An Efficient Numerical Approach for Determining the Dispersion Characteristics of Dual-Mode Elliptical-Core Optical Fibers

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Abstract—Based on vectorial formulations which combine the surface integral equation method and the finite-element method, a novel numerical approach is proposed for calculating the dispersion coefficients of dual-mode elliptical-core fibers with arbitrary refractive index profiles. By differentiating the original formulations involving the propagation constant \( \beta \) and the guided mode fields \( H_x \) and \( H_y \) once and twice with respect to the normalized frequency \( V \), the new formulations for \( \{d\beta/dV, dH_x/dV, dH_y/dV\} \) and \( \{d^2\beta/dV^2, d^2H_x/dV^2, d^2H_y/dV^2\} \) are obtained, respectively. Once \( \{\beta, H_x, H_y\} \) is solved through the eigenvalue procedure which dominates the computing time, only a few matrix manipulations are required to obtain \( \{d\beta/dV, dH_x/dV, dH_y/dV\} \) and \( \{d^2\beta/dV^2, d^2H_x/dV^2, d^2H_y/dV^2\} \). Some numerical examples are examined to see the influence of different refractive index distributions with dips on the dispersions of the four nondegenerate LP11 modes for elliptical-core fibers.

I. INTRODUCTION

DUAL-MODE elliptical-core (e-core) optical fibers have been used extensively in many sensors and optical devices such as intermodal couplers, modal filters, and acoustooptic frequency shifter [1]–[3]. Recently, a chromatic dispersion compensation technique that employs the dual-mode fiber with small ellipticity was proposed by Poole et al. [4]. This technique is based on the principle of converting the LP01 mode from the communication fiber operated near 1.55 \( \mu \)m to the LP11 modes in an e-core compensating fiber and using the large negative dispersions of the LP11 modes close to their cutoff wavelengths to compensate for the dispersion in the communication fiber. It has been shown that the dispersion of the LP11 mode for the circular-core compensator is sensitive to the variation of the refractive index profile using a scalar waveguide theory [5], but, to our knowledge, the dispersion of the higher-order modes for the e-core compensator has not been studied theoretically in detail. Because the e-core compensator is operated near the cutoff wavelength, the difference in the dispersion among the four nondegenerate LP11 modes, which include the first-higher-order modes of \( x \) and \( y \) polarizations (LP11,LP21,LP31,LP41), and the second-higher-order modes of \( x \) and \( y \) polarizations (LP11,LP21,LP31,LP41), is significant. Rigorous vectorial waveguide theory is thus needed in order to investigate the dispersions of the four nondegenerate LP11 modes for the e-core fibers. Some related analysis results have recently been reported by Poole et al. [6]. In this paper, a novel numerical approach based on the vectorial waveguide theory is proposed to efficiently calculate the dispersions of the e-core fibers.

The computation of dispersion coefficient requires the first and second derivatives of the propagation constant with respect to wavelength. A simple way to evaluate the dispersion is the direct numerical calculation, based on finite differences, of the first and second derivatives from the propagation constant versus wavelength data. However, such simple method requires large computational efforts and may result in great errors [7]. Some improved numerical approaches based on the scalar theory for circular optical fibers, such as that using the Rayleigh quotient to obtain the first derivative of the propagation constant [8], that solving three differential equations which include the first and second derivatives of the propagation constant [9], and that based on the matrix perturbation method [10], have been proposed. Besides, based on the variational finite-element formulation of the scalar theory, an algorithm [11], in which formulas for the first and second derivatives of the propagation constant with respect to \( V \) are explicitly derived by intrinsically differentiating the scalar wave equation with respect to \( V \) and by using a reaction formula, has been presented for calculating the dispersions of optical fibers. This algorithm which eliminates the need of numerical differentiation and is efficient in computational effort can only treat the dispersion of circular-core fiber with circularly symmetric refractive index profile (1-D problem).

