Perturbation calculation of the axial anomaly of a Ginsparg-Wilson lattice Dirac operator

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A recent proposal suggests that even if a Ginsparg-Wilson lattice Dirac operator does not possess any topological zero modes in topologically nontrivial gauge backgrounds, it can reproduce the correct axial anomaly for sufficiently smooth gauge configurations, provided that it is exponentially local, doubler-free, and has the correct continuum behavior. In this paper, we calculate the axial anomaly of this lattice Dirac operator in weak coupling perturbation theory, and show that it recovers the topological charge density in the continuum limit.

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I. INTRODUCTION

Recently, one of us (T.W.C.) has constructed [1] a Ginsparg-Wilson Dirac operator that is γ_5 Hermitian, exponentially local, doubler-free, and has the correct continuum behavior, but it does *not* possess any topological zero modes for topologically nontrivial background gauge fields. This suggests that one might have the option to turn off the topological zero modes of a Ginsparg-Wilson lattice Dirac operator, without affecting its physical behavior (axial anomaly, fermion propagator, etc.) at least for the topologically trivial gauge sector. Therefore it is interesting to verify explicitly that it indeed reproduces the continuum axial anomaly for smooth gauge backgrounds, in the framework of weak coupling perturbation theory which is amenable to analytic calculations.

The Ginsparg-Wilson lattice Dirac operator proposed in Ref. [1] is

$$D = a^{-1} D_c (1 + r D_c)^{-1}, \quad r > 0, \quad (1)$$

with

$$D_c = \sum_{\mu} \gamma^{\mu} T^{\mu}, \quad T^{\mu} = f t^{\mu} f, \quad (2)$$

$$f = \left(\frac{2c}{\sqrt{t^2 + w^2} + w} \right)^{1/2}, \quad t^2 = - \sum_{\mu} t^{\mu} t^{\mu}. \quad (3)$$

Here $\gamma^{\mu} t^{\mu}$ is the naive lattice fermion operator and $-w$ is the Wilson term with a negative mass $-c$ ($0 < c < 2$):

$$t^{\mu}(x, y) = \frac{1}{2} [U_{\mu}(x) \delta_{x+\hat{\mu}, y} - U_{\mu}^{\dagger}(y) \delta_{x-\hat{\mu}, y}], \quad (4)$$

$$U_{\mu}(x) = \exp \left[i a g A_{\mu} \left(x + \frac{a}{2} \hat{\mu} \right) \right], \quad (5)$$

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma_{\mu} \\ \sigma_{\mu}^{\dagger} & 0 \end{pmatrix}, \quad (6)$$

$$\sigma_{\mu} \sigma_{\nu}^{\dagger} + \sigma_{\nu} \sigma_{\mu}^{\dagger} = 2 \delta_{\mu\nu}, \quad (7)$$

$$w(x, y) = c - \frac{1}{2} \sum_{\mu} [2 \delta_{x, y} - U_{\mu}(x) \delta_{x+\hat{\mu}, y} - U_{\mu}^{\dagger}(y) \delta_{x-\hat{\mu}, y}], \quad 0 < c < 2, \quad (8)$$

where the Dirac, color, and flavor indices have been suppressed. Note that in Eq. (1), we do not fix $r = 1/2c$, but keep it as a free parameter, since we intend to show that *the axial anomaly is independent of r in the continuum limit*.

Evidently, the lattice Dirac operator (1) is γ_5 Hermitian,

$$D^{\dagger} = \gamma_5 D \gamma_5, \quad (9)$$

and satisfies the Ginsparg-Wilson relation [2]

$$D \gamma_5 + \gamma_5 D = 2 r a D \gamma_5 D. \quad (10)$$

In the free fermion limit, D is exponentially-local, doubler-free, and has correct continuum behavior [1]. These properties should be sufficient to guarantee that D can reproduce the correct axial anomaly in a smooth gauge background.

In the next section, we calculate the axial anomaly of Eq. (1) in weak coupling perturbation theory, and show that, in the continuum limit, it recovers the topological charge density of the gauge background

$$\frac{g^2}{32\pi^2} \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \text{tr}(F_{\mu\nu} F_{\lambda\sigma}),$$

where

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + i g [A_{\mu}, A_{\nu}].$$

Before we proceed to the perturbation calculations in the next section, we first clarify that even though a Ginsparg-Wilson lattice Dirac operator can reproduce the continuum axial anomaly for sufficiently smooth gauge configurations, it does *not* necessarily imply that it could possess topological zero modes for topologically nontrivial gauge backgrounds. This can be seen as follows.

