Virtual Auxiliary Termination for Multiport Scattering Matrix Measurement Using Two-Port Network Analyzer

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Abstract—Almost all methods for measuring the scattering matrix of an n -port device with the use of a two-port vector network analyzer (VNA) require one to terminate the other $n - 2$ ports in the fully characterized auxiliary terminations and prefer auxiliary terminations with small or moderate reflection coefficients. In this paper, a technique is presented to measure the auxiliary terminations indirectly. It not only eases the measurement of auxiliary terminations, but also makes the concept of virtual auxiliary termination realizable. In practice, examples of virtual auxiliary terminations can be the inherent connectors or bonding pads of the test device. Also studied is the remedy for the numerical pitfall possibly coexisting with the use of strongly reflecting auxiliary terminations such as virtual auxiliary terminations. The scattering matrix of a four-port circulator is then acquired accordingly from measurements using a two-port VNA and virtual auxiliary terminations.

Index Terms—Auxiliary termination, multiport network, scattering matrix measurement.

I. INTRODUCTION

WHEN IT comes to n -port scattering matrix (S -matrix) measurement with the use of a two-port vector network analyzer (VNA), one can follow the mathematical definition [1] and perform C_2^m combinations of two-port measurements with the other $n - 2$ unused ports terminated in perfectly matched loads, where $C_2^m = n(n - 1)/2$. Nonetheless, a matched load is not always accessible to applications such as millimeter wave or broadband. By taking advantage of the formulas derived in [2], the following two paragraphs are given to explain how an imperfectly matched load degrades the accuracy of the measured S -parameters, and to introduce the nomenclature in this paper.

For an n -port device-under-test (DUT), the true n -port S -matrix and its element in the p th row and q th column are denoted $\bar{S}$ and S_{pq} , respectively. For each two-port measurement, the set of number of ports $\{1, 2, \dots, n\}$ is divided into two disjoint subsets, which are $J = \{j_1, j_2\}$ for the two measured ports and $K = \{k_1, k_2, \dots, k_{n-2}\}$ for the other $n - 2$ terminated ports. Therefore, the possible combinations of J and K are both C_2^m . The measured two-port S -matrix is given by [2]

$$\bar{S}^{mj_1j_2} = \bar{S}_{JJ} + \bar{S}_{JK}(\bar{\Gamma}_{KK}^{-1} - \bar{S}_{KK})^{-1}\bar{S}_{KJ}. \quad (1)$$

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In (1),

$$\bar{\Gamma}_{KK} = \begin{bmatrix} \Gamma_{k_1} & 0 & 0 & \cdots & 0 \\ 0 & \Gamma_{k_2} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \Gamma_{k_{n-3}} & 0 \\ 0 & 0 & \cdots & 0 & \Gamma_{k_{n-2}} \end{bmatrix}. \quad (2)$$

The diagonal elements in (2) are the reflection coefficients of the auxiliary terminations, which are connected in sequence to the terminated ports. $\bar{S}_{JJ}$, $\bar{S}_{JK}$, $\bar{S}_{KJ}$, and $\bar{S}_{KK}$ are partitioned matrices of the true n -port S -matrix and are defined as (3), shown at the bottom of the following page.

Taking a three-port network with $J = \{1, 3\}$ as measured ports and the other port $K = \{2\}$ terminated in Γ_2 as an example, we can write the measured two-port S -matrix according to (1)–(3) as

$$\begin{aligned} \bar{S}^{m13} &\triangleq \begin{bmatrix} S_{11}^{m13} & S_{12}^{m13} \\ S_{21}^{m13} & S_{22}^{m13} \end{bmatrix} \\ &= \begin{bmatrix} S_{11} & S_{13} \\ S_{31} & S_{33} \end{bmatrix} \\ &\quad + \begin{bmatrix} S_{12} \\ S_{32} \end{bmatrix} (\Gamma_2^{-1} - S_{22})^{-1} [S_{21} \quad S_{23}]. \end{aligned} \quad (4)$$

It is observable that four elements of the true three-port S -matrix can be obtained as the first matrix on the right side of (4) if $\Gamma_2 = 0$. Otherwise, the imperfectly matched load Γ_2 causes mismatch-induced errors and makes the measured scattering parameters (S -parameters) deviate from the true ones.