In this paper, stemming from the same principle as in [11] and based on vectorial formulations which combine the surface integral equation method (SIEM) and the finite-element method (FEM) [12], a novel numerical approach is proposed for calculating the waveguide dispersions of e-core fibers with arbitrary refractive index profiles (2-D problem). First, the propagation constant \( \beta \) and the guided mode fields \( H_x \) and \( H_y \) are solved by the combined formulations of SIEM and FEM and the continuity requirement of \( E_z \) and
different cy-power profiles and the index dip effect, on the boundary rounding medium (cladding region) with an elliptical cross-section techniques have been proposed for optical waveguides with homogeneous cladding, required in obtaining $\{\beta, H_z, H_y\}$ and Green function in the homogeneous cladding, and $P$. The solution procedure described by matrix representation is derived by differentiating the original formulations twice with respect to $V$. Matching the continuity of $E_x$ and $H_z$ at the boundary, $\{\beta, H_z, H_y\}$ is obtained. Similarly, the formulations about $\{\beta, H_z, H_y\}$ with the solved $\{\beta, H_z, H_y\}$ and $\{\beta, H_z, H_y\}$ being substituted can be derived by differentiating the original formulations twice with respect to $V$ and $\{\beta, H_z, H_y\}$ is obtained by matching the continuity requirement of $E_x$ and $H_z$ at the boundary. When $\beta$, $\beta$, and $\beta$ are obtained, the dispersion can be calculated by the definition. Most of the computational efforts are spent in solving $\{\beta, H_z, H_y\}$. Only some matrix manipulations are required in obtaining $\{\beta, H_z, H_y\}$ and $\{\beta, H_z, H_y\}$ once $\{\beta, H_z, H_y\}$ is solved.

The detailed mathematical formulations about $\{\beta, H_z, H_y\}$, $\{\beta, H_z, H_y\}$, and $\{\beta, H_z, H_y\}$ are presented in Section II. The solution procedure described by matrix representation is discussed in Section III. In Section IV, as a numerical example, the influence of the refractive index distributions, including different $\alpha$-power profiles and the index dip effect, on the dispersions of the four nondegenerate LP$_{11}$ modes for the e-core compensators is investigated. The vectorial mode field patterns of these four modes are also presented. Section V gives the conclusions.

II. FORMULATIONS

As described in [12], based on a full-wave formulation, a combined method employing the surface integral equation method and the finite-element technique can treat the propagation characteristics of inhomogeneous optical waveguides with arbitrary cross-sections. Although similar numerical techniques have been proposed for optical waveguides with arbitrary cross-sections [13], [14], they can only treat the propagation characteristics of homogeneous waveguides.

Fig. 1 gives the sketch of an inhomogeneous dielectric waveguide (core region) embedded in a homogeneous surrounding medium (cladding region) with an elliptical cross-section. In the homogeneous cladding with the core-cladding boundary $\Gamma$, the transverse magnetic field $F(=H_x$ or $H_y$) and its normal derivative $dF/dn(=dH_x/dn or dH_y/dn)$ on $\Gamma$ can be related through a surface integral equation

$$
\frac{1}{2} F(\vec{r}) = P \int_{\Gamma} F(\vec{r'}) \frac{dG(\vec{k}, \vec{r}, \vec{r'})}{dn} d\vec{r'} - d_0 \int_{\Gamma} G(\vec{k}, \vec{r}, \vec{r'}) \frac{dF(\vec{r'})}{dn} d\vec{r'} \tag{1}
$$

where $k_d$ is related to the wave number in free space $k_0$ and the propagation constant of the guided mode $\beta$ by $k_d = (\beta^2 - k_0^2$)1/2 with $\epsilon_c$ being the relative permittivity of the homogeneous cladding, $\vec{r}$ and $\vec{r'}$ are the position vectors in the 2-D vector space with $\vec{r}$ on $\Gamma$ as shown in Fig. 1, $G = (1/2\pi) K_0(k_d|\vec{r} - \vec{r'}|)$ with $K_0$ being the modified Bessel function of the second kind and order zero denotes the 2-D Green function in the homogeneous cladding, and $P \int$ denotes the Cauchy principal value integral with the singularity at the point of $\vec{r} = \vec{r'}$ being removed. How the $P \int$ integral is approximated will be described in next section. Through the integral equation, the transverse magnetic field at an arbitrary position $\vec{r}$ in the cladding region can be described by the field on $\Gamma$ and its normal derivative along $\vec{n}$. When $F(\vec{r})$ is chosen as the field on the boundary $\Gamma$, a complete relation between $F$ and $dF/dn$ on $\Gamma$ for the cladding region is established in (1).

In the inhomogeneous core region with the relative permittivity distribution $\epsilon_c(x, y)$, the magnetic fields of the guided modes satisfy the following source-free equation

$$
k_0^2 \vec{H} - \nabla \times \left( \frac{1}{\epsilon_c} \nabla \times \vec{H} \right) = 0. \tag{2}
$$

