Explicit Birefringence Analysis for Anisotropic Fibers

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Abstract—A hyper-perturbation approach is employed to find the Taylor expansion formulas for the dispersion curves of arbitrarily indexed optical fibers. These formulas are then used to give analytically the birefringence characteristics of optical fibers with biaxially anisotropic core and cladding. The dependence of the fiber birefringence on the index difference, index anisotropy, and the normalized frequency is explicitly determined. Simple, accurate analytic formulas have been presented for anisotropic fibers with step- and parabolic-index profiles. Numerical results are in good agreement with the literature results, whenever available.

I. INTRODUCTION

IN reality, all conventional single-mode fibers support two orthogonal polarizations. Bends and twists, unavoidable for the practical fibers, have the property of coupling energy between these two degenerate modes. For sophisticated systems like coherent communication systems, sensory systems based on interferometric principles, or systems utilizing polarization-dependent devices, these changes in the state of polarization would lead to signal fading and noise. To overcome this, isotropic fibers with noncircular geometry and fibers with anisotropic material have been proposed to remove the polarization degeneracy and reduce the polarization mode conversion [1]. Most of the high birefringence fibers have a built-in stress-applying components which makes the core and/or cladding biaxially anisotropic.

Several sophisticated numerical methods have been proposed to investigate the birefringence characteristics of anisotropic optical fibers. Under the weakly guiding assumption, Lindell and Oksanen proposed a variational analysis for step- [2] and parabolic- [3] index fibers, which relies much on the choice of trial functions and still suffers from worse accuracy for fibers operating in near cutoff or well-guided region. Wu and Chen presented a general variational reaction analysis [4] which is applicable for arbitrarily indexed fibers even when the index difference is large, in the price of a huge computational load. Recently, Chen derived a very tedious perturbation analysis [5] which is applicable for step-index fibers only and cannot yield good results in the near cutoff region.

In this paper, we present a novel hyper-perturbation approach to find the Taylor expansion of the dispersion relation

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such that the propagation constant can be explicitly expressed by the power series of frequency. First of all, the conventional finite element method or the variational reaction theory [6] is employed to solve the propagation constant and modal field of an arbitrarily indexed optical fibers. The expansion coefficients up to very high orders can then be calculated directly via a few simple matrix manipulations once the eigen-solution is solved at a certain frequency. The whole dispersion curve can be approximated to a high accuracy by the expansion formulas at few distinct frequencies. Given the explicit expressions for propagation constants, it becomes straightforward to determine the birefringence characteristics for weakly guiding fibers with anisotropic core and/or cladding.

II. HYPER-PERTURBATION THEORY

Consider an optical fiber with the refractive-index profile

$$\begin{aligned} n^2(x) &= n_d^2 + (n_r^2 - n_d^2) \cdot f(R) \quad \text{for } R = r/a \leq 1 \\ &= n_d^2 \quad \text{for } r > a \end{aligned} \quad (1)$$

where a is the core radius, n_d is the cladding index, n_r is the index maximum in the core, and function $f(R)$ defines the profile shape. As the index difference tends to zero, it is well known that the propagation characteristics can be determined from the LP_{mn} mode which satisfies the scalar-wave equation

$$\frac{1}{R} \frac{d}{dR} \left(R \frac{d\psi}{dR} \right) + \left[V^2 \cdot f(R) - W^2 - \left(\frac{m}{R} \right)^2 \right] \psi = 0 \quad (2)$$

under the boundary condition

$$\frac{1}{\psi} \frac{d\psi}{dR} \Big|_{R=1} \equiv Z_m(W) = \frac{W \cdot K'_m(W)}{K_m(W)}.$$

Here, the normalized frequency V and the parameter W are defined by

$$V = ka\sqrt{n_r^2 - n_d^2} \quad \text{and} \quad W = a\sqrt{\beta^2 - n_d^2 \cdot k^2}$$

respectively, $K_m(W)$ is the m th order modified Bessel function of the second kind, and the prime ' denotes the differentiation with respect to the argument W .