Thanks to the efforts of pioneering researchers, many mathematical formulas for correcting the mismatch-induced errors have been developed to resolve the predicament of measuring multiport devices with the use of a two-port VNA [2]–[11]. Among them, each method asks one to deploy auxiliary terminations according to its own strategy. However, the need for all [2]–[9] or partial [10], [11] auxiliary terminations to be fully characterized in terms of their complex reflection coefficients is in common. In addition, we observe that most of the methods prefer auxiliary terminations with small or moderate reflection coefficients. It is believed that they can alleviate possible numerical difficulties by those auxiliary terminations. Thus, there are two issues worthy of our attention. Firstly, depending on the types of the ports of the DUT, measuring the reflection coefficient of auxiliary termination may be easy or tedious. For instance, it could be beneficial for not directly probing auxiliary terminations like surface mount devices. Secondly, in contrast to

the processes of soldering surface mount resistors for a printed circuit board (PCB) and fabricating poly resistors for a monolithic microwave integrated circuit (MMIC), which suffer from large variation, the process of implementing strongly reflecting termination like a short-circuited or an open-circuited transmission line is repeatable and reliable for both applications. The applicability of existing methods is then limited, as they do not function normally with the use of auxiliary terminations having large reflection coefficients. Only recently have these two issues been tackled in [11]. It proposes a method to acquire the S -matrix of a multistage line coupler using a two-port VNA and short-circuited reflective terminations with $|\Gamma| \approx 0.6 - 0.8$, in which all but one are unknown.

Inspired by an attempt to deal with the first issue mentioned above, we propose an indirect measurement technique to circumvent those direct measurements of auxiliary terminations. It is later learned that the synergy of this technique and the method in [2] allows one to dispense with the use of auxiliary terminations. In other words, the DUT can have its unused ports left unconnected even though it may give large reflection coefficients. Since the unused ports have no terminations to be specified, they are denoted as virtual auxiliary terminations in this paper. The inherent subminiature A (SMA) connector of the DUT is an example of virtual auxiliary termination. It should be noted that the reflection coefficient of a virtual auxiliary termination usually cannot be measured directly.

Although the concept of virtual auxiliary termination is verified by reconstructing the three-port S -matrix of a Mini-Circuits D17I 17 dB directional coupler in [12], numerical experiments to be conducted in Section II reveal that the utilization of virtual auxiliary terminations could entrap the method in [2] into numerical pitfalls for low-loss DUTs at certain frequency points. A remedy for the numerical difficulties is then studied. The practical applicability of virtual auxiliary termination is demonstrated by reconstructing the four-port S -matrix of a nonreciprocal component, i.e., Narda COF-2040 circulator, in Section III. A successful fulfillment of using only virtual auxiliary terminations can simplify the measurement arrangement and procedures, as well as broaden the applicability of the method in [2].

II. FORMULATION

Here, a technique for indirectly measuring an auxiliary termination is described. The concept of virtual auxiliary terminations and how their utilization impacts the accuracy of the method in [2] are elucidated in detail. A corresponding remedy

based on Newton's method and simple moving average is then formulated.

A. Indirect Measurement of Auxiliary Termination

When the third to the n th ports of an n -port network are terminated in $\Gamma_3, \Gamma_4, \dots, \Gamma_n$, one can measure ports 1 and 2 and obtain the two-port S -matrix $\tilde{S}^{m12}$. One then terminates the second port in Γ_2 and measures port 1 to get the one-port S -parameter S^{m1} . By performing signal flow graph analysis, Γ_2 can be calculated from the elements of $\tilde{S}^{m12}$ and S^{m1} as [12]

$$\Gamma_2 = \left(S_{22}^{m12} + \frac{S_{12}^{m12} S_{21}^{m12}}{S^{m1} - S_{11}^{m12}} \right)^{-1}. \quad (5)$$

Therefore, through proper permutations, all Γ_i can be acquired without directly measuring them.

B. Virtual Auxiliary Termination

According to [2], the S -matrix of an n -port network can be reconstructed using C_2^n combinations of two-port measurements with the other $n - 2$ unused ports terminated in fully characterized auxiliary terminations. It is given by

$$\tilde{S}^{\text{rec}} = \bar{\Gamma}^{-1} - \bar{\Sigma}^{-1}. \quad (6)$$

In (6),

$$\bar{\Gamma} = \begin{bmatrix} \Gamma_1 & 0 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \Gamma_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \Gamma_n \end{bmatrix} \quad (7)$$

is an $n \times n$ diagonal matrix, whose diagonal elements are the reflection coefficients of the auxiliary terminations in which ports are invariably terminated while not being connected to the VNA. The elements of $\bar{\Sigma}$ to be expressed in Section II-C are related to the reflection coefficients of the auxiliary terminations and the measured two-port S -matrices so that the reconstructed S -matrix $\tilde{S}^{\text{rec}}$ is mathematically identical to the true n -port S -matrix $\tilde{S}$.

