + r_4 s_1 t_4 + r_3 s_4 t_2 + r_4 s_2 t_2, d'_k) \end{aligned}$$

and

$$\begin{aligned} & (r_1, r_2, r_3, r_4, d'_i) \circ \left[(s_1, s_2, s_3, s_4, d'_j) \circ (t_1, t_2, t_3, t_4, d'_k) \right] \\ &= (r_1, r_2, r_3, r_4, d'_i) \circ (s_1 t_3 + s_2 t_1, s_1 t_4 + s_2 t_2, s_3 t_3 + s_4 t_1, s_3 t_4 + s_4 t_2, d'_k) \\ &= (r_1 s_3 t_3 + r_2 s_1 t_3 + r_1 s_4 t_1 + r_2 s_2 t_1, r_1 s_3 t_4 + r_2 s_1 t_4 + r_1 s_4 t_2 + r_2 s_2 t_2, \\ & \quad r_3 s_3 t_3 + r_4 s_1 t_3 + r_3 s_4 t_1 + r_4 s_2 t_1, r_3 s_3 t_4 + r_4 s_1 t_4 + r_3 s_4 t_2 + r_4 s_2 t_2, d'_k). \end{aligned}$$

Since both computation sequences give the same rational form, the associativity holds. $\square$

According to Lemma 3.1, the three substitution operations in $d'_4 = (-c_4 e_3, d_4, 0, 1, d'_3) \circ (-c_3 e_2, d_3, 0, 1, d'_2) \circ (-c_2 e_1, d_2, 0, 1, d'_1) \circ (d_1, 0, 1, 1, d'_0)$ can be performed in any order. Fig. 1 shows a computation tree for d'_4 . The white nodes in the leftmost stage 0 are used for preparing input data only. The black nodes in stages 1 and 2 perform the substitution operation. The single node in stage 3 assigns zero to the pseudo variable d'_0 to obtain the desired result d'_4 .

If we combine the computation trees for d'_i , $1 \leq i \leq 8$, we have an overlaid tree network, as shown in Fig. 2, for computing all the suffix values of d'_8 . The white nodes in stages 1, 2, and 3 are used for transmitting data only. The pseudo-variables are not real data and hence are omitted there. Note that there are $\log n + 2$ stages in this network, meaning that the computation time is only $O(\log n)$. If each node is implemented by a processor, the processor number is as large as $n(\log n + 2)$, which is going to be reduced to $\Theta(n/\log n)$ in the later sections.

Fig. 1. A computation tree of d'_4 .

Fig. 2. A network for computing suffix values of d'_8 .

Now for the computation of b'_k in (2.2), the CF for b'_4/b'_3 , say, is given by

$$\begin{aligned} X_4 &= \frac{b'_4}{b'_3} \\ &= \frac{b_4 - (c_4/d'_3)b'_3}{b'_3} \\ &= (-c_4/d'_3) + \frac{b_4}{b'_3}. \end{aligned}$$

When x is of the form $r_1 + r_2/r_3$, we define $N(x)$ to be the value $r_1 r_3 + r_2$. The sub-CFs for the above CF are

$$\begin{aligned} X_4 &= (-c_4/d'_3) + \frac{b_4}{N(X_3)}, \\ X_3 &= (-c_3/d'_2) + \frac{b_3}{N(X_2)}, \\ X_2 &= (-c_2/d'_1) + \frac{b_2}{N(X_1)}, \\ X_1 &= \frac{b_1}{1 + N(X_0)}, \end{aligned}$$

where $N(X_0) = 0$, $N(X_1) = b_1$, and $b'_i = -(c_i/d_{i-1})b'_{i-1} + b_i = N(X_i)$ for all i .