By making the dot product of the left-hand side of (2) with an arbitrary vector function $\vec{H}^c$, which is independent of $\vec{H}$, and integrating the scalar product over the entire space, the differential equation, (2) can be transformed into the variational-equation formulations. After some manipulations as shown in [12], the variational equations with the arbitrary fields $\vec{H}^c$ being chosen as $H_x^c$ and $H_y^c$, respectively, are

$$
\int \left( (k_0^2 \epsilon_c - \beta^2) H_x H_x^c - \frac{\partial H_x}{\partial x} \frac{\partial H_x^c}{\partial x} - \frac{\partial H_x}{\partial y} \frac{\partial H_x^c}{\partial y} \right) dx dy = 0 \tag{3a}
$$

and

$$
\int \left( (k_0^2 \epsilon_c - \beta^2) H_y H_y^c - \frac{\partial H_y}{\partial x} \frac{\partial H_y^c}{\partial x} - \frac{\partial H_y}{\partial y} \frac{\partial H_y^c}{\partial y} \right) dx dy = 0. \tag{3b}
$$

By making a variation of (3a) and (3b) with respect to $H_x$ and $H_y$, one obtains the differential equation of $H_x$ and $H_y$ in (2). For guided modes, the boundary condition that $E_x$ and $E_y$ are continuous at the boundary $\Gamma$ should be satisfied. From the Maxwell equations $j \omega E = \nabla \times \vec{H}$ and $\nabla \cdot \vec{H} = 0, E_z$ and $H_z$ can be expressed as

$$
j \omega E_z = \frac{1}{\epsilon} \left( \frac{\partial H_x}{\partial t} - \frac{\partial H_y}{\partial n} \right). \tag{4a}
$$

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where $\epsilon = \epsilon_0 \epsilon_c$ or $\epsilon = \epsilon_0 \epsilon_d$ with $\epsilon_0$ being the permittivity of the vacuum, $H_n$ and $H_t$ are the magnetic fields normal to and tangential to the boundary, respectively, and $\partial / \partial n$ and $\partial / \partial t$ denote the partial derivatives along the normal and the tangential directions, respectively. Explicitly, $H_n = H_x \cos(\theta) + H_y \sin(\theta)$ and $H_t = H_x \sin(\theta) - H_y \cos(\theta)$, where $\theta$ is the angle between $\hat{n}$ and the $x$ direction. By discretizing (1) and (3) with the boundary element method and the finite element method, respectively, the normal derivative fields and the tangential directions, respectively. Explicitly, and denote the partial derivatives along the normal and the tangential derivatives in (4a) and (4b) which are now in terms of (1) and (3) with the boundary element method and the finite element method, respectively, in (4a) and (4b) which are now in terms of $H_x$ and $H_y$ only, the propagation characteristics of the guided modes can be obtained.

The integral equations. (1) and (3) and the continuity requirement (4) on $\Gamma$ give a complete description of the electromagnetic behaviour of the guided mode ($\beta, H_x, H_y$). When we perturb (1), (3), and (4) with respect to the normalized frequency $V$, similar descriptions about the electromagnetic behaviour of the perturbed field ($\beta, H_x, H_y$) can be constructed. Described in the following is the details for the perturbation process.

To calculate the first derivative of $\beta$ with respect to the normalized frequency $V$, we differentiate (perturb) (5), (6), and (7) with respect to $V$. In doing so, we obtain that $\beta$ at the cladding boundary should satisfy the integral equation

$$
\frac{1}{2} \beta = P \int_{\Gamma} F(\beta) \frac{dG(k_d, \beta, \vec{r})}{dn} d\vec{r}
- \int_{\Gamma} G(k_d, \beta, \vec{r}) \frac{dF(\beta)}{dn} d\vec{r}
+ P \int_{\Gamma} F(\beta) \frac{dG(k_d, \beta, \vec{r})}{dn} d\vec{r}
- \int_{\Gamma} G(k_d, \beta, \vec{r}) \frac{dF(\beta)}{dn} d\vec{r}
(5)
$$

that $H_x, H_y$, and $\beta$ in the core region can be related as

$$
\int (k_0^2 \epsilon_c - \beta^2) H_x H_y^* - \frac{\partial H_x}{\partial x} \frac{\epsilon_c}{\epsilon_y} \frac{\partial H_y}{\partial y} + \int (2k_0 \epsilon_0 \epsilon_c - 2 \beta \beta) H_x H_y^* dxdy = 0
(6a)
$$

and

$$
\int (k_0^2 \epsilon_c - \beta^2) H_y H_x^* - \frac{\partial H_y}{\partial x} \frac{\epsilon_c}{\epsilon_y} \frac{\partial H_x}{\partial y} + \int (2k_0 \epsilon_0 \epsilon_c - 2 \beta \beta) H_y H_x^* dxdy = 0
(6a)
$$

where $\epsilon = \epsilon_0 \epsilon_c$ or $\epsilon = \epsilon_0 \epsilon_d$ with $\epsilon_0$ being the permittivity of the vacuum, $H_n$ and $H_t$ are the magnetic fields normal to and tangential to the boundary, respectively, and $\partial / \partial n$ and $\partial / \partial t$ denote the partial derivatives along the normal and the tangential directions, respectively. Explicitly, $H_n = H_x \cos(\theta) + H_y \sin(\theta)$ and $H_t = H_x \sin(\theta) - H_y \cos(\theta)$, where $\theta$ is the angle between $\hat{n}$ and the $x$ direction. By discretizing (1) and (3) with the boundary element method and the finite element method, respectively, in (4a) and (4b) which are now in terms of $H_x$ and $H_y$ only, the propagation characteristics of the guided modes can be obtained.