Consider a gauge configuration with positive definite topological charge $Q > 0$, and its topological charge density $\rho(x)$ is *very smooth* for all x ,

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$$\int d^4x \rho(x) = Q. \quad (11)$$

Then, in principle, $\rho(x)$ can be decomposed into two parts,

$$\rho(x) = \rho_w(x) + \rho_0(x), \quad (12)$$

where $\rho_w(x)$ is the “winding number density,” which is positive for all x , and it is the *smoothest* particular solution of Eq. (11),

$$\int d^4x \rho_w(x) = Q, \quad \rho_w(x) > 0 \quad \forall x, \quad (13)$$

and $\rho_0(x)$ is the solution of the homogeneous equation (11),

$$\int d^4x \rho_0(x) = 0. \quad (14)$$

If one assumes that the amount of the topological charge Q is bounded by a constant times the square root of the space-time volume, then in the infinite volume and continuum limit, $\rho_w(x)$ becomes infinitesimally small at any x ; thus it becomes *unobservable*. In other words, even if one calculates the axial anomaly $\mathcal{A}(x)$ [Eq. (15)] of a Ginsparg-Wilson lattice Dirac operator, and shows that it agrees with $\rho(x) \simeq \rho_0(x)$ in the infinite volume and continuum limit; one still cannot be sure whether the integral $\int d^4x \mathcal{A}(x)$ is equal to the topological charge Q or not.

In order to have a definite answer to this question, one needs to perform a *nonperturbative* calculation of the axial anomaly on a *finite* lattice. Otherwise, the “winding number density” $\rho_w(x)$ could hardly be detected at a site. However, to our knowledge, there is no previous *nonperturbative analytic* calculations of the axial anomaly for any Ginsparg-Wilson lattice Dirac operator on a *finite* lattice. Therefore, in this paper, we do not intend to tackle the difficult problem of performing a *nonperturbative analytic* calculation of the axial anomaly on a *finite* lattice.

Nevertheless, from the argument presented in Ref. [1], it is clear that Eq. (1) does not possess any topological zero modes. Thus, it follows that, on a finite lattice, the axial anomaly for at least some of the lattice sites is not in agreement with the topological charge density, since its sum over all sites is equal to the zero index, rather than the nonzero topological charge Q of the gauge background. Therefore, for any *topologically nontrivial* gauge background, we have no difficulties to understand why the axial anomaly of Eq. (1) cannot be exactly equal to the topological charge density for all sites on a *finite* lattice. However, in the infinite volume and continuum limit, $\rho_w(x) \rightarrow 0$, then the axial anomaly $\mathcal{A}(x)$ can appear to be in agreement with the topological charge density $\rho(x)$ for sufficiently smooth gauge backgrounds, even though the integral $\int d^4x \mathcal{A}(x)$ is always zero. Thus, for a sufficiently smooth gauge background (except for any possible “winding” at the infinity), we can calculate the axial anomaly (at any point except the infinity) of the lattice Dirac operator (1) in weak coupling perturbation theory, and

show that in the continuum limit, it agrees with the topological charge density of the gauge background, regardless of the global topological charge.

II. PERTURBATION CALCULATION OF THE AXIAL ANOMALY

The axial anomaly¹ of a flavor-singlet of Ginsparg-Wilson lattice fermions can be written as [3]

$$\begin{aligned} \mathcal{A}(x) &= \text{tr}[\gamma_5(1 - raD)(x, x)] \\ &= \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{i(p-q) \cdot x} \text{tr}[\gamma_5(1 - raD)(p, q)], \end{aligned} \quad (15)$$

where the trace runs over the Dirac, color, and flavor space. Now we expand D in power series of the gauge coupling (through the link variables)

$$D = D_0 + D_1 + D_2 + D_3 + D_4 + O(g^5), \quad (16)$$

where D_n denotes the terms containing the factor g^n , the gauge coupling to the n th power. Then we have

$$\begin{aligned} 1 - raD &= e_0 - e_0 d_1 e_0 - e_0(d_2 - d_1 e_0 d_1) e_0 \\ &\quad - e_0(d_3 - d_1 e_0 d_2 - d_2 e_0 d_1 + d_1 e_0 d_1 e_0 d_1) e_0 \\ &\quad - e_0(d_4 - d_1 e_0 d_3 - d_3 e_0 d_1 - d_2 e_0 d_2 + d_1 e_0 d_1 e_0 d_2 \\ &\quad + d_1 e_0 d_2 e_0 d_1 + d_2 e_0 d_1 e_0 d_1 - d_1 e_0 d_1 e_0 d_1 e_0 d_1) e_0 \\ &\quad + O(g^5), \end{aligned} \quad (17)$$

where

$$e_0 = 1 - raD_0 \equiv (1 + d_0)^{-1}, \quad (18)$$

$$d_1 = r \sum_{\mu} \gamma_{\mu} (f_1 t_0^{\mu} f_0 + f_0 t_1^{\mu} f_0 + f_0 t_0^{\mu} f_1), \quad (19)$$

$$\begin{aligned} d_2 &= r \sum_{\mu} \gamma_{\mu} (f_2 t_0^{\mu} f_0 + f_0 t_2^{\mu} f_0 + f_0 t_0^{\mu} f_2 + f_1 t_1^{\mu} f_0 + f_1 t_0^{\mu} f_1 \\ &\quad + f_0 t_1^{\mu} f_1), \end{aligned} \quad (20)$$

$$d_3 = r \sum_{\mu} \gamma_{\mu} \sum_{i=0}^3 \sum_{j=0}^{3-i} (f_i t_j^{\mu} f_{3-i-j}), \quad (21)$$

$$d_4 = r \sum_{\mu} \gamma_{\mu} \sum_{i=0}^4 \sum_{j=0}^{4-i} (f_i t_j^{\mu} f_{4-i-j}), \quad (22)$$

$$f = f_0 + f_1 + f_2 + f_3 + f_4 + O(g^5), \quad (23)$$

¹It is understood that the axial anomaly (15) at a lattice site has to be normalized by the volume of the unit cell, a^4 , when it is compared with the topological charge density in the continuum.