Since substituting $(s_1 + s_2 N(X_j))/(s_3 + s_4 N(X_j))$ into X_i in $(r_1 + r_2 N(X_i))/(r_3 + r_4 N(X_i))$, i.e. substituting $s_1 + s_2 N(X_j)$ into $N(X_i)$, gives $((r_1 + r_2 s_1) + r_2 s_2 N(X_j))/((r_3 + r_4 s_1) + r_4 s_2 N(X_j))$, the substitution operation here is defined as:

$$\begin{aligned} &(r_1, r_2, r_3, r_4, N(X_i)) \circ (s_1, s_2, s_3, s_4, N(X_j)) \\ &= (r_1 + r_2 s_1, r_2 s_2, r_3 + r_4 s_1, r_4 s_2, N(X_j)). \end{aligned}$$

This operation uses 4 multiplications and 2 additions and can be verified to be associative too. The problem of solving the recurrence in (2.2) becomes the suffix computations of X_n , so the overlaid tree network in Fig. 2 can be used again.

Similarly, from (2.3), the sub_CFs for x_{n-3}/x_{n-2} , say, is given by

$$Y_{n-3} = (-e_{n-3}/d'_{n-3}) + \frac{b'_{n-3}/d'_{n-3}}{N(Y_{n-2})},$$

$$Y_{n-2} = (-e_{n-2}/d'_{n-2}) + \frac{b'_{n-2}/d'_{n-2}}{N(Y_{n-1})},$$

$$Y_{n-1} = (-e_{n-1}/d'_{n-1}) + \frac{b'_{n-1}/d'_{n-1}}{N(Y_n)},$$

$$Y_n = \frac{b'_n/d'_n}{1 + N(Y_0)},$$

where $N(Y_{n+1}) = 0$, and $x_i = N(Y_i)$ for all i . The problem of solving the recurrence in (2.3) becomes the suffix computations of Y_i .

4 Three-phase algorithm on unshuffle network

In this section, we propose a three-phase algorithm on the unshuffle network for computing all the suffix values of d'_n , X_n and Y_1 . For saving space, we only discuss the suffix computations of d'_n . By adapting appropriate operations in the processors, the suffix computations of X_n and Y_1 can be similarly performed on the same network with the same time complexity. Previously, two approaches have been proposed for some tree-like computations. Based on a depth-width tradeoff in overlaid tree structure, Carlson and Sugla [2,3] presented a perfect shuffle network to solve the first-order linear recurrence in $O(n/p)$ time using $\Theta(p \log p)$ processors. Meijer and Akl [14] presented an optimal binary tree network for computing prefix sums. Our algorithm is based on Stone's unshuffle data routing [19].

In Fig. 2, there is one white node in the first stage, two in the second stage and four in the third stage. The multi-stage unshuffle network which can simulate that network is depicted in Fig. 3, where $S_{i,j}$ and $P_{i,j}$, $0 \leq i \leq 7$, $1 \leq j \leq 3$, are simple devices and processors respectively. Let $q_0 q_1 \cdots q_{m-1}$ ($m = \log n$) be the binary representation of i . The output line of $P_{i,j}$ is

Fig. 3. The multi-stage unshuffle network.

Fig. 4. The single-stage unshuffle network.

Fig. 5. The three-phase unshuffle network.

connected to $S_{q_{m-1}q_0q_1 \dots q_{m-2},j+1}$ to provide the unshuffle data routing. The simple device $s_{i,j}$ is capable of producing two copies of its input and sends them to $P_{i,j}$ and $P_{i+1,j}$ except that the bottom $S_{11 \dots 1,j}$ has only one receiver. For tracing the messages passed in the network, each simple device is labelled by a number inside the parentheses to indicate where its data originate.

Due to the unshuffle connections from the processors to the simple devices in Fig. 3, the function of white nodes in a stage of Fig. 2 can be controlled by a masking mechanism. For instance, $P_{0,1}$ in stage 1 is masked by a cross mark to merely transmit the input to the next stage; $P_{0,2}$ and $P_{4,2}$ in stage 2 are masked also. Similarly, in stage 3, $P_{0,3}$, $P_{2,3}$, $P_{4,3}$, $P_{6,3}$ are masked.