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$$
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- \int_{\Gamma} G(k_d, \beta, \vec{r}) \frac{dF(\beta)}{dn} d\vec{r}
+ P \int_{\Gamma} F(\beta) \frac{dG(k_d, \beta, \vec{r})}{dn} d\vec{r}
- \int_{\Gamma} G(k_d, \beta, \vec{r}) \frac{dF(\beta)}{dn} d\vec{r}
(5)
$$

that $H_x, H_y$, and $\beta$ in the core region can be related as

$$
\int (k_0^2 \epsilon_c - \beta^2) H_x H_y^* - \frac{\partial H_x}{\partial x} \frac{\epsilon_c}{\epsilon_y} \frac{\partial H_y}{\partial y} + \int (2k_0 \epsilon_0 \epsilon_c - 2 \beta \beta) H_x H_y^* dxdy = 0
(6a)
$$

and

$$
\int (k_0^2 \epsilon_c - \beta^2) H_y H_x^* - \frac{\partial H_y}{\partial x} \frac{\epsilon_c}{\epsilon_y} \frac{\partial H_x}{\partial y} + \int (2k_0 \epsilon_0 \epsilon_c - 2 \beta \beta) H_y H_x^* dxdy = 0
(6a)
$$

where $\epsilon = \epsilon_0 \epsilon_c$ or $\epsilon = \epsilon_0 \epsilon_d$ with $\epsilon_0$ being the permittivity of the vacuum, $H_n$ and $H_t$ are the magnetic fields normal to and tangential to the boundary, respectively, and $\partial / \partial n$ and $\partial / \partial t$ denote the partial derivatives along the normal and the tangential directions, respectively. Explicitly, $H_n = H_x \cos(\theta) + H_y \sin(\theta)$ and $H_t = H_x \sin(\theta) - H_y \cos(\theta)$, where $\theta$ is the angle between $\hat{n}$ and the $x$ direction. By discretizing (1) and (3) with the boundary element method and the finite element method, respectively, in (4a) and (4b) which are now in terms of $H_x$ and $H_y$ only, the propagation characteristics of the guided modes can be obtained.

The integral equations. (1) and (3) and the continuity requirement (4) on $\Gamma$ give a complete description of the electromagnetic behaviour of the guided mode ($\beta, H_x, H_y$). When we perturb (1), (3), and (4) with respect to the normalized frequency $V$, similar descriptions about the electromagnetic behaviour of the perturbed field ($\beta, H_x, H_y$) can be constructed. Described in the following is the details for the perturbation process.

To calculate the first derivative of $\beta$ with respect to the normalized frequency $V$, we differentiate (perturb) (5), (6), and (7) with respect to $V$. In doing so, we obtain that $\beta$ at the cladding boundary should satisfy the integral equation

$$
\frac{1}{2} \beta = P \int_{\Gamma} F(\beta) \frac{dG(k_d, \beta, \vec{r})}{dn} d\vec{r}
- \int_{\Gamma} G(k_d, \beta, \vec{r}) \frac{dF(\beta)}{dn} d\vec{r}
+ P \int_{\Gamma} F(\beta) \frac{dG(k_d, \beta, \vec{r})}{dn} d\vec{r}
- \int_{\Gamma} G(k_d, \beta, \vec{r}) \frac{dF(\beta)}{dn} d\vec{r}
(5)
$$

that $H_x, H_y$, and $\beta$ in the core region can be related as

$$
\int (k_0^2 \epsilon_c - \beta^2) H_x H_y^* - \frac{\partial H_x}{\partial x} \frac{\epsilon_c}{\epsilon_y} \frac{\partial H_y}{\partial y} + \int (2k_0 \epsilon_0 \epsilon_c - 2 \beta \beta) H_x H_y^* dxdy = 0
(6a)
$$