$$t^\mu = t_0^\mu + t_1^\mu + t_2^\mu + t_3^\mu + t_4^\mu + O(g^5). \quad (24)$$

Evidently some terms in Eq. (17) do not contribute to the axial anomaly, since they only have a factor of a product less than four distinct γ matrices; thus they vanish after taking the trace with γ_5 in Eq. (15). Note that each term of d_n only has one factor of γ matrix, and e_0 contributes at most one factor of γ matrix [see Eq. (33)]. Thus, the terms e_0 and $e_0 d_n e_0$ (for any n) can be dropped from the series, when Eq. (17) is substituted into Eq. (15).

Then the axial anomaly at the order g^2 is due to the fourth term in Eq. (17),

$$\begin{aligned} \mathcal{A}_2(x) = & \int_p \int_q \int_k e^{i(p-q) \cdot x} \\ & \times \text{tr}[\gamma_5 e_0(p) d_1(p, k) e_0(k) d_1(k, q) e_0(q)], \end{aligned} \quad (25)$$

where

$$\int_p \equiv \int \frac{d^4 p}{(2\pi)^4},$$

etc.

Similarly, substituting higher-order terms of Eq. (17) into Eq. (15) gives the axial anomaly at orders g^3 and g^4 . Although there seems to be many terms involving higher-order vertices (d_2, d_3 , etc.) at higher orders, the nonvanishing contributions at orders g^3 and g^4 only come from terms $e_0 d_1 e_0 d_1 e_0 d_1 e_0$ and $e_0 d_1 e_0 d_1 e_0 d_1 e_0 d_1 e_0$, respectively, as we shall show later. At this point, it is instructive to identify $e_0 = (\mathbb{1} + d_0)^{-1}$ [Eq. (18)] with the free fermion propagator, d_1 [Eq. (19)] with the “quark-quark-gluon” vertex, d_2 [Eq. (20)] with the “quark-quark-gluon-gluon” vertex, etc. Then the expression $\text{tr}(\gamma_5 e_0 d_1 e_0 d_1 e_0)$ contributing to the axial anomaly at $O(g^2)$ [Eq. (25)] corresponds to the triangle diagram coupling the axial current to two external gauge bosons through an internal fermion loop. Similarly, the expressions $\text{tr}(\gamma_5 e_0 d_1 e_0 d_1 e_0 d_1 e_0)$ and $\text{tr}(\gamma_5 e_0 d_1 e_0 d_1 e_0 d_1 e_0 d_1 e_0)$, contributing to the axial anomaly at $O(g^3)$ and $O(g^4)$, correspond to the quadrilateral (box) and the pentagon diagrams coupling the axial current to three and four external gauge

bosons, respectively. These are exactly the Feynman diagrams² that contribute to the axial anomaly in continuum non-Abelian gauge theories [5]. Note that even though higher-order vertices are allowed at finite lattice spacing, symmetry forbids their contributions to the axial anomaly, thus only the “quark-quark-gluon” vertex d_1 can enter the fermion loop contributing to the axial anomaly. In the following three subsections, we evaluate the axial anomaly at orders g^2 , g^3 , and g^4 respectively.

A. The axial anomaly at order g^2

To evaluate Eq. (25), first we obtain d_1 [Eq. (19)] by expanding f and t^μ in power series of the gauge coupling, Eqs. (23) and (24). Consider

$$h = \sqrt{t^2 + w^2} + w = h_0 + h_1 + h_2 + h_3 + h_4 + O(g^5), \quad (26)$$

which satisfies the identity $f \cdot f \cdot h = 2c$, i.e.,

$$(f_0 + f_1 + \dots)(f_0 + f_1 + \dots)(h_0 + h_1 + \dots) = 2c. \quad (27)$$

To order g , it gives

$$f_0 f_0 h_0 = 2c,$$

$$f_1 f_0 h_0 + f_0 f_1 h_0 + f_0 f_0 h_1 = 0.$$

Solving these two equations, we obtain

$$\begin{aligned} f_0(p, k) &= \left(\frac{2c}{\sqrt{t_0^2(p) + w_0^2(p)} + w_0(p)} \right)^{1/2} \delta^4(p - k) \\ &\equiv f_0(p) \delta^4(p - k), \end{aligned} \quad (28)$$

$$f_1(p, k) = -\frac{1}{2c} \frac{f_0^2(p) f_0^2(k)}{f_0(p) + f_0(k)} h_1(p, k), \quad (29)$$

where

$$t_0^2(p) = -\sum_\mu t_0^\mu(p) t_0^\mu(p) = \sum_\mu \sin^2(p_\mu a). \quad (30)$$