This method imposes no restriction on the deployment of auxiliary terminations, except each port should be terminated in the same auxiliary termination while not being connected to the VNA. Accordingly, a logical and handy choice of auxiliary terminations is to use the inherent connectors for components with coaxial-type connectors or probe pads for the MMIC. In other words, the DUT leaves its unused ports with no connection to

$$\begin{bmatrix} \tilde{S}_{JJ} & | & \tilde{S}_{JK} \\ \hline - & + & - \\ \tilde{S}_{KJ} & | & \tilde{S}_{KK} \end{bmatrix} = \begin{bmatrix} S_{j_1 j_1} & S_{j_1 j_2} & | & S_{j_1 k_1} & S_{j_1 k_2} & \cdots & S_{j_1 k_{n-2}} \\ S_{j_2 j_1} & S_{j_2 j_2} & | & S_{j_2 k_1} & S_{j_2 k_2} & \cdots & S_{j_2 k_{n-2}} \\ \hline S_{k_1 j_1} & S_{k_1 j_2} & | & S_{k_1 k_1} & S_{k_1 k_2} & \cdots & S_{k_1 k_{n-2}} \\ S_{k_2 j_1} & S_{k_2 j_2} & | & S_{k_2 k_1} & S_{k_2 k_2} & \cdots & S_{k_2 k_{n-2}} \\ \vdots & \vdots & | & \vdots & \vdots & \ddots & \vdots \\ S_{k_{n-2} j_1} & S_{k_{n-2} j_2} & | & S_{k_{n-2} k_1} & S_{k_{n-2} k_2} & \cdots & S_{k_{n-2} k_{n-2}} \end{bmatrix} \quad (3)$$

the specified terminations. Instead of knowing the reflection coefficients of these auxiliary terminations in advance, one can acquire them by following the procedures of developing (5). By this approach, one can leave those ports not connected to the VNA unconnected during each two-port measurement. These terminations are then dubbed as the virtual auxiliary terminations. Note that the virtual auxiliary terminations usually have reflection coefficients close to one. Studies of the possible computation difficulties are presented in the following.

C. Accuracy Analysis

Since both calculations of $\bar{\Sigma}$ and (6) involve matrix inversion, it could be numerically difficult under certain circumstances [3], [4], [13]–[15]. Judging by [15], one can tell the accuracy of the method in [2] from the two-norm condition number of $\bar{\Sigma}$. A matrix with a large condition number is known as ill conditioned, one whose solution to an inverse matrix is overly sensitive to perturbations in its elements. In our case, numerical perturbations mostly originate from the measurement errors and the finite precision of the computer.

Before the discussion of the condition number of $\bar{\Sigma}$, the calculation of $\bar{\Sigma}$ itself is given in the following. For a two-port measurement with the measured ports $J = \{j_1, j_2\}$ and the terminated ports $K = \{k_1, k_2, \dots, k_{n-2}\}$, $\bar{\Gamma}$ in (7) can be partitioned accordingly as

$$\begin{bmatrix} \bar{\Gamma}_{JJ} & | & \bar{0} \\ \hline \bar{0} & | & \bar{\Gamma}_{KK} \end{bmatrix} = \begin{bmatrix} \Gamma_{j_1} & 0 & | & 0 & 0 & \cdots & 0 \\ 0 & \Gamma_{j_2} & | & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & | & \Gamma_{k_1} & 0 & \cdots & 0 \\ 0 & 0 & | & 0 & \Gamma_{k_2} & \ddots & \vdots \\ \vdots & \vdots & | & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & | & 0 & \cdots & 0 & \Gamma_{k_{n-2}} \end{bmatrix}. \quad (8)$$

By defining [2],

$$\bar{\sigma}_{JJ} = (\bar{\Gamma}_{JJ}^{-1} - \bar{S}^{m_{j_1 j_2}})^{-1} \quad (9)$$

and substituting (1) into (9), one can show that [2]

$$\bar{\sigma}_{JJ} = \left(\bar{\Gamma}_{JJ}^{-1} - \bar{S}_{JJ} - \bar{S}_{JK} (\bar{\Gamma}_{KK}^{-1} - \bar{S}_{KK})^{-1} \bar{S}_{KJ} \right)^{-1}. \quad (10)$$