and

$$
\int (k_0^2 \epsilon_c - \beta^2) H_y H_x^* - \frac{\partial H_y}{\partial x} \frac{\epsilon_c}{\epsilon_y} \frac{\partial H_x}{\partial y} + \int (2k_0 \epsilon_0 \epsilon_c - 2 \beta \beta) H_y H_x^* dxdy = 0
(6a)
$$
Given \( V \)
- Match boundary continuity of \((E_z, H_z)\) with (4) \( \Rightarrow \) solve (1) and (3)
- then with \( V \) and \( \{E_x, H_y, \beta \} \)
- Match boundary continuity of \((E_z, H_z)\) with (7) \( \Rightarrow \) \( \{\beta, H_z, H_y\} \)
- then with \( V, \{H_z, H_y, \beta \}, \{\beta, H_z, H_y\} \)
- Match boundary continuity of \((E_z, H_z)\) with (11) \( \Rightarrow \) \( \{\beta, H_z, H_y\} \)

and

\[
\begin{align*}
\int (k_0^2 c - \beta^2) \frac{\partial H_y}{\partial x} H_y^* \, dx &+ \left( \frac{\partial H_x}{\partial y} \frac{\partial H_y}{\partial x} - \frac{\partial H_y}{\partial y} \frac{\partial H_x}{\partial x} \right) \, dx dy \\
+ \frac{\partial}{\partial y} \left( \frac{\partial H_y}{\partial x} \frac{\partial}{\partial x} \frac{\partial H_y}{\partial y} \right) \, dx dy &+ \int 2(k_0^2 c - \beta^2 + k_0 \eta c - \beta \eta) H_y H_y^* \, dx dy \\
+ \int 4(k_0^2 c - \beta \eta) H_y H_y^* \, dx dy &= 0
\end{align*}
\]

for \( \beta \) as shown at the top of the page. In the following, the steps of the numerical procedures are described in detail with the matrix-form representations.

Steps 1 and 2 describe the numerical procedures for solving the propagation characteristics of the guided modes \( \{\beta, H_z, H_y\} \) for a given normalized frequency \( V \). Steps 3 and 4 describe the procedures for solving the first-order perturbed propagation characteristics \( \{\beta, H_z, H_y\} \) of the guided modes for the given \( V \) and the solved \( (\beta, H_z, H_y) \) in steps 1 and 2. Steps 5 and 6 describe the procedures for solving the second-order perturbed propagation characteristics \( \{\beta, H_z, H_y\} \) of the guided modes for the given \( V \) and the solved \( (\beta, H_z, H_y) \) in steps 1 and 2 and \( (\beta, H_z, H_y) \) in steps 3 and 4.

Step 1: Express the normal derivatives of the fields on the boundary \( \Gamma \) in terms of the boundary fields for the cladding and core regions by discretizing (1) and (3), respectively. Based on the boundary-element method with the pulse bases for (1), the relation between the boundary fields and their normal derivatives on the cladding boundary for the cladding region can be expressed as

\[
\frac{1}{2} f_i = \sum_{j=1}^{M} f_j G_{ij} \ell_j - \sum_{j=1}^{M} f_j^* G_{ij} \ell_j
\]

for \( i = 1, 2, \ldots, M \), where \( M \) is the total number of divided segments on \( \Gamma \), and \( \ell_j \) is the arc length of the \( j \)-th segment. \( f_j \) and \( f_j^* \), which are both unknowns, denote the transverse magnetic field and its normal derivative, respectively, on the boundary \( \Gamma \) at node \( j \). Similarly, the matrix elements \( G_{ij} \) and \( G_{ij}^* \) denote the values of the Green function and its normal derivative, respectively, on the boundary \( \Gamma \) between node \( i \) and node \( j \). Note that the first summation in (14) excludes the point \( i = j \) due to the fact that in the Cauchy principal integral the singular value \( G_{ij}^* \) for \( i = j \) is removed. By some linear algebra operations on the \( M \) equations for \( i = 1, 2, \ldots, M \) in (14), a matrix form representation of (14) can be written as

\[
[f_n]_{M \times 1} = [M_a][f]_{M \times 1}
\]
By employing the FEM with the triangular elements and linear bases for both the unknown fields \((H_x, H_y, \text{ and } H_z)\) and taking the variation on the arbitrary fields (see [12] for details), (3a) and (3b) are discretized as

\[
\begin{bmatrix}
    h_x^n \\
    h_y^n
\end{bmatrix}_{2(N-M) \times 1} = [Q]\begin{bmatrix}
    h_x^d \\
    h_y^d
\end{bmatrix}_{2(N-M) \times 1}
\] (16)

where \(N\) is the entire number of nodes in the core region including the nodes on \(\Gamma\), \(h_x^d = [h_x^d_{11}, h_x^d_{22}, \ldots, h_x^d_{N-M, N-M}]^T\) denotes the column vector of nodal unknowns inside the core region (excluding the nodes on \(\Gamma\)) with \(j\) polarization \((j = x, y)\), and \([Q]\) is a matrix determined from (3a) as \([M_d]\) in (15) is derived from (1).