Similarly, we obtain

$$h_1(p, k) = \frac{w_1(p, k)[w_0(p) + w_0(k)] - \sum_\mu t_1^\mu(p, k)[t_0^\mu(p) + t_0^\mu(k)]}{\sqrt{t_0^2(p) + w_0^2(p)} + \sqrt{t_0^2(k) + w_0^2(k)}} + w_1(p, k). \quad (31)$$

Here we recall some well-known basic formulas in weak coupling perturbation theory (see the Appendix):

$$t^\mu(p, k) = t_0^\mu(p) \delta^4(p - k) + t_1^\mu(p, k) + O(g^2),$$

$$w(p, k) = w_0(p) \delta^4(p - k) + w_1(p, k) + O(g^2),$$

where

²See, for example, Fig. 22.2 in Ref. [4].

$$t_0^\mu(p) = i \sin(p_\mu a),$$

$$w_0(p) = c - \sum_\mu [1 - \cos(p_\mu a)],$$

$$t_1^\mu(p, k) = g \tilde{A}_\mu(p - k) \partial_\mu t_0^\mu \left(\frac{p + k}{2} \right),$$

$$w_1(p, k) = \sum_\mu g \tilde{A}_\mu(p - k) \partial_\mu w_0 \left(\frac{p + k}{2} \right),$$

$$\tilde{A}_\mu(p - k) = \sum_x e^{-i(p-k) \cdot [x + (a/2)\hat{\mu}]} A_\mu \left(x + \frac{a}{2}\hat{\mu} \right).$$

Therefore, the axial anomaly (25) at the order g^2 can be written as

$$\begin{aligned} \mathcal{A}_2(x) = & g^2 \int_p \int_q e^{i(p-q) \cdot x} \sum_{\mu, \nu} \int_k \text{tr} \{ \tilde{A}_\mu(p - k) \tilde{A}_\nu(k - q) \} \\ & \times \text{tr} [\gamma_5 e_0(p) d_{1,\mu}(p, k) e_0(k) d_{1,\nu}(k, q) e_0(q)], \end{aligned} \quad (32)$$

where

$$e_0(p) = 1 - ra D_0(p) = \frac{1}{1 + d_0(p)} = \frac{1 - rf_0^2(p) \gamma \cdot t_0(p)}{1 + r^2 f_0^4(p) t_0^2(p)} \equiv b(p) + \sum_\mu \gamma^\mu c^\mu(p), \quad (33)$$

$$d_0(p) = rf_0^2(p) \gamma \cdot t_0(p) \equiv \sum_\mu \gamma^\mu d_0^\mu(p), \quad (34)$$

$$d_1(p, k) = g \sum_\mu \tilde{A}_\mu(p - k) d_{1,\mu}(p, k), \quad (35)$$

$$d_{1,\mu}(p, k) = r \left[f_{1,\mu}(p, k) \gamma \cdot t_0(k) f_0(k) + f_0(p) \gamma \cdot t_0(p) f_{1,\mu}(p, k) + f_0(p) \gamma^\mu \partial^\mu t_0^\mu \left(\frac{p + k}{2} \right) f_0(k) \right], \quad (36)$$

$$f_{1,\mu}(p, k) = -\frac{1}{2c} \frac{f_0^2(p) f_0^2(k)}{f_0(p) + f_0(k)} \left[\partial_\mu w_0 \left(\frac{p + k}{2} \right) + \frac{\partial_\mu w_0((p + k)/2) [w_0(p) + w_0(k)] - \partial_\mu t_0^\mu((p + k)/2) [t_0^\mu(p) + t_0^\mu(k)]}{\sqrt{t_0^2(p) + w_0^2(p)} + \sqrt{t_0^2(k) + w_0^2(k)}} \right]. \quad (37)$$

In the limit $k = p$, Eqs. (37) and (36) reduce to

$$f_{1,\mu}(p, p) = \partial_\mu f_0(p), \quad (38)$$

$$d_{1,\mu}(p, p) = \partial_\mu [rf_0(p) \gamma \cdot t_0(p) f_0(p)] = \partial_\mu d_0(p). \quad (39)$$

To evaluate the integral in Eq. (32), we change the variables $p \rightarrow p + k$ and $q \rightarrow q + k$; then Eq. (32) becomes

$$\mathcal{A}_2(x) = g^2 \int_p \int_q e^{i(p-q) \cdot x} \sum_{\mu, \nu} \text{tr} \{ \tilde{A}_\mu(p) \tilde{A}_\nu(-q) \} G_{\mu\nu}(p, q), \quad (40)$$

where

$$\begin{aligned} G_{\mu\nu}(p, q) = & \int_k \text{tr} [\gamma_5 e_0(p + k) d_{1,\mu}(p + k, k) e_0(k) \\ & \times d_{1,\nu}(k, q + k) e_0(q + k)]. \end{aligned}$$

In the following, we show that

$$\begin{aligned} G_{\mu\nu}(p, q) = & \sum_{\lambda, \sigma} p_\lambda q_\sigma \frac{\partial}{\partial p_\lambda} \frac{\partial}{\partial q_\sigma} G_{\mu\nu}(p, q) \Big|_{p, q=0} + O(a) \\ = & -M(c) \sum_{\lambda, \sigma} 4p_\lambda q_\sigma \epsilon_{\mu\nu\lambda\sigma} + O(a) \end{aligned} \quad (41)$$

which, when substituted into Eq. (40), leads to the axial anomaly at the order g^2 ,

$$\begin{aligned} \mathcal{A}_2(x) = & g^2 M(c) \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \text{tr} \{ (\partial_\mu A_\nu - \partial_\nu A_\mu) \\ & \times (\partial_\lambda A_\sigma - \partial_\sigma A_\lambda)(x) \}, \end{aligned} \quad (42)$$

where $M(c)$ is a coefficient that tends to $1/32\pi^2$ (for $0 < c < 2$) in the continuum limit.