Consider the integration of (2) multiplied with an arbitrary test function ψ^a . By employing the integration by part and imposing the boundary condition, we can obtain a reaction formula for the guided mode [6]

$$\langle \psi^a | \mathcal{H} | \psi \rangle = \int_0^1 \left\{ \left[V^2 f - W^2 - \left(\frac{m}{R} \right)^2 \right] \psi^a \cdot \psi \right.$$

$$-\frac{d\psi^a}{dR} \cdot \frac{d\psi}{dR} \Bigg\} R dR + Z_m(W) \cdot \psi^a(1) \cdot \psi(1). \quad (3)$$

Equation (3) will be zero for arbitrary field ψ^a if W and ψ are the desired eigen-solution. By choosing suitable mesh division and basis functions in the finite element method, (3) can be reduced to a matrix equation. Given a frequency V_0 , we can search for zero determinant of the matrix, [7, ch. 11] to locate the associated eigenvalue W_0 and solve the modal field ψ_0 . In general, the determinant is a nonlinear function of W and V such that the eigen-search process has to be repeated each time the frequency is changed.

Conversely, we can treat the frequency V as a perturbation and expand the eigenvalue W and modal field ψ near the frequency V_0 by

$$W = W_0 + \sum_{n=1}^{\infty} W_n \cdot (V - V_0)^n \quad (4)$$

$$\psi = \psi_0 + \sum_{n=1}^{\infty} \psi_n \cdot (V - V_0)^n \quad (5)$$

where

$$W_n = \frac{1}{n!} \frac{d^n W}{dV^n} \Big|_{V=V_0} \quad \text{and} \quad \psi_n = \frac{1}{n!} \frac{d^n \psi}{dV^n} \Big|_{V=V_0}$$

are the expansion coefficient and the expansion field, respectively. Since the operator $\mathcal{H}$ in (3) is dependent on V as well as W , it can also be expanded as

$$\mathcal{H}(V, W) = \mathcal{H}_0 + \sum_{n=1}^{\infty} \mathcal{H}_n \cdot (V - V_0)^n \quad (6)$$

where $\mathcal{H}_0 = \mathcal{H}(V_0, W_0)$ and the expansion operator $\mathcal{H}_n$ is defined by

$$\mathcal{H}_n = \frac{1}{n!} \frac{d^n \mathcal{H}}{dV^n} \Big|_{V=V_0, W=W_0} \equiv \mathcal{H}_{nv} + \frac{\partial \mathcal{H}}{\partial W} W_n. \quad (7)$$

It should be noted that the operator $\mathcal{H}_{nv}$ depends on V_0 , W_0 , and the expansion coefficients $W_1, W_2, \dots$, up to W_{n-1} but not W_n . For example, it equals $\partial \mathcal{H} / \partial V$ as $n = 1$ and $\frac{1}{2} \partial^2 \mathcal{H} / \partial V^2 + W_1 \cdot \partial^2 \mathcal{H} / \partial V \partial W$ as $n = 2$. Since the operator $\mathcal{H}$ defined in (3) is expressed explicitly in terms of V and W , the derivatives in (7) can be taken *analytically*.

By substituting (4)–(6) into the reaction formula (3) and neglecting all the terms with second power of $(V - V_0)$ or higher, we obtain the first order formula

$$\langle \psi^a | \mathcal{H}_0 | \psi_1 \rangle = - \langle \psi^a | \mathcal{H}_1 | \psi_0 \rangle. \quad (8)$$

If the arbitrary field ψ^a in (8) is replaced by the modal field ψ_0 , it is not difficult to prove that the left-hand side (LHS) in (8) is zero. Then, the first-order derivative of W , i.e., W_1 , can be found by

$$W_1 = - \left\langle \psi_0 \left| \frac{\partial \mathcal{H}}{\partial V} \right| \psi_0 \right\rangle \Bigg/ \left\langle \psi_0 \left| \frac{\partial \mathcal{H}}{\partial W} \right| \psi_0 \right\rangle. \quad (9)$$

Once the eigen-solution (W_0, ψ_0) is obtained, the terms in (9) can be calculated by numerical integration. In this way,

the derivative W_1 is determined without taking any numerical differentiation. Equation (9) is nothing more than the conventional formula given in the Rayleigh quotient [8] or the first-order perturbation theory [9].