The relation between the associated fields and their normal derivatives on the core boundary \(\Gamma\) can be obtained by the finite difference with (16) [12] as

\[
\begin{bmatrix}
    h_x^{n,x} \\
    h_y^{n,x} \\
\end{bmatrix}_{2M \times 1} = [M_c]\begin{bmatrix}
    h_x^x \\
    h_y^y \\
\end{bmatrix}_{2M \times 1}
\] (17)

where \(h_x^{n,x}\) is the column vector with its elements being the normal derivatives of \(h_x^x, h_y^y\), respectively, of \(j\)-polarized fields on the core boundary \(\Gamma\). For the sake of convenience, in the following discussion we define \(h = [h_x^x, h_y^y, h_x^y, h_y^y]^T\).

Step 2: Solve for \(\beta\) and \(h\). By substituting (15) and (17) in (4) and enforcing the continuity requirement of \(E_z\) and \(H_z\) at the core-cladding boundary \(\Gamma\), the relation between \(h_x^d\) and \(h_x^y\) can be expressed as

\[
[R(\beta)]_{2M \times 2M} \begin{bmatrix}
    h_x^d \\
    h_y^d
\end{bmatrix}_{2M \times 1} = 0.
\] (18)

For guided modes, the eigenvalue \(\beta\) is determined from \(\det[R(\beta)] = 0\) and the eigenvector \([h_x^d, h_y^d]^T\) is nontrivial. The bisection method serves to locate the desired \(\beta\). Because step 1 is repeated at each trial in the bisection root search, it is the most time-consuming step. Once \(\beta\) and \([h_x^d, h_y^d]^T\) with the constraint

\[
\sum_{i=1}^{M} (h_x^{d,2} + h_y^{d,2}) = 1
\] (19)

are obtained, \([h_x^{n,d}, h_y^{n,d}]^T\) and \([h_x^e, h_y^e]^T\) can be found from (15) and (16), respectively. The reason for the constraint on \([h_x^d, h_y^d]^T\) with normalized norm will become clear in the next few steps.

Step 3: Discretize (5) and (6) and obtain the relation between \(\tilde{f_n}\) and \([f]\) and the relation between \([h_x^{n,e}, h_y^{n,e}]^T\) and \([h_x^e, h_y^e]^T\). By the same method with the same basis functions and adopting the same nodes as in step 1, (5) and (6) can be transformed into

\[
[r_n]_{M \times 1} = [M_d][\tilde{f}]_{M \times 1} + \beta[\gamma_d] + [\delta_d]
\] (20)

and

\[
\begin{bmatrix}
    h_x^x \\
    h_y^y
\end{bmatrix}_{2(N-M) \times 1} = [Q]\begin{bmatrix}
    h_x^d \\
    h_y^d
\end{bmatrix}_{2(N-M) \times 1} + \beta[\gamma_q] + [\delta_q]
\] (21)

respectively, and by the finite difference with (21) as in step 1 \([h_x^{n,e}, h_y^{n,e}]^T\) is expressed as

\[
\begin{bmatrix}
    h_x^{n,e} \\
    h_y^{n,e}
\end{bmatrix}_{2M \times 1} = [M_c]\begin{bmatrix}
    h_x^e \\
    h_y^e
\end{bmatrix}_{2M \times 1} + \beta[\gamma_q] + [\delta_q]
\] (22)

where the column vectors \([\gamma_d]\) and \([\delta_d]\) are determined by the solved \(\beta\), the boundary nodal fields \([f_n, f]\) in steps 1-2, and the first derivatives of the Green function and its normal derivative on the cladding boundary \(\Gamma\) \((G, dg/dn)\) with respect to \(V\), and the column vectors \([\gamma_q]\) and \([\delta_q]\) are determined by \(\beta\) and the nodal fields \(h\) in the core region.

Step 4: Solve \(\beta\) and \(h\). By substituting (20) and (22) into (7) and enforcing the continuity requirement of \(E_z\) and \(H_z\) at the boundary \(\Gamma\), one obtains

\[
[R(\beta)]_{2M \times 2M} \begin{bmatrix}
    h_x^d \\
    h_y^d
\end{bmatrix}_{2M \times 1} + \beta[\gamma_d] + [\delta_d] = 0.
\] (23)

In (23), there are \(2M + 1\) unknowns \((h_x^d, h_y^d, \text{ and } \beta)\), but only \(2M\) equations. By differentiating (19) with respect to \(V\), another constraint

\[
\sum_{i=1}^{M} (h_x^{d,i} h_y^{d,i} + h_y^{d,i} h_y^{d,i}) = 0
\] (24)

is obtained. Only one solution of \(\{\beta, h_x^d, h_y^d\}\) can be solved from (23) and (24).