We expand $G_{\mu\nu}(p, q)$ in power series of p and q ,

$$\begin{aligned}
G_{\mu\nu}(p,q) &= G_{\mu\nu}(0,0) + \sum_{\lambda} p_{\lambda} \frac{\partial}{\partial p_{\lambda}} G_{\mu\nu}(p,q) \Big|_{p,q=0} \\
&+ \sum_{\sigma} q_{\sigma} \frac{\partial}{\partial q_{\sigma}} G_{\mu\nu}(p,q) \Big|_{p,q=0} \\
&+ \sum_{\lambda,\sigma} p_{\lambda} q_{\sigma} \frac{\partial}{\partial p_{\lambda}} \frac{\partial}{\partial q_{\sigma}} G_{\mu\nu}(p,q) \Big|_{p,q=0} + O(a).
\end{aligned} \tag{43}$$

It is easy to see that the zeroth order and the first-order terms in Eq. (43) vanish, by symmetry and the basic properties of γ matrices,

$$\begin{aligned}
\text{tr}(\gamma_5) &= \text{tr}(\gamma_5 \gamma^{\alpha}) \\
&= \text{tr}(\gamma_5 \gamma^{\alpha} \gamma^{\beta}) \\
&= \text{tr}(\gamma_5 \gamma^{\alpha} \gamma^{\beta} \gamma^{\sigma}) \\
&= \text{tr}(\gamma_5 \gamma^{\alpha} \gamma^{\beta} \gamma^{\lambda} \gamma^{\sigma} \gamma^{\delta}) = 0,
\end{aligned} \tag{44}$$

$$\text{tr}(\gamma_5 \gamma^{\alpha} \gamma^{\beta} \gamma^{\lambda} \gamma^{\sigma}) = \text{tr}(\mathbb{1}) \epsilon_{\alpha\beta\lambda\sigma}. \tag{45}$$

Explicitly,

$$\begin{aligned}
G_{\mu\nu}(0,0) &= \int_k \text{tr} \{ \gamma_5 e_0(k) \partial_{\mu} d_0(k) e_0(k) \partial_{\nu} d_0(k) e_0(k) \} \\
&= \text{tr}(\mathbb{1}) \int_k \epsilon_{\alpha\beta\delta\sigma} c^{\alpha}(k) c^{\delta}(k) b(k) \partial_{\mu} d_0^{\beta}(k) \partial_{\nu} d_0^{\sigma}(k),
\end{aligned} \tag{46}$$

where Eqs. (33), (34), (39), (44), and (45) have been used. Note that repeated indices are summed over in Eqs. (46), (50), (51), (54), and (55). Evidently the integrand in Eq. (46) vanishes due to contraction of the completely antisymmetric tensor $\epsilon_{\alpha\beta\delta\sigma}$ with the symmetric tensor $c^{\alpha}(k) c^{\delta}(k)$, i.e.,

$$\sum_{\alpha\delta} \epsilon_{\alpha\beta\delta\sigma} c^{\alpha}(k) c^{\delta}(k) = 0. \tag{47}$$

Hence, we have

$$G_{\mu\nu}(0,0) = 0.$$

Next consider the first-order term of the power series (43),

$$\begin{aligned}
&\frac{\partial}{\partial p_{\lambda}} G_{\mu\nu}(p,q) \Big|_{p,q=0} \\
&= \int_k \{ \text{tr} [\gamma_5 e_0(k) \partial_p^{\lambda} d_{1,\mu}(p,k) |_{p=k} e_0(k) d_{1,\nu}(k,k) e_0(k)] \\
&\quad + \text{tr} [\gamma_5 \partial_{\lambda} e_0(k) d_{1,\mu}(k,k) e_0(k) d_{1,\nu}(k,k) e_0(k)] \}
\end{aligned} \tag{48}$$