While (9) signifies that $\bar{\sigma}_{JJ}$ can be completely known from measurements, (10) relates it to the true n -port S -matrix. In the meantime, one can also define [2]

$$\begin{aligned} \bar{\Sigma} &= (\bar{\Gamma}^{-1} - \bar{S})^{-1} \\ &= \left(\begin{bmatrix} \bar{\Gamma}_{JJ} & \bar{0} \\ \bar{0} & \bar{\Gamma}_{KK} \end{bmatrix}^{-1} - \begin{bmatrix} \bar{S}_{JJ} & \bar{S}_{JK} \\ \bar{S}_{KJ} & \bar{S}_{KK} \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} \bar{\Gamma}_{JJ}^{-1} - \bar{S}_{JJ} & -\bar{S}_{JK} \\ -\bar{S}_{KJ} & \bar{\Gamma}_{KK}^{-1} - \bar{S}_{KK} \end{bmatrix}^{-1}. \end{aligned} \quad (11)$$

Fig. 1. Port enumeration of the four-port circulator.

The formula for matrix inversion in [16] is then applied to express (11) in the form of

$$\begin{aligned} \bar{\Sigma} &= \begin{bmatrix} \bar{\Sigma}_{JJ} & \bar{\Sigma}_{JK} \\ \bar{\Sigma}_{KJ} & \bar{\Sigma}_{KK} \end{bmatrix} \\ &= \begin{bmatrix} \left(\bar{\Gamma}_{JJ}^{-1} - \bar{S}_{JJ} - \bar{S}_{JK} (\bar{\Gamma}_{KK}^{-1} - \bar{S}_{KK})^{-1} \bar{S}_{KJ} \right)^{-1} & \bar{\Delta}_1 \\ \bar{\Delta}_2 & \bar{\Delta}_3 \end{bmatrix}. \end{aligned} \quad (12)$$

As $\bar{\Delta}_1$, $\bar{\Delta}_2$ and $\bar{\Delta}_3$ are of no significance for the following discussion, their explicit expressions are not shown. By making a comparison between (10) and (12), the identity between $\bar{\sigma}_{JJ}$ and $\bar{\Sigma}_{JJ}$ concludes that $\bar{\sigma}_{JJ}$ is a partitioned matrix of $\bar{\Sigma}$. Consequently, one can piece together $\bar{\Sigma}$ from C_2^n combinations of $\bar{\sigma}_{JJ}$, which can be evaluated from the measured two-port S -matrices and the reflection coefficients of auxiliary terminations by using (9).

Through trying to expand (10), one finds that the elements of $\bar{\Sigma}$ are nonlinear complex functions of S -parameters of the n -port network and reflection coefficients of the auxiliary terminations. It is difficult, if not impossible, to figure out what $\bar{\Gamma}$ tends to make $\bar{\Sigma}$ ill conditioned by interpreting and deciphering the equations. In contrast, it is practicable to perform quantitative analyses by conducting computer simulation of measurement.

Designating a Narda COF-2040 four-port circulator, which operates between 2–4 GHz as the experimental sample, we illustrate its port enumeration in Fig. 1 and prepare its S -matrices at different frequencies for the ensuing simulations by following the mathematical definition of the S -parameter [1] and performing six combinations of two-port measurements with unused ports terminated in the loads of an Agilent 85052D calibration kit.

The first computer simulation is to study how different reflection coefficients of auxiliary terminations affect the condition numbers of $\bar{\Sigma}$. Although auxiliary terminations are permitted to be all identical, partially distinct, or all distinct, the case of being all identical is of interest and is simulated. Defining the average power dissipation of an n -port network as

$$P_{d,avg} = -10 \log_{10} \left(\frac{1}{n} \sum_{q=1}^n \sum_{p=1}^n |S_{pq}|^2 \right) \quad (13)$$

Fig. 2. Average power dissipation of the circulator.

and with Fig. 2 depicting that of the circulator, we group the S -matrices of the circulator into three types. The corresponding features of the condition number of $\bar{\Sigma}$ are individually studied from their sampled S -matrices.

The type 1 S -matrix located between 1–1.5 GHz is high loss. The S -matrix sampled at 1.2 GHz is shown in (14) at the bottom of this page.

The type 2 S -matrix residing between 2–4 GHz is in-band and has characteristics of low loss, good matching, low backward leakage, high isolation, and high forward transmission. Consequently, the measurement of type 2 S -matrix requires careful arrangement due to the high dynamic range in the S -parameters. The S -matrix sampled at 2.95 GHz is shown in (15) at

the bottom of this page. The type 3 S -matrix lying between 4.5–5 GHz is low loss. The S -matrix sampled at 4.75 GHz is shown in (16) at the bottom of this page.