Step 5: Discretize (9) and (10) and obtain the relation between \([f_n]\) and \([f]\) and the relation between \([h_x^{n,e}, h_y^{n,e}]^T\) and \([h_x^e, h_y^e]^T\). By the same method with the same basis functions and adopting the same nodes as in steps 1 and 3, (9) and (10) can be transformed into

\[
[r_n]_{M \times 1} = [M_d][\tilde{f}]_{M \times 1} + \beta[\gamma_d] + [\delta_d]
\] (25)

and

\[
\begin{bmatrix}
    h_x^x \\
    h_y^y
\end{bmatrix}_{2(N-M) \times 1} = [Q]\begin{bmatrix}
    h_x^d \\
    h_y^d
\end{bmatrix}_{2(N-M) \times 1} + \beta[\gamma_q] + [\delta_q]
\] (26)

respectively, and by the finite difference with (26) as in steps 1 and 3 \([h_x^{n,e}, h_y^{n,e}]^T\) is expressed as

\[
\begin{bmatrix}
    h_x^{n,e} \\
    h_y^{n,e}
\end{bmatrix}_{2M \times 1} = [M_c]\begin{bmatrix}
    h_x^e \\
    h_y^e
\end{bmatrix}_{2M \times 1} + \beta[\gamma_q] + [\delta_q]
\] (27)

where \([\gamma_d]\) and \([\delta_d]\) are determined by the first and second derivatives of the Green function and its normal derivative on the cladding boundary \(\Gamma\) \((G, dg/dn, G, dg/dn)\) with respect to \(V\), and the solved \(\beta, \beta, [f_n, f]\), and \([f_n, f]\) in steps 1-4, \([\gamma_q]\), and \([\delta_q]\) are determined by \(\beta, \beta, h\), and \(h\) which have been solved in steps 1-4.
Step 6: Solve $\tilde{\beta}$ and $\tilde{h}$. By substituting (25) and (27) into (11) and enforcing the continuity requirement of $\bar{E}_z$ and $\bar{H}_z$ at the boundary $\Gamma'$, one obtains

$$[R(\beta)]_{2M \times 2M} \begin{bmatrix} h_x^{iM} \\ h_y^{iM} \end{bmatrix} + \tilde{\beta}[\gamma'] + [\delta'] = 0. \quad (28)$$

By differentiating (24) again with respect to $V$, the constraint becomes

$$\sum_{i=1}^{M} (h_{d,i}^x, \tilde{h}^{x'}_{d,i}, h_{d,i}^y, \tilde{h}^{y'}_{d,i}, h_{d,i}^z, \tilde{h}^{z'}_{d,i}) = 0 \quad (29)$$

Only one solution of $\{\tilde{\beta}, \tilde{h}_x, \tilde{h}_y\}$ can be solved from (28) and (29).

Based on the full-wave vectorial formulations of the SIEM and FEM, the approach described above can efficiently calculate the first and second derivatives of the propagation constants with respect to $V$ of the guided modes for optical waveguides with inhomogeneous core and uniform cladding. Most of the computational time is spent in the first two steps when searching for $\beta$ and $h$. In steps 3–6, only some matrix generations, matrix multiplications, and matrix inversions are required.

IV. RESULTS AND DISCUSSIONS

Consider a dual-mode e-core fiber with the major and minor axis radii being $a$ and $b$, respectively, as shown in Fig. 1. The refractive index profile is defined as

$$n^2(R) = \begin{cases} n_1^2 - (n_1^2 - n_2^0) R^\alpha & \text{for } R \leq 1 \\ n_2^0 & \text{elsewhere} \end{cases}$$

where $R = (x^2/a^2 + y^2/b^2)^{1/2}$. Note that $\epsilon_e(R) = n_e^2(R)$ and $\epsilon_m = n_m^2$. When $\alpha = 1, 2, \text{ and } \infty$, the index profile is a triangular, parabolic, and step distribution, respectively. The change of the vectorial magnetic field patterns of the four higher-order LP$_{11}$ modes as the fiber deforms from a circle ($a/b = 1$) to an ellipse ($a/b = 1.05$) is shown in Fig. 2 with the parameters being $\Delta = 10\%$, $\alpha = (2\pi/\lambda)\sqrt{n_1^2 - n_2^0} = 4$ and $\alpha = 2$. Fig. 2(a), (b), (c), and (d) shows the patterns changing from TE$_{01}$ to LP$_{11,2}^e$, TM$_{01}$ to LP$_{11,2}^e$, HE$_{11}^e$ to LP$_{11,2}^{odd}$ and HE$_{21}^e$ to LP$_{11,2}^{odd}$, respectively. Each arrow represents the orientation and the relative strength of the magnetic field at the point specified by the arrow root. It can be seen that the small deformation of core geometry results in the linearly polarized field patterns with two lobe orientations.