Given the eigen-solution (W_0, ψ_0) as well as W_1 , the operator $\mathcal{H}_1$ as defined in (7) is available now. Since (8) is valid for arbitrary field ψ^a , it can be employed to find the first-order expansion field ψ_1 . By the finite element method choosing the same mesh division and basis functions, (8) becomes a matrix equation for the nodal unknowns corresponding to ψ_1 . Here, the matrix corresponding to the operator $\mathcal{H}_0$ is already available, while the right-hand side (RHS) vector is contributed by the term $\langle \psi^a | \mathcal{H}_1 | \psi_0 \rangle$. It should be noted that the matrix is singular at the eigen-solution and the modal field ψ_0 is determined up to an arbitrary constant. Hence, an additional boundary condition is required to solve the matrix equation for ψ_1 uniquely. A simple way is to choose that one of the nodal values, usually the boundary one, of the modal field ψ_0 is a constant. Then, we achieve an essential boundary condition for (8), e.g., ψ_1 is enforced to be zero at that node.

Similar process can be repeated to calculate the higher order expansion coefficients. Substituting (4)–(6) into the reaction formula (3) and equating the terms with power $(V - V_0)^n$, we now obtain the following recursive relation:

$$\langle \psi^a | \mathcal{H}_0 | \psi_n \rangle = - \sum_{i=1}^n \langle \psi^a | \mathcal{H}_i | \psi_{n-i} \rangle \quad (10)$$

which holds for arbitrary field ψ^a . If ψ^a is chosen as the modal field ψ_0 , we can still show that the LHS of (10) is zero. Hence, the expansion coefficient W_n can be found by

$$W_n = - \left\{ \langle \psi_0 | \mathcal{H}_{nv} | \psi_0 \rangle + \sum_{i=1}^{n-1} \langle \psi_0 | \mathcal{H}_i | \psi_{n-i} \rangle \right\} \Bigg/ \left\langle \psi_0 \left| \frac{\partial \mathcal{H}}{\partial W} \right| \psi_0 \right\rangle. \quad (11)$$

On the other hand, (10) can be employed to solve the n th order expansion field ψ_n from the lower order expansions W_i and ψ_i ($i = 0, 1, \dots, n-1$) as well as the just solved W_n . Again, the matrix representation for $\mathcal{H}_0$ is singular and the essential boundary condition requires that $\psi_n = 0$ at a particular node.

III. EXPLICIT DISPERSION FORMULAS

Based on the hyper-perturbation theory together with the finite element method, a Fortran program is tailored on IBM/PC to solve the various order expansion coefficients for guided LP modes of optical fibers. The solution region is uniformly divided into M elements, inside each of which the quadratic basis is employed. Since the index profile may be of arbitrary shape $f(R)$, the element integral is evaluated by the four-point Gaussian Legendre quadrature. Given an operating frequency V_0 , the program first searches for the eigenvalue W_0 and then calculates the Taylor expansion coefficients of the dispersion curve up to an arbitrary order.

Since the algorithm involves numerical integration instead of differentiation, the calculated expansion coefficients W_n converge very fast as M increases. Numerical analysis shows

that $M = 5$ is enough to make W_n converge with relative error smaller than 10^{-4} for step-index fibers. More divisions are required for graded index fibers. For example, $M = 7$ should be chosen to achieve a similar accuracy for parabolic-index fibers.

Given the profile shape, the dispersion characteristics for guided LP modes can be uniquely determined from the parameter W versus the normalized frequency V , say $W = F(V)$. Listed here are the explicit expressions of the dispersion relations of fundamental LP modes for step- and parabolic-index fibers in the usual frequency range $0 \leq V \leq 10$. The absolute error of the results given by the following formulas is smaller than 10^{-4} .

A. Step-Index Fibers

$$W \cong 0. \quad \text{for } 0 \leq V \leq 0.45$$

$$W = 0.0004848 + 0.015520y + 0.20189y^2 + 1.2852y^3 \\ + 3.2657y^4 - 4.6273y^5 - 32.642y^6 + 43.885y^7 \\ + 265.40y^8;$$

$$y \equiv V - 0.5 \quad \text{for } 0.45 \leq V \leq 0.6$$

$$W = 0.0244853 + 0.28626y + 1.0647y^2 + 0.58050y^3 \\ - 3.3021y^4 + 1.3074y^5 + 12.072y^6 - 25.746y^7 \\ - 5.7815y^8;$$