The two-port measurements are then simulated by partitioning the sampled S -matrices properly and substituting these partitioned matrices and the specified reflection coefficients of auxiliary terminations into (1). After six combinations of $\bar{\sigma}_{JJ}$ are evaluated from (9), $\bar{\Sigma}$ is acquired and its two-norm condition numbers for various reflection coefficients of auxiliary terminations over the Smith chart are depicted for the sampled S -matrices in Figs. 3–5.

The condition number distribution pattern of the type 1 sample shows only one peak, whereas those of types 2 and 3 have four relatively higher peaks. After running more simulations at other frequencies for each type of S -matrix, we observe that the resulting patterns are similar to those given in Figs. 3–5. Moreover, even though peaks of the patterns change positions for different cases, they never move inward on the Smith chart.

Three implications can then be drawn from the results of the first experimental simulation. Firstly, the method in [2] prefers auxiliary terminations with small reflection coefficients. Secondly, when auxiliary terminations with large reflection coefficients are taken, it may encounter numerical difficulty whose destructiveness depends inversely on the power dissipation of the DUT. Thirdly, since the reflection coefficient of virtual auxiliary termination is close to that of an open circuit, which is at the furthest right point in the Smith chart in Fig. 4(b), the second sampled S -matrix is a specimen of the numerical difficulty for using virtual auxiliary terminations.

The second computer simulation explores the compatibility between virtual auxiliary terminations and the method in [2]. Assuming the reflection coefficients of virtual auxiliary terminations to be unity, we evaluate the corresponding condition numbers of $\bar{\Sigma}$ with the given S -matrices of the circulator from

$$\bar{S}_1 = \begin{bmatrix} 0.53\angle 55^\circ & 0.32\angle 81^\circ & 0.10\angle -119^\circ & 0.18\angle -29^\circ \\ 0.43\angle 150^\circ & 0.52\angle 59^\circ & 0.05\angle 171^\circ & 0.09\angle -98^\circ \\ 0.09\angle -97^\circ & 0.18\angle -25^\circ & 0.45\angle 63^\circ & 0.42\angle 63^\circ \\ 0.05\angle 165^\circ & 0.09\angle -123^\circ & 0.50\angle 148^\circ & 0.45\angle 57^\circ \end{bmatrix} \quad (14)$$

$$\bar{S}_2 = \begin{bmatrix} 0.016\angle -172^\circ & 0.022\angle 154^\circ & 0.025\angle 17^\circ & 0.955\angle -90^\circ \\ 0.975\angle 91^\circ & 0.020\angle -132^\circ & 0.001\angle 85^\circ & 0.018\angle -16^\circ \\ 0.020\angle -35^\circ & 0.957\angle -80^\circ & 0.009\angle -101^\circ & 0.018\angle -178^\circ \\ 0.0007\angle -90^\circ & 0.011\angle -51^\circ & 0.975\angle 92^\circ & 0.019\angle 154^\circ \end{bmatrix} \quad (15)$$

$$\bar{S}_3 = \begin{bmatrix} 0.23\angle 84^\circ & 0.56\angle 121^\circ & 0.31\angle -30^\circ & 0.65\angle -167^\circ \\ 0.85\angle 20^\circ & 0.23\angle 105^\circ & 0.16\angle 102^\circ & 0.34\angle -34^\circ \\ 0.35\angle -30^\circ & 0.66\angle -143^\circ & 0.22\angle 118^\circ & 0.54\angle 123^\circ \\ 0.15\angle 77^\circ & 0.30\angle -35^\circ & 0.87\angle 24^\circ & 0.22\angle 83^\circ \end{bmatrix} \quad (16)$$

Fig. 3. Condition number distribution patterns in: (a) 3-D plot and (b) contour plot of the circulator at 1.2 GHz.

1 to 5 GHz. In Fig. 6, one observes that peaks with a condition number greater than 30 appear almost periodically within the operation bandwidth of the circulator. This simulation study manifests the necessity of a remedy for those ill-conditioned points.