From Eq. (36), it is easy to see that

$$\partial_p^{\lambda} d_{1,\mu}(p,k) |_{p=k} = \sum_{\sigma} \gamma^{\sigma} h_{\lambda\mu}^{\sigma}, \tag{49}$$

where each term has one factor of the γ matrix, and the explicit expression of $h_{\lambda\mu}^{\sigma}$ is not required for our purposes. Then the first term and the second term of the integrand (48) both vanish,

$$\begin{aligned}
&\text{tr} [\gamma_5 e_0(k) \gamma \cdot h_{\lambda\mu}(k) e_0(k) \partial_{\nu} d_0(k) e_0(k)] \\
&= \text{tr}(\mathbb{1}) \epsilon_{\alpha\beta\delta\sigma} c^{\alpha}(k) c^{\delta}(k) b(k) h_{\lambda\mu}^{\beta}(k) \partial_{\nu} d_0^{\sigma}(k) = 0
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
&\text{tr} [\gamma_5 \partial_{\lambda} e_0(k) \partial_{\mu} d_0(k) e_0(k) \partial_{\nu} d_0(k) e_0(k)] \\
&= \text{tr}(\mathbb{1}) (\epsilon_{\delta\alpha\beta\sigma} + \epsilon_{\delta\alpha\sigma\beta}) c^{\beta}(k) b(k) \partial_{\lambda} c^{\delta}(k) \\
&\quad \times \partial_{\mu} d_0^{\alpha}(k) \partial_{\nu} d_0^{\sigma}(k) \\
&\quad + \text{tr}(\mathbb{1}) \epsilon_{\alpha\beta\sigma\delta} c^{\beta}(k) c^{\delta}(k) \partial_{\lambda} b(k) \partial_{\mu} d_0^{\alpha}(k) \partial_{\nu} d_0^{\sigma}(k) \\
&= 0,
\end{aligned} \tag{51}$$

where Eqs. (33), (34), (39), (49), (44), (45), and (47) have been used. Thus

$$\frac{\partial}{\partial p_{\lambda}} G_{\mu\nu}(p,q) \Big|_{p,q=0} = 0.$$

Similarly, we have

$$\frac{\partial}{\partial q_{\sigma}} G_{\mu\nu}(p,q) \Big|_{p,q=0} = 0. \tag{52}$$

Now consider the second order term of the power series (43),

$$\begin{aligned}
\frac{\partial}{\partial p_{\lambda}} \frac{\partial}{\partial q_{\sigma}} G_{\mu\nu}(p,q) \Big|_{p,q=0} &= \int_k \{ \text{tr} [\gamma_5 \partial_{\lambda} e_0(k) d_{1,\mu}(k,k) e_0(k) \partial_q^{\sigma} d_{1,\nu}(k,q) |_{q=k} e_0(k)] \\
&\quad + \text{tr} [\gamma_5 e_0(k) \partial_p^{\lambda} d_{1,\mu}(p,k) |_{p=k} e_0(k) d_{1,\nu}(k,k) \partial_{\sigma} e_0(k)] \\
&\quad + \text{tr} [\gamma_5 e_0(k) \partial_p^{\lambda} d_{1,\mu}(p,k) |_{p=k} e_0(k) \partial_q^{\sigma} d_{1,\nu}(k,q) |_{q=k} e_0(k)] \\
&\quad + \text{tr} [\gamma_5 \partial_{\lambda} e_0(k) d_{1,\mu}(k,k) e_0(k) d_{1,\nu}(k,k) \partial_{\sigma} e_0(k)] \}.
\end{aligned} \tag{53}$$

Evidently, the first three terms of the integrand are zero, by symmetry. Explicitly, the first term is equal to

$$\begin{aligned} & \text{tr}[\gamma_5 \partial_\lambda e_0(k) \partial_\mu d_0(k) e_0(k) \gamma \cdot h_{\sigma\nu}(k) e_0(k)] \\ &= \text{tr}(1)(\epsilon_{\delta\alpha\beta\gamma} + \epsilon_{\delta\alpha\gamma\beta}) c^\beta(k) b(k) \partial_\lambda c^\delta(k) \partial_\mu d_0^\alpha(k) \\ & \quad \times h_{\sigma\nu}^\gamma(k) \\ &+ \text{tr}(1) \epsilon_{\alpha\beta\gamma\delta} c^\beta(k) c^\delta(k) \partial_\lambda b(k) \partial_\mu d_0^\alpha(k) h_{\sigma\nu}^\gamma(k) \\ &= 0, \end{aligned} \quad (54)$$

the second term is also zero since it has the same form of the first, and the third term is

$$\begin{aligned} & \text{tr}[\gamma_5 e_0(k) \gamma \cdot h_{\lambda\mu}(k) e_0(k) \gamma \cdot h_{\sigma\nu}(k) e_0(k)] \\ &= \text{tr}(1) \epsilon_{\alpha\beta\delta\gamma} c^\alpha(k) c^\delta(k) b(k) h_{\lambda\mu}^\beta(k) h_{\sigma\nu}^\gamma(k) = 0, \end{aligned} \quad (55)$$

where Eqs. (33), (34), (39), (49), (44), (45), and (47) have been used. Thus, Eq. (53) becomes

$$\begin{aligned} & \frac{\partial}{\partial p_\lambda} \frac{\partial}{\partial q_\sigma} G_{\mu\nu}(p, q) \Big|_{p, q=0} \\ &= \int_k \text{tr}\{\partial_\lambda [\gamma_5 e_0(k)] \partial_\mu d_0(k) e_0(k) \partial_\nu d_0(k) \partial_\sigma e_0(k)\}. \end{aligned} \quad (56)$$