$$y \equiv V - 0.7 \quad \text{for } 0.6 \leq V \leq 0.8$$

$$W = 0.202365 + 0.84859y + 0.63048y^2 - 0.73150y^3 \\ + 0.37318y^4 + 0.47363y^5 - 1.5775y^6 + 2.2912y^7 \\ - 1.6084y^8;$$

$$y \equiv V - 1 \quad \text{for } 0.8 \leq V \leq 1.2$$

$$W = 0.718197 + 1.12619y + 0.078285y^2 - 0.12854y^3 \\ + 0.12172y^4 - 0.093802y^5 + 0.060164y^6 \\ - 0.028270y^7 + 0.0019594y^8;$$

$$y \equiv V - 1.5 \quad \text{for } 1.2 \leq V \leq 1.7$$

$$W = 1.290213 + 1.14819y - 0.0085133y^2 - 0.019350y^3 \\ + 0.019055y^4 - 0.013432y^5 + 0.0083050y^6 \\ - 0.0047200y^7 + 0.0024839y^8;$$

$$y \equiv V - 2 \quad \text{for } 1.7 \leq V \leq 2.4$$

$$W = 2.972755 + 1.09426y - 0.015769y^2 + 0.0021716y^3 \\ - 0.10373 \times 10^{-3}y^4 - 0.91300 \times 10^{-4}y^5 + 0.55821 \\ \times 10^{-4}y^6 - 0.23555 \times 10^{-4}y^7 + 0.87318 \times 10^{-5}y^8;$$

$$y \equiv V - 3.5 \quad \text{for } 2.4 \leq V \leq 4.5$$

$$W = 6.677591 + 1.03597y - 0.0040003y^2 + 0.44097 \\ \times 10^{-3}y^3 - 0.47791 \times 10^{-4}y^4 + 0.50237 \times 10^{-5}y^5 \\ - 0.50031 \times 10^{-6}y^6 + 0.44664 \times 10^{-7}y^7 \\ - 0.30529 \times 10^{-8}y^8;$$

$$y \equiv V - 7 \quad \text{for } 4.5 \leq V \leq 10.$$

B. Parabolic-Index Fibers

$$W \cong 0. \quad \text{for } 0 \leq V \leq 0.55$$

$$W = 0.005082 + 0.011860y + 0.11299y^2 + 0.53227y^3 \\ + 1.0427y^4 - 0.78234y^5 - 5.2432y^6 \\ + 3.5557y^7 + 23.234y^8;$$

$$y \equiv V - 0.7 \quad \text{for } 0.55 \leq V \leq 0.8$$

$$W = 0.0328590 + 0.26385y + 0.66751y^2 + 0.21966y^3 \\ - 0.99872y^4 + 0.38446y^5 + 1.6633y^6 \\ - 2.8108y^7 + 0.10073y^8;$$

$$y \equiv V - 1 \quad \text{for } 0.8 \leq V \leq 1.2$$

$$W = 0.318137 + 0.80983y + 0.32464y^2 - 0.27216y^3 \\ + 0.13437y^4 + 0.013353y^5 - 0.12003y^6 \\ + 0.15788y^7 - 0.12466y^8;$$

$$y \equiv V - 1.5 \quad \text{for } 1.2 \leq V \leq 1.8$$

$$W = 1.28718 + 1.03570y + 0.019159y^2 - 0.021734y^3 \\ + 0.013645y^4 - 0.0070347y^5 + 0.0032285y^6 \\ + 0.0013322y^7 + 0.47687 \times 10^{-3}y^8;$$

$$y \equiv V - 2.5 \quad \text{for } 1.8 \leq V \leq 3.1$$

$$W = 2.84847 + 1.03590y - 0.0054901y^2 - 0.22435 \\ \times 10^{-4}y^3 + .39110 \times 10^{-3}y^4 - 0.18864 \times 10^{-3}y^5 \\ + 0.67623 \times 10^{-4}y^6 - 0.21548 \times 10^{-4}y^7 \\ + 0.64511 \times 10^{-5}y^8;$$

$$y \equiv V - 4.0 \quad \text{for } 3.1 \leq V \leq 5$$

$$W = 5.91672 + 1.01344y - 0.0020187y^2 + 0.26653 \\ \times 10^{-3}y^3 - 0.28924 \times 10^{-4}y^4 + 0.21441 \times 10^{-5}y^5 \\ + 0.20382 \times 10^{-7}y^6 - 0.48528 \times 10^{-7}y^7 \\ + .12792 \times 10^{-7}y^8;$$