D. Remedy

In (1), a measured two-port S -matrix $\bar{S}^{mj_1j_2}$ is related to the true n -port S -matrix $\bar{S}$ of the DUT and the reflection coefficients $\bar{\Gamma}_{KK}$ of auxiliary terminations. With C_2^n combinations of two-port measurements, there are $4C_2^n$ equations available, which can be written concisely in the form of a system of non-linear equations as

$$\begin{aligned} f_1(S_{11}, S_{12}, \dots, S_{nn}, \Gamma_1, \Gamma_2, \dots, \Gamma_n, S_{11}^{m12}) &= 0 \\ f_2(S_{11}, S_{12}, \dots, S_{nn}, \Gamma_1, \Gamma_2, \dots, \Gamma_n, S_{12}^{m12}) &= 0 \\ &\vdots \\ f_{4C_2^n}(S_{11}, S_{12}, \dots, S_{nn}, \Gamma_1, \Gamma_2, \dots, \Gamma_n, S_{22}^{m(n-1)n}) &= 0. \end{aligned} \quad (17)$$

Among these $4C_2^n$ measured two-port S -parameters, the reflection term of each port is measured $n - 1$ times with different ports terminated in auxiliary terminations. For example, S_{11}^{m12} , S_{11}^{m13} , and S_{11}^{m14} are the three reflection terms measured

Fig. 4. Condition number distribution patterns in: (a) 3-D plot and (b) contour plot of the circulator at 2.95 GHz.

from the first port of a four-port DUT. The repetitions of reflection terms sum up to $4C_2^n - n^2$ redundant equations in (17). By keeping only one of the reflection terms of each port and dropping the others, the number of equations is then identical to that of the unknowns, which herein are the elements of $\bar{S}$.

The reduced version of (17) is arranged as

$$\begin{aligned} g_1(S_{11}, S_{12}, \dots, S_{nn}, \Gamma_1, \Gamma_2, \dots, \Gamma_n, S_{11}^{m12}) &= 0 \\ g_2(S_{11}, S_{12}, \dots, S_{nn}, \Gamma_1, \Gamma_2, \dots, \Gamma_n, S_{12}^{m12}) &= 0 \\ &\vdots \\ g_{n^2}(S_{11}, S_{12}, \dots, S_{nn}, \Gamma_1, \Gamma_2, \dots, \Gamma_n, S_{22}^{m(n-1)n}) &= 0. \end{aligned} \quad (18)$$

These equations are rational functions concerning each element of the true S -matrix, therefore they are analytic over the complex plane, except for those points making the denominator equal to 0 [17]. The existence of the partial derivatives hints that Newton's method is a candidate for solving (18). By using the vector notation and defining

$$\bar{X} = [S_{11} \ S_{12} \ \cdots \ S_{nn}]^t \quad (19)$$

$$\bar{G} = [g_1 \ g_2 \ \cdots \ g_{n^2}]^t \quad (20)$$

Fig. 5. Condition number distribution patterns in: (a) 3-D plot and (b) contour plot of the circulator at 4.75 GHz.

Fig. 6. Computer-simulated condition numbers of the circulator with the reflection coefficients of auxiliary terminations to be unity.

and

$$\bar{J} = \begin{bmatrix} \frac{\partial g_1}{\partial S_{11}} & \frac{\partial g_1}{\partial S_{12}} & \cdots & \frac{\partial g_1}{\partial S_{nn}} \\ \frac{\partial g_2}{\partial S_{11}} & \frac{\partial g_2}{\partial S_{12}} & \cdots & \frac{\partial g_2}{\partial S_{nn}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_{n^2}}{\partial S_{11}} & \frac{\partial g_{n^2}}{\partial S_{12}} & \cdots & \frac{\partial g_{n^2}}{\partial S_{nn}} \end{bmatrix} \quad (21)$$

the solution is denoted by [18]

$$\bar{X}^{\text{new}} = \bar{X}^{\text{old}} - [\bar{J}(\bar{X}^{\text{old}})]^{-1} \bar{G}(\bar{X}^{\text{old}}) \quad (22)$$

which is an iteration operation to calculate $\bar{X}^{\text{new}}$ from $\bar{X}^{\text{old}}$. The attribute of starting with an initial guess and iterating (22) to approach the true solution makes the success of Newton's method greatly dependent on the initial guess and the convergence criterion.

If the threshold of the condition number is set appropriately, the S -matrix reconstructed using (6) before the onset of numerical difficulty is reliable and is a good initial guess due to the continuity property of S -parameters and the smooth variation of the condition number, as shown in Fig. 6. Since the magnitudes of the measured S -parameters could differ by as much as many tens of decibels, the relative error

$$\bar{R}_G = \begin{bmatrix} \frac{|g_1|}{|S_{11}^{m12}|} & \frac{|g_2|}{|S_{12}^{m12}|} & \cdots & \frac{|g_{n^2}|}{|S_{22}^{m(n-1)n}|} \end{bmatrix}^t \quad (23)$$

is adopted to establish the convergence criterion as $\max(\bar{R}_G) < \epsilon$.