From Eqs. (33) and (34), it is easy to see that $e_0(k)$ also satisfies the Ginsparg-Wilson relation

$$\gamma_5 + e_0^{-1}(k) \gamma_5 e_0(k) = 2 \gamma_5 e_0(k), \quad (57)$$

which, after differentiation with respect to k_λ , gives

$$[\partial_\lambda e_0^{-1}(k)] \gamma_5 e_0(k) + e_0^{-1}(k) \gamma_5 [\partial_\lambda e_0(k)] = 2 \partial_\lambda [\gamma_5 e_0(k)]. \quad (58)$$

Substituting Eq. (58) and $d_0(k) = e_0^{-1}(k) - 1$ [from Eq. (18)] into the integrand of Eq. (56), then the integrand becomes

$$\begin{aligned} & \frac{1}{2} \text{tr}\{\partial_\lambda e_0^{-1}(k) \gamma_5 e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) \partial_\nu e_0^{-1}(k) \partial_\sigma e_0(k)\} \\ &+ \frac{1}{2} \text{tr}\{e_0^{-1}(k) \gamma_5 \partial_\lambda e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) \partial_\nu e_0^{-1}(k) \\ & \quad \times \partial_\sigma e_0(k)\}, \end{aligned}$$

which can be further reduced to

$$\begin{aligned} & -\frac{1}{2} \text{tr}\{\gamma_5 \partial_\mu e_0(k) \partial_\nu e_0^{-1}(k) \partial_\sigma e_0(k) \partial_\lambda e_0^{-1}(k)\} \\ &+ \frac{1}{2} \text{tr}\{\gamma_5 \partial_\lambda e_0(k) \partial_\mu e_0^{-1}(k) \partial_\nu e_0(k) \partial_\sigma e_0^{-1}(k)\}, \end{aligned} \quad (59)$$

where the identities

$$e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) = -\partial_\mu e_0(k), \quad (60)$$

$$e_0^{-1}(k) \partial_\mu e_0(k) e_0^{-1}(k) = -\partial_\mu e_0^{-1}(k), \quad (61)$$

have been used. By symmetry, it is obvious that the contribution of the second expression of Eq. (59) is equal to that of the first one; thus Eq. (56) becomes

$$\begin{aligned} I_{\mu\nu\lambda\sigma} &= \frac{\partial}{\partial p_\lambda} \frac{\partial}{\partial q_\sigma} G_{\mu\nu}(p, q) \Big|_{p, q=0} \\ &= \int \frac{d^4 k}{(2\pi)^4} \text{tr}\{\gamma_5 \partial_\mu e_0(k) \partial_\nu e_0^{-1}(k) \partial_\lambda e_0(k) \partial_\sigma e_0^{-1}(k)\} \\ &= \int \frac{d^4 k}{(2\pi)^4} \text{tr}\left\{\gamma_5 \partial_\mu \left(\frac{1}{1+d_0(k)}\right) \partial_\nu d_0(k) \right. \\ & \quad \times \partial_\lambda \left(\frac{1}{1+d_0(k)}\right) \partial_\sigma d_0(k)\Big\}, \end{aligned} \quad (62)$$

where the domain of integration is the four-torus, $\mathcal{T}_4 = \otimes_{i=1}^4 [-\pi/a, \pi/a]$, in which the end points $(\pm \pi/a)$ in each direction are identified to be the same point. In the limit $a \rightarrow 0$, $\mathcal{T}_4 \rightarrow \otimes_{i=1}^4 (-\infty, \infty)$, which is invariant under the transformation

$$k_\mu \rightarrow -\frac{1}{r^2 k_\mu}, \quad \mu = 1, \dots, 4, \quad (63)$$

for any $r \neq 0$.

Now we change the variables according to relation (63); then Eq. (62) becomes

$$\begin{aligned} I_{\mu\nu\lambda\sigma} &= \int \frac{d^4 k}{(2\pi)^4} \text{tr}\left\{\gamma_5 \partial_\mu \left(\frac{1}{1+d_0(K)}\right) \partial_\nu d_0(K) \right. \\ & \quad \times \partial_\lambda \left(\frac{1}{1+d_0(K)}\right) \partial_\sigma d_0(K)\Big\}, \end{aligned} \quad (64)$$

where the arguments of d_0 are $K_\mu = -(r^2 k_\mu)^{-1}$. Note that $dk_\mu \partial_\mu$ is invariant under the transformation (63).