Since all the interconnection patterns between the stages of the multi-stage unshuffle network are the same, we can compress the stages all together to obtain a single-stage unshuffle network, as shown in Fig. 4, which has only n processors. After each iteration of execution, the processors feed the intermediate results back to the simple devices as inputs for the next iteration. The masking on the network at different iterations is exactly what we have described for different stages of the multi-stage network.

In the following, a modified three-phase unshuffle network is proposed to reduce the processor number further. We start with an unshuffle network of k simple devices and k processors, $k \leq n$, and change the unidirectional connections from the simple devices to the processors to be bidirectional. To each processor P_i , attach a serial memory [15], which consists of a linearly connected array of memory cells $C_{i,j}$, $0 \leq j \leq n/k - 1$, as depicted in Fig. 5 for $n = 8$ and $k = 4$. In each execution cycle, the attached memory shifts all its data one position left, P_i executes on the data shifted from $C_{i,0}$ and writes its result to $C_{i,n/k-1}$.

According to the value of the mask register M_i resided in P_i , which is set by the masking mechanism, P_i takes one of the following actions:

- (1) $M_i = 0$: P_i transmits the content of its working register Q_i to S_i . (Note that if only one-way connection from S_i to P_i is allowed, sending data from P_i to S_i requires $O(\log k)$ time.)
- (2) $M_i = 1$: P_i receives data from S_i and transmits it to $S_{q_{m-1}q_0q_1 \dots q_{m-2},j}$, where $m = \log k$.
- (3) $M_i = 2$: P_i performs the substitution operation on its two input data and transmits the result to $S_{q_{m-1}q_0q_1 \dots q_{m-2},j}$.

- (4) $M_i = 3$: Except P_0 which receives input data from S_0 , P_i receives input data from S_{i-1} . The routing mechanism is forced to stop and the network goes to another phase of computation.

Initially, n input data (quadruples) are evenly divided into k pipes, each of them is stored in a local memory. Algorithm 4.1 expresses how the network works.

Algorithm 4.1

Phase 1 (Local computations).

Each P_i sequentially computes n/k suffix values from its corresponding pipe of data and stores them in its local memory. The register Q_i has the final rational form computed from the pipe of data and transmits it to the register R_i in S_i .

Phase 2 (Macro suffix computations).

All the processors work together using the unshuffle routing mechanism for suffix computations. After $\log k$ steps, the working register R_i holds the rational form of $d'_{(i+1)n/k}$, for $0 \leq i \leq k-1$.

Phase 3 (Adaptation).

Each P_i except P_0 receives the rational form from S_{i-1} , sequentially modifies all the suffix values calculated in phase 1 by substituting them into the received rational form.

Phase 1 and phase 3 both contain n/k steps. Phase 2 contains $\log k$ steps. So the three-phase algorithm takes $O(n/k + \log k)$ time in total to finish computing all the suffix values of d'_n .

We use $d'_8 (= d_8 + \frac{-c_8 e_7}{d_7} \dots \frac{-c_2 e_1}{d_1})$ as an example to illustrate the algorithm. Let $n = 8$ and $k = 4$. Initially, the 8 input data (quadruples) for d'_8 are evenly divided into 4 pipes, each containing 2 data. The cells $C_{0,0}$ and $C_{0,1}$ have the data $(d_1, 0, 1, 1)$ and $(-c_2 e_1, d_2, 0, 1)$ respectively. The other cells $C_{i,j}$ $1 \leq i \leq 3$, $0 \leq j \leq 1$, have the data $(-c_{2i+j+1} e_{2i+j}, d_{2i+j+1}, 0, 1)$. We denote $d_i + \frac{-c_i e_{i-1}}{d_{i-1}} \dots \frac{-c_2 e_1}{d_1 + d_0}$ by $F_{i,1}$ and $d_i + \frac{-c_i e_{i-1}}{d_{i-1} + \dots} \dots \frac{-c_j e_{j-1}}{d'_{j-1}}$ by $F_{i,j}$ for $i \geq j \geq 2$.

Phase 1.