To check the correctness of this novel numerical approach, we compare the dispersions, which are calculated by our approach, of the LP$_{11}^{even}$ modes for the e-core fiber with the results by Poole et al. [6] which are also based on a vectorial waveguide theory. The e-core parameters used in [6] are $\epsilon_e = 2(a-b)/(a+b) = 10\%$ and $\alpha = \infty$ (step index distribution). The cutoff wavelength of the LP$_{11,2}^{even}$ mode is designed to be at 1600 nm for the e-core fiber. Fig. 3 shows the dispersions of the LP$_{11,2}^{even}$ and LP$_{11,2}^{odd}$ modes versus wavelength for three different index steps $\Delta = 0.5\%$, $2\%$, and $4\%$, respectively.
The index distribution on the waveguide dispersions of the TE_{01}, TE_{11}, and HE_{21} modes for the circular fiber with $\Delta = 2\%$. The radius $b = 2.2 \mu m$ is chosen such that the cutoff wavelength is 1.623 $\mu m$ for the HE_{21} mode. It can be seen that the absolute value of the dispersion increases as the $\alpha$ increases with the maximum occurring at $\alpha = \infty$. The FEM results under the scalar theory [11] are also shown in this figure and they are found to be close to the dispersions of the TM_{01} mode. Because the compensator is designed for operating near the cutoff wavelength, the difference in the dispersion of these three modes is significant and the dispersion behaviours for different $\alpha$-powers cannot be sufficiently predicted by the scalar theory.

The effect of the refractive-index dip on the dispersion characteristics of the $e$-core compensator is also investigated for $\alpha = 2$ and $\alpha = \infty$ cases. Assume that in the presence of the dip the index profile can be written as

$$n^2(R) = \begin{cases} n_0^2(R) - (n_1^2 - n_2^2) p \left(e^{-R^2/d^2} - e^{-1/d^2} \right) & \text{for } R \leq 1 \\ n_2^2 & \text{elsewhere} \end{cases}$$

where $n_0^2(R)$, which can be any $\alpha$-power distribution, is the refractive index profile in the absence of the dip and $p$ and $d$ define the fractional dip depth and the fractional dip width, respectively. A case of the profile with $n_0^2(R)$ being the step distribution ($\alpha = \infty$) is shown in Fig. 5. The dispersions of the four LP_{11} modes versus wavelength from 1.53 $\mu m$ to 1.56 $\mu m$ for three different dip parameters ($d = p = 0$, $d = p = 0.2$, and $d = p = 0.3$) with $\alpha = \infty$, $a/b = 1.05$, $\Delta = 2\%$ and $b = 2.137 \mu m$ are shown in Fig. 6. For each pair of curves the upper curve is for $x$ polarization and the lower curve is for $y$ polarization for each case with different dip parameter. It can be seen that the index dip causes the increase of the dispersions of the LP_{11} modes. The reason is that due to the dip the fractional power guiding in the core decreases and the compensator is operated much near the cutoff wavelength. The influence of the dip on the dispersions of the compensator with $\alpha = 2$, $a/b = 1.05$, $\Delta = 2\%$, and $b = 3.06 \mu m$ is shown in Fig. 7. The dip effect of increasing the dispersion values is the same as in the previous case ($\alpha = \infty$). Note that the dispersion discussed in Figs. 3, 5, and 6 is the summation of the waveguide dispersion calculated by the approach described in Section III and the material dispersion obtained from the empirical Sellmeier equation with 13% GeO_{2} doping in silica for the core and pure silica for the cladding [15]. Compared with the waveguide dispersion, the contribution of the material dispersion is relatively small for the $e$-core compensator which is operated near the cut-off wavelength.

### V. Conclusion

Based on the full-wave vectorial formulations of combining the surface integral equation method (SIEM) and the finite-element method (FEM), a novel approach for determining the dispersion characteristics of the dual-mode elliptical-core fibers with arbitrary index profile in the core is proposed without resorting to numerical differentiation. Most computing
efforts are spent in solving the propagation constant \( \beta \) and the guided mode fields \( H_x \) and \( H_y \). Once they are obtained, only some matrix manipulations are required to obtain the first and second derivatives of the propagation constant with respect to \( V \). By this proposed approach, the influence of the \( \alpha \) - powers and the index dip effect, on the dispersions of the four nondegenerate LP_{11} modes for the \( e \)-core compensators are investigated as numerical examples. The vectorial field patterns of these four modes for the \( e \)-core fibers have also been presented.

**REFERENCES**


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