After taking into account the measurement errors possibly involved in the measured two-port S -parameters and the matrix inversion involved in (22), it is proposed to perform a simple moving average to remove the glitches at certain frequencies. A u -point simple moving average is the unweighted mean of the datum of interest and its $u - 1$ neighboring ones, and is the simplest form of a low-pass filter from the perspective of discrete-time signal processing [19]. It is given as

$$x^{\text{avg}}(v) = \frac{1}{u} \sum_{w=-(u-1)/2}^{(u-1)/2} x(v-w) \quad (24)$$

where v and w are the indices, and u is an odd number. In analogy to the design of a low-pass filter, u is chosen to be reasonably large to make the curve of the S -parameter smooth and not to distort those segments occupying the valleys of condition numbers in Fig. 6, which are reliably reconstructed using (6).

III. EXPERIMENT

The following experiment is conducted to demonstrate the applicability of virtual auxiliary termination. The DUT is a Narda COF-2040 four-port circulator, which is the experimental sample in Section II. It has four SMA female connectors. In the experiment, its four-port S -matrix is going to be reconstructed from measuring two-port S -matrices while leaving the other two ports unconnected as the virtual auxiliary terminations. The reason for using a four-port circulator in this experiment is that it is a nonreciprocal device with S -parameters having a large dynamic range to verify the applicability of our approach.

One needs to measure six combinations of two-port S -matrices, namely, $\bar{S}^{m12}$, $\bar{S}^{m13}$, $\bar{S}^{m14}$, $\bar{S}^{m23}$, $\bar{S}^{m24}$, and $\bar{S}^{m34}$, to get $\bar{\Sigma}$. Additionally, the one-port S -parameters for characterizing the unknown reflection coefficients of virtual auxiliary terminations are acquired by connecting one port to the VNA while

Fig. 7. Typical reflection coefficient of the virtual auxiliary termination.

Fig. 8. Condition number of the circulator terminated in virtual auxiliary terminations.

leaving the other three ports unconnected. The unknown reflection coefficients of the virtual auxiliary terminations at the other three ports are then derived based on the procedures of developing (5). For example, if the one-port S -parameter of the first port is measured and denoted as S^{m1} , the reflection coefficients of the virtual auxiliary terminations at the second, third, and fourth ports are given by

$$\Gamma_2 = \left(S_{22}^{m12} + \frac{S_{12}^{m12} S_{21}^{m12}}{S^{m1} - S_{11}^{m12}} \right)^{-1} \quad (25)$$

$$\Gamma_3 = \left(S_{22}^{m13} + \frac{S_{12}^{m13} S_{21}^{m13}}{S^{m1} - S_{11}^{m13}} \right)^{-1} \quad (26)$$

and

$$\Gamma_4 = \left(S_{22}^{m14} + \frac{S_{12}^{m14} S_{21}^{m14}}{S^{m1} - S_{11}^{m14}} \right)^{-1}. \quad (27)$$

Since the four connectors of the DUT are the same type of female SMA, the one-port S -parameter measured from any port is algebraically capable of characterizing the reflection coefficient of the virtual auxiliary terminations.

Fig. 9. Reflection coefficient S_{11} in: (a) magnitude and (b) phase.

However, measuring the one-port S -parameters four times from four ports individually and evaluating the 12 possible values of the reflection coefficient help get a refined one and improve the accuracy of the calculation followed.

With $\bar{\Gamma}$ and $\bar{\Sigma}$ in hand, (6) is then carried out to reconstruct the four-port S -matrix of the DUT, which is called the result of the first step in the process of reconstruction. Once arriving at this stage, one reconstructs the S -matrices for those frequency points with condition numbers low enough. The remedy described in Section II-D is then applied for those ill-conditioned points by executing Newton's method given in (22) and the simple moving average given in (24) with an appropriate u value to acquire the final results.

In this experiment, each two- or one-port measurement produces a set of raw data for further processing. As a result, measurement precision plays a crucial role in lowering the numerical error. Special attention should be paid in the arrangements of calibration and measurement [20]–[22].

Fig. 10. Forward transmission coefficient S_{21} in: (a) magnitude and (b) phase.

IV. RESULTS

As expected, the reflection coefficient of the virtual auxiliary termination is identified as being close to unity after comprehensively surveying the 12 possible values. Typical results of the virtual auxiliary termination are shown in Fig. 7.

Fig. 8 shows the condition numbers of the DUT when terminating in virtual auxiliary terminations. The similarity between the pattern of Fig. 8 and that of Fig. 6 validates the conclusion from the computer simulation in Section II.