$$y \equiv V - 7 \quad \text{for } 5 \leq V \leq 10.$$

IV. DIRECT BIREFRINGENCE ANALYSIS

Without loss of generality, consider an optical fiber with the permittivity tensor

$$[\epsilon] = \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix} \quad (12)$$

where

$$n_i^2 = n_{di}^2 + (n_{ri}^2 - n_{di}^2) \cdot f_i(R) \\ = n_{di}^2 \cdot [1 + \Delta_i \cdot f_i(R)]$$

for $i = x, y, z$. Here, $f_i(R)$ defines the profile shape in the principal axes. It has the maximum of unity in the core and is zero in the cladding. In practical applications, the profile anisotropy can be neglected by assuming $f_i(R) = f(R)$. In terms of the permittivity model (12), the cladding as well as

core can be anisotropic, while the index profile can be step, parabolic, or arbitrary.

Under the weakly guiding assumption, usually made for optical fibers, the LP modes are linearly polarized such that only the permittivity in the polarization axis is relevant. The dispersion curve $W = F(V)$ will be identical to that for isotropic fibers, if W and V are now replaced by

$$W_i = a\sqrt{\beta_i^2 - n_{di}^2 \cdot k^2}, \quad V_i = ka\sqrt{n_{ri}^2 - n_{di}^2} \quad (13)$$

where $i = x$ and y for LP modes polarized in x and y directions, respectively. The function F is dependent on the profile shape $f(R)$ only. Its explicit expression has been given in the above session for step and parabolic-index fibers, and can be found similarly by the present approach for arbitrarily indexed fibers.

One of the most important parameters for the anisotropic fibers is the normalized polarization birefringence defined by [2]

$$B = \frac{(\beta_x - \beta_y)}{(\beta_x + \beta_y)/2} = \frac{(n_{dx}^2 - n_{dy}^2)k^2 + (W_x^2 - W_y^2)/a^2}{(\beta_x + \beta_y)^2/2} \quad (14)$$

In the weakly guidance limit, β_i in the denominator can be replaced by $k \cdot n_{di}$ or $k \cdot n_{ri}$ ($i = x$ or y). Employing the relations $W_x = F(V_x)$ and $W_y = F(V_y)$ and noting that V_x is very close to V_y , (14) can be reduced to a simple formula

$$B = B_0 + (B_\infty - B_0) \cdot G(V); \quad G(V) = F(V) \cdot F'(V)/V \quad (15)$$

where

$$B_0 = \frac{(n_{dx} - n_{dy})}{(n_{dx} + n_{dy})/2} \quad \text{and} \quad B_\infty = \frac{(n_{rx} - n_{ry})}{(n_{rx} + n_{ry})/2}$$

are the normalized birefringence at low and high-frequency limits, respectively. The function $G(V)$, varying from zero at low frequency limit to unity at high frequency limit, defines the frequency dependence of birefringence characteristics. Since the explicit expression for $F(V)$ is available, it becomes straightforward to find the birefringence characteristics for anisotropic fibers with arbitrary index profile.

V. NUMERICAL RESULTS

For circular fibers with anisotropic core but isotropic cladding, the permittivity model in (12) can be assumed [2]

$$n_{dx} = n_{dy} = n_{dz} = n_d \quad \text{and} \quad \Delta_x = A_x \Delta, \quad \Delta_y = A_y \Delta, \quad \Delta_z = \Delta.$$

The axial normalized frequency V_i now is related to the normalized frequency V by

$$V_i = \sqrt{A_i} V, \quad (i = x, y); \quad V = kan_d \sqrt{\Delta}.$$

Then, the normalized propagation constants for both polarizations will be

$$b_i \equiv \left[(\beta_i / n_d k)^2 - 1 \right] / \Delta = W_i^2 / V^2 = F^2(\sqrt{A_i} V) / V^2 \quad (16)$$

Fig. 1. Birefringence characteristic function $G(V)$ for step and parabolic-index fibers in the limit of weak guidance and perturbational anisotropy.

and the normalized polarization birefringence by (15) becomes

$$B = \frac{\Delta \cdot q}{2} G(V) \quad (17)$$

where $q = A_x/A_y - 1$ is the anisotropy parameter.