Each individual P_i computes the two forms $F_{2i+1,2i+1}$ and $F_{2i+2,2i+1}$ and shifts them back to the serial memory sequentially. That is, P_0 computes all the suffix values of $F_{2,1}$; P_1 computes all the suffix values of $F_{4,3}$, and so on.

The cell $C_{i,j}$ saves $F_{2i+j+1,2i+1}$. After 2 steps in this phase, we have

$$\begin{aligned} C_{3,0} &= F_{7,7}, & C_{3,1} &= F_{8,7}, \\ C_{2,0} &= F_{5,5}, & C_{2,1} &= F_{6,5}, \\ C_{1,0} &= F_{3,3}, & C_{1,1} &= F_{4,3}, \\ C_{0,0} &= F_{1,1}, & C_{0,1} &= F_{2,1}, \end{aligned}$$

and

$$Q_3 = F_{8,7}, \quad Q_2 = F_{6,5}, \quad Q_1 = F_{4,3}, \quad Q_0 = F_{2,1}.$$

Then, for all i , M_i becomes 0 and the content of Q_i is transmitted to R_i .

Phase 2.

The following macro suffix computations are controlled by the unshuffle routing mechanism. During the first iteration, $M_0 = 1$ and $M_i = 2$ for $i = 1, 2, 3$. Once the P_i 's complete their

substitution operations, the contents of R_i 's become

$$\begin{aligned} R_3 &= F_{8,7} \circ F_{6,5} = F_{8,5}, \\ R_2 &= F_{6,5} \circ F_{4,3} = F_{6,3}, \\ R_1 &= F_{4,3} \circ F_{2,1} = F_{4,1}, \\ R_0 &= F_{2,1}. \end{aligned}$$

During the second iteration, $M_i = 2$ for all i except $i \equiv 0 \pmod{2}$ for which $M_i = 1$. Once the P_i 's complete their substitution operations again, the contents of R_i 's become

$$\begin{aligned} R_3 &= F_{8,5} \circ F_{4,1} = F_{8,1}, \\ R_2 &= F_{6,3} \circ F_{2,1} = F_{6,1}, \\ R_1 &= F_{4,1}, \\ R_0 &= F_{2,1}. \end{aligned}$$

Let k be 2^h . In general, during the j th iteration, $1 \leq j \leq h$, $M_i = 2$ for all i , except $i \equiv 0 \pmod{(2^{h+1-j})}$ for which $M_i = 1$. After this phase of computation, R_i saves $F_{2(i+1),1}$, a macro suffix value.

Phase 3.

During the first step of this phase, $M_i = 3$, each P_i except P_0 receives input data from S_{i-1} . The contents of Q_i 's become

$$Q_3 = F_{6,1}, \quad Q_2 = F_{4,1}, \quad Q_1 = F_{2,1}, \quad Q_0 = F_{2,1}.$$

The unshuffle routing mechanism is forced to stop and each P_i sequentially modifies the contents of $C_{i,j}$'s, $j = 0, 1$, by performing the substitution operation $Q_i \circ C_{i,j}$. The final results are

$$\begin{aligned} C_{3,0} &= F_{7,7} \circ F_{6,1} = F_{7,1}, & C_{3,1} &= F_{8,7} \circ F_{6,1} = F_{8,1}, \\ C_{2,0} &= F_{5,5} \circ F_{4,1} = F_{5,1}, & C_{2,1} &= F_{6,5} \circ F_{4,1} = F_{6,1}, \\ C_{1,0} &= F_{3,3} \circ F_{2,1} = F_{3,1}, & C_{1,1} &= F_{4,3} \circ F_{2,1} = F_{4,1}, \\ C_{0,0} &= F_{1,1}, & C_{0,1} &= F_{2,1}. \end{aligned}$$

5 Cost optimality

As discussed in Section 4, all the suffix values of d'_n can be computed in $O(n/k + \log k)$ time using k processors. For the subsequent parts of the computation, namely computing the suffix values of X_n and Y_1 , the algorithm works as well if we properly adjust the substitution operations in the processors. Preparing the input data from one part of the computation to another takes $O(n/k)$ time, n/k is the size of each data pipe. So we have the following lemma.