The port enumeration of the DUT illustrated in Fig. 1 suggests that elements of each column of the S -matrix can be categorized into four operation coefficients as reflection, forward transmission, isolation, and backward leakage. Although the operation coefficients of one port are not identical to those of another port, they do behave in a similar way. For the sake of brevity, only the operation coefficients of the first port are presented in Figs. 9–12.

Fig. 9–12 shows the benchmark, first step result based on (6), and final result. The benchmark for verifying the results is

Fig. 11. Isolation coefficient S_{31} in: (a) magnitude and (b) phase.

the four-port S -matrix prepared for the computer simulation in Section II. They are obtained from two-port measurements with other two ports terminated with loads of a calibration kit. The aim of showing results of the first step is twofold. Although it shows that the utilization of virtual auxiliary termination possibly suffers from numerical difficulty, it serves to specify the threshold of the condition number by clarifying the interrelation between the magnitude of the condition number and the extent of S -parameters runaway. From observing the results of the first step in Figs. 9–12 and the condition number in Fig. 8, the coincidence between the soar of the condition number and the runaway of the reconstructed S -parameters is apparent.

For those ill-conditioned S -parameters, we set the threshold of the condition number to be 5 and apply Newton's iteration given in (22) maximally seven times to meet the convergence criterion of $\max(\bar{R}_G) < 10^{-9}$. The simple moving average defined in (24) is then performed to give the final results shown in Fig. 9–12. Each solid line in these figures consists of 801 data points, and that in Fig. 10 and those in other figures are smoothed using (24) with $u = 11$ and $u = 21$, respectively. In

Fig. 12. Backward leakage coefficient S_{41} in: (a) magnitude and (b) phase.

most situations, the final results show good agreement with the benchmark.

V. CONCLUSION

A technique has been described to measure the auxiliary terminations indirectly. In practice, it could be beneficial for not directly probing unknown auxiliary terminations. Based on this technique and the method in [2], the concept of virtual auxiliary termination is introduced and applied in the multiport S -matrix measurement with the use of a two-port VNA and leaving the rest of the ports of the DUT unconnected. The two-norm condition number of $\tilde{\Sigma}$ is studied as an indicator for one to tell the accuracy of the result evaluated from (6). If the calculation of (6) suffers from numerical difficulty at certain frequencies, it is complemented by a remedy that uses both Newton's method given in (22) to solve those points iteratively and the simple moving average given in (24) to remove the remaining glitches in the curve of the S -parameter.

The formulation and approach presented in this paper can acquire the S -matrix of an n -port network with the use of a

two-port VNA and virtual auxiliary terminations. By reviewing the DUTs on which we have experimented, we then give the following observation on a three-level difficulty in implementing virtual auxiliary terminations. The three-port S -matrix of a Mini-Circuits D171 17-dB directional coupler can almost be reconstructed by (6) due to having peak values of the condition numbers lower than 40 over the operational bandwidth [12]. Using (6) and (22) enables reconstructing of the three-port S -matrix of a DiTom three-port circulator D3C2040, though not shown here. The peak values of the condition numbers of this device are below 90 from 1 to 5 GHz. For the four-port circulator Narda COF-2040 given in Section III, the moving average operation defined in (24) is further applied to remove the remaining glitches after performing (6) and (22). In other words, moving average is a complementary approach to (6) and (22) for better reconstructing of the S -matrix of a DUT with the highest level of difficulty, which is observed as a four-port device with the peak values of the condition numbers greater than 70.

Bearing in mind that one may distort the S -parameters of a DUT by mistaking rapidly changing S -parameters for the remaining glitches and misusing moving average, two possible cases are discussed concerning what condition numbers the sharpest segment of the S -parameters is with. For the case with the condition number lower than 5, the sharpest segment is reliably reconstructed by (6) and surely not the remaining glitches. The u in (24) is then chosen to be reasonably large to remove the suspected glitches occurring near the peaks of the condition numbers and not to distort the sharpest segment. As the sharpest segment is with the condition numbers larger than 5, it is possibly masked by the suspected glitches and turns out to be distorted. To prevent this scenario from happening, *a priori* knowledge about the characteristics of the DUT is helpful to distinguish the rapidly changing magnitudes or phases from the remaining glitches.

In this paper, the practical applicability of virtual auxiliary termination is demonstrated by an experiment for measuring a four-port circulator. The final results show the agreement with those of the benchmark, and should be a creditable S -matrix of the DUT, by which one can acquire the characteristics of the DUT and use it in system design. It then leads to the ease of multiport network characterization by leaving the unused ports unconnected as the virtual auxiliary terminations.

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