Fig. 1 shows the birefringence characteristic curve $G(V)$, or $2B/\Delta q$, for both the step- and parabolic-index fibers. The results for the step-index fiber are in good agreement with those in Fig. 9 of [2] except for smaller V , say $V < 1.2$. The discrepancy is expected since the functional in [2] does not work well at those frequencies.

For circular fibers with anisotropic core and cladding, one may assume the permittivity model in (12) with [2], [3]

$$n_{dx}^2 = A_x n_d^2, \quad n_{dy}^2 = A_y n_d^2, \quad n_{dz}^2 = n_d^2; \quad \text{and} \quad \Delta_x = \Delta_y = \Delta_z = \Delta.$$

Then, the normalized propagation constant will be given by

$$b_i \equiv \frac{\left[(\beta_i / n_d k)^2 - 1 \right]}{\Delta} = \frac{(n_{di} / n_d)^2 + (W_i / n_d k a)^2 - 1}{\Delta} = \frac{A_i - 1}{\Delta} + \frac{1}{V^2} \cdot F^2(\sqrt{A_i} V) \quad \text{for } i = x \text{ or } y. \quad (18)$$

The normalized polarization birefringence by (15) now becomes

$$B = B_0 = A_x/A_y - 1 = q \quad (19)$$

almost independent of the frequency.

Fig. 2 shows the normalized propagation constant b_x versus V for step-index fibers. The parameters are chosen as $A_x = 1.0002$, 1.0 , 0.9998 and $\Delta = 0.004$, 0.006 , 0.01 by fixing $A_y = 1$. The solid curve gives b_x for the isotropic case that $A_x = A_y = 1$. As far as the range with $V > 1.2$ is concerned, whence the method in [2] is reliable, the present results are in good agreement with those in Figs. 10 and 11 in [2].

Fig. 3 shows the normalized propagation constant b_x versus V for parabolic-index fibers. For the sake of comparison with [3, Fig. 5], the parameter A_x is chosen 1.0002 , 1.00002 , 1.0 , and 0.9998 by fixing $A_y = 1.0$, and $\Delta = 0.004$. Since the modal field distribution is more complicated in the parabolic-index fibers than those in step-index fibers, it becomes very

Fig. 2. Dependence of the dispersion relation on the index difference $\Delta = 0.004, 0.006, 0.01$, and the anisotropy parameter $A_x = 1.0002, 1.0, 0.9998$ for step-index fibers.

Fig. 3. Dependence of the dispersion relation on the anisotropy parameter $A_x = 1.0002, 1.0000, 1.0$, and 0.9998 for parabolic-index fibers.

difficult to find the suitable trial functions required in the variational analysis [3]. The computed results in [3] are not reliable for near cutoff modes nor for well guided modes. In the midfrequency range with $0.2 < b < 0.7$, our results can be compared with those in [3] with reasonable agreement.

Finally, consider step-index circular fibers with more general birefringence model. Without loss of generality, assume that x is the principal axis such that $n_{rx} \geq n_{ry}$ and $n_{dx} \geq n_{dy}$. The index difference and the index anisotropy, critical in the propagation characteristics, are defined as

$$\begin{aligned} d_{rd} &= 1 - n_{dx}/n_{rx} \quad \text{and} \quad d_{xyr} = 1 - n_{ry}/n_{rx}, \\ d_{xyd} &= 1 - n_{dy}/n_{dx}. \end{aligned} \quad (20)$$

Now, let the generalized parameters V and b_i ($i = x, y$) be defined by [5]

$$V = kan_{dx}\sqrt{\Delta_x} \quad \text{and} \quad b_i = [(\beta_i/n_{dx}k)^2 - 1]/\Delta_x.$$

Then, the normalized b_i are

$$b_x = F^2(V)/V^2 \quad \text{and} \quad b_y = F^2(\sqrt{A_y}V)/V^2 + b_{y0} \quad (21)$$

where

$$\begin{aligned} A_y &= \frac{n_{ry}^2 - n_{dy}^2}{n_{rx}^2 - n_{dx}^2} \cong \frac{d_{xyd} - d_{xyr}}{d_{rd}} + 1 \quad \text{and} \\ b_{y0} &= -\frac{n_{dx}^2 - n_{dy}^2}{n_{rx}^2 - n_{dx}^2} \cong -\frac{d_{xyd}}{d_{rd}}. \end{aligned}$$

Fig. 4. Dispersion relation of biaxial fibers.