Lemma 5.1. *A tridiagonal system can be solved in $O(n/k + \log k)$ time on an unshuffle network of k processors.*

The performance of a parallel algorithm can be measured by

$$\text{Cost} = \text{Processor Number} \times \text{Execution Time}.$$

Given a problem, if the cost of a parallel algorithm matches the sequential time lower bound within a constant, the parallel algorithm is said to be cost-optimal. In the case of solving the

tridiagonal system, since there are n values to be produced, the sequential time lower bound is clearly $\Omega(n)$.

Theorem 5.1 determines the number of processors that is required to minimize both the cost and the execution time.

Theorem 5.1. *For solving tridiagonal system on the unshuffle network, we can use $\Theta(n/\log n)$ processors to achieve the optimal cost $\Omega(n)$ and the computing time $O(\log n)$.*

Proof. By Lemma 5.1, the cost of our three-phase computation model with $k (= 2^h)$ processors is $O(k(n/k + \log k)) = O(n + k \log k)$. Because $k \log k$ is an increasing function of k , the solution k satisfying $k \log k = cn$ (for some constant c) is the number of processors needed at most. The above equation can be written as

$$2^h h = cn = n'.$$

For applying the bootstrapping method [8], we initially let

$$h = \log n' - \log h. \quad (5.1)$$

Therefore,

$$h = O(\log n').$$

Substituting this approximation into (5.1) yields the following better result:

$$h = \log n' + O(\log \log n').$$

Substituting this approximation again into (5.1) improves the result as follows:

$$\begin{aligned} h &= \log n' - \log \left(\log n' \left(1 + O \left(\frac{\log \log n'}{\log n'} \right) \right) \right) \\ &= \log n' - \log \log n' - \log \left(1 + O \left(\frac{\log \log n'}{\log n'} \right) \right). \end{aligned} \quad (5.2)$$

By Taylor's expansion, we know

$$e^{\frac{\log \log n'}{\log n'}} = 1 + O \left(\frac{\log \log n'}{\log n'} \right).$$

That is,

$$\frac{\log \log n'}{\log n'} = \log \left(1 + O \left(\frac{\log \log n'}{\log n'} \right) \right).$$

Hence, (5.2) can be written as

$$h = \log \frac{n'}{\log n'} + O \left(\frac{\log \log n'}{\log n'} \right).$$

Recovering k from $k = 2^h$,

$$\begin{aligned} k &= \frac{n'}{\log n'} + O \left(2^{\frac{\log \log n'}{\log n'}} \right) \\ &= \Theta(n/\log n). \end{aligned}$$

It is clear that when $k = \Theta(n/\log n)$ processors are used in the network, the computing time is $O(n/k + \log k) = O(\log n)$. $\square$

6 Concluding remarks

In this paper we have first transformed the solution of a tridiagonal system into three recurrence equations. The recursive-doubling method [12,17] is an alternative approach to solve these recurrence equations through a rather tricky divide-and-conquer reformulation of the recurrences into ones which use two indices. Our approach is simply doing their CF expansions and applying the straightforward substitution concept. Not only the associativity of the substitution brings out the parallelism for the computation, but also it supplies the idea of three-phase computation to reduce the processor number. An analysis technique presented in [6] can be used to verify that our method is numerically stable when the coefficient matrix A is symmetric and diagonally dominant.

The substitution scheme used in this paper was originally presented in [5] for the parallel computation of general continued fractions. It allows all the first n prefix values of any continued fraction to be computed in $O(\log n)$ time using $\Theta(n/\log n)$ processors. We also embed a self-substitution concept into the scheme to derive a sequential $O(\log n)$ algorithm to fast compute periodic CFs [4]. Its applications include the approximation of quadratic surd numbers and the solving of second-order linear recurrences with constant coefficients.

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