The normalized polarization birefringence is thus given by (15) with parameters

$$\begin{aligned} B_0 &= \frac{d_{xyd}}{1 - d_{xyd}/2} \cong d_{xyd} \quad \text{and} \\ B_\infty &= \frac{d_{xyr}}{1 - d_{xyr}/2} \cong d_{xyr}. \end{aligned} \quad (22)$$

In all the numerical calculations, n_{rx} is set to 1.5 while other indices are varied by changing d_{xyr} , d_{xyd} , and d_{rd} . Here, the mode polarized mainly in the x -direction is more fundamental, whose normalized propagation constant is denoted by b_x . Curves of b_x versus V for various values of d_{xyd} and d_{xyr} are indistinguishable, as depicted by (21) and shown in Fig. 4. Also shown in the figure are the curves for b_y which do vary with d_{xyd} and d_{xyr} . This figure can be compared with [5, Fig. 4] which is based on a tedious perturbation analysis. Both are in agreement.

Of the particular interest is the difference $b_x - b_y$, which is related to the normalized polarization birefringence B defined in (14) by $b_x - b_y \cong B/d_{rd}$. In Figs. 5–7, the polarization birefringence is plotted as a function of V with d_{xyd} and d_{xyr} as parameters. The results are of the same parameters and can be compared with those in [5, Figs. 5–7]. It should be recalled that the present analysis, with explicit dispersion relation available, is simple and direct. Also, the results for a large frequency range of $0 < V < 10$ can be obtained without any difficulty.

In Fig. 5, $d_{xyr} \cong B_\infty$ is kept constant while $d_{xyd} \cong B_0$ varies. Since the cladding anisotropy is chosen larger than the core anisotropy here, the difference $b_x - b_y$ decreases from its maximum d_{xyd}/d_{rd} at the low frequency limit and tends to its high-frequency limit d_{xyr}/d_{rd} . The behavior can be explained easily in terms of (15). For fibers operating near cutoff, fields spread out to the cladding region such that the cladding anisotropy d_{xyd} should and do have a strong influence on the birefringence characteristics. On the other hand, for fibers operating in the well-guided region, fields are concentrated in the core region such that d_{xyr} is dominant. This is also shown in Fig. 6 where d_{xyd} is kept constant while d_{xyr} varies. In Fig. 7, $d_{xyd} = d_{xyr}$ and they vary simultaneously. For these cases, the difference $b_x - b_y$ is almost independent of V .

Fig. 5. Birefringence of biaxial fibers with $d_{rd} = 0.005$ and $d_{xyr} = 0.0001$.

Fig. 6. Birefringence of biaxial fibers with $d_{rd} = 0.005$ and $d_{xyd} = 0.0001$.

Fig. 7. Birefringence of biaxial fibers with $d_{rd} = 0.005$ and $d_{xyr} = d_{xyd}$.

VI. DISCUSSION AND CONCLUSIONS

A hyper-perturbation theory is applied to find the expansion coefficients of the dispersion relation up to arbitrary orders. After the conventional finite element analysis to search for the eigenvalues, these expansion coefficients are directly calculated by a few simple matrix vector multiplications. Without any numerical differentiation involved, the results of high

accuracy can be achieved with few division elements only. With these coefficients, an expansion formula valid for a certain frequency range is available. The explicit dispersion relation, which is very convenient for practical applications, can then be made up from the expansion formulas at several distinct frequencies.

Given the explicit dispersion relation of single-mode optical fibers, the paper presents a simple birefringence analysis for optical fibers with biaxially anisotropic core and/or cladding. The polarization birefringence between the two fundamental modes depends critically on the index difference and the index anisotropy. For fibers operating near cutoff, the cladding anisotropy determines the birefringence characteristics. On the other hand, the core anisotropy is dominant for fibers operating in the well-guided region. For the frequencies in between, the polarization birefringence lies somewhat between the above two limits and is governed by the birefringence characteristic function which can be found from the derivative of the dispersion relation of the corresponding isotropic fibers.

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