Therefore, our ansatz of evaluating Eq. (64) in the continuum limit is to substitute $d_0(K)$ by $d_0^{-1}(k)$ in the integrand of Eq. (64), since $d_0(k) \rightarrow i r a k$ as $a \rightarrow 0$, and the trace with γ_5 picks out four distinct γ matrices, one from each of the four factors with partial derivatives. Then Eq. (64) reads as

$$\begin{aligned} I_{\mu\nu\lambda\sigma} &= \int \frac{d^4 k}{(2\pi)^4} \text{tr}\left\{\gamma_5 \partial_\mu \left(\frac{d_0(k)}{1+d_0(k)}\right) \partial_\nu d_0^{-1}(k) \right. \\ & \quad \times \partial_\lambda \left(\frac{d_0(k)}{1+d_0(k)}\right) \partial_\sigma d_0^{-1}(k)\Big\} \\ &= \int \frac{d^4 k}{(2\pi)^4} \text{tr}\{\gamma_5 \partial_\mu h(k) \partial_\nu h^{-1}(k) \partial_\lambda h(k) \partial_\sigma h^{-1}(k)\}, \end{aligned} \quad (65)$$

where

$$h(k) = \frac{d_0(k)}{1 + d_0(k)}.$$

The integral (65) can be evaluated by first removing an infinitesimal ball B_ϵ of radius ϵ from the origin ($k=0$) of the four-torus $\mathcal{T}_4$, then performing the integration, and finally taking $\epsilon \rightarrow 0$, i.e.,

$$I_{\mu\nu\lambda\sigma} = \frac{1}{16\pi^4} \lim_{\epsilon \rightarrow 0} \int_{\mathcal{T}_4 \setminus B_\epsilon} d^4k \times \text{tr}\{\gamma_5 \partial_\mu h(k) \partial_\nu h^{-1}(k) \partial_\lambda h(k) \partial_\sigma h^{-1}(k)\} \quad (66)$$

$$= \frac{1}{16\pi^4} \lim_{\epsilon \rightarrow 0} \int_{\mathcal{T}_4 \setminus B_\epsilon} d^4k \times \partial_\mu \text{tr}\{\gamma_5 h(k) \partial_\nu h^{-1}(k) \partial_\lambda h(k) \partial_\sigma h^{-1}(k)\}, \quad (67)$$

where the ∂_μ operation in Eq. (67) produces Eq. (66), plus three terms that are symmetric in $\mu\nu$, $\mu\lambda$, and $\mu\sigma$, respectively; hence none one of these three terms contributes to the integral.

Then according to the Gauss theorem, the volume integral over $\mathcal{T}_4 \setminus B_\epsilon$ can be expressed as a surface integral on the surface S_ϵ of the ball B_ϵ , provided that the integrand is continuous in $\mathcal{T}_4 \setminus B_\epsilon$. The last condition is satisfied since $h(k) = raD_0(k)$ is analytic, and $h(k) \neq 0$ (free of species doublings, for $0 < c < 2$) for any $k \in \mathcal{T}_4 \setminus B_\epsilon$ [1]. Thus Eq. (67) becomes

$$I_{\mu\nu\lambda\sigma} = \frac{1}{16\pi^4} \lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} d^3s n_\mu \times \text{tr}\{\gamma_5 h(k) \partial_\nu h^{-1}(k) \partial_\lambda h(k) \partial_\sigma h^{-1}(k)\}, \quad (68)$$

where n_μ is the μ th component of the outward normal vector on the surface S_ϵ . Since $d_0(k) \rightarrow ira\mathbf{k}$ as $k \rightarrow 0$, we can set $d_0(k) = ira\mathbf{k}$ on the surface S_ϵ and obtain

$$\begin{aligned} I_{\mu\nu\lambda\sigma} &= \frac{1}{16\pi^4} \lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} d^3s n_\mu \\ &\times \text{tr}\left\{\gamma_5 \left(\frac{\mathbf{k}}{1 + r^2 a^2 k^2}\right) \frac{\gamma_\nu}{k^2} \left(\frac{\gamma_\lambda}{1 + r^2 a^2 k^2}\right) \frac{\gamma_\sigma}{k^2}\right\}, \\ &= \frac{1}{16\pi^4} \text{tr}(1) \epsilon_{\mu\nu\lambda\sigma} \lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} d^3s \frac{n_\mu k_\mu}{k^4 (1 + r^2 a^2 k^2)^2}, \end{aligned} \quad (69)$$

where we have used Eqs. (44) and (45), and the property

$$\int_{S_\epsilon} d^3s n_\mu k_\nu F(k^2) = \delta_{\mu\nu} \int_{S_\epsilon} d^3s n_\mu k_\mu F(k^2).$$

Finally, the result of the integral (69) is

$$\begin{aligned} I_{\mu\nu\lambda\sigma} &= -\frac{N_f}{4\pi^4} \epsilon_{\mu\nu\lambda\sigma} \lim_{\epsilon \rightarrow 0} \frac{1}{(1 + r^2 a^2 \epsilon^2)^2} \\ &\times \int_0^{2\pi} d\phi \int_0^\pi d\theta_2 \sin \theta_2 \int_0^\pi d\theta_1 \sin^2 \theta_1 \cos^2 \theta_1 \\ &= -\frac{N_f}{8\pi^2} \epsilon_{\mu\nu\lambda\sigma}, \end{aligned} \quad (70)$$

where N_f denotes the number of fermion flavors. Note that Eq. (70) is *independent* of the parameter r in D [Eq. (1)]. From Eqs. (70) and (41), we obtain

$$M(c) = \frac{N_f}{32\pi^2}, \quad 0 < c < 2, \quad (71)$$

and the axial anomaly (42) at order g^2 ,

$$\begin{aligned} \mathcal{A}_2(x) &= \frac{g^2 N_f}{32\pi^2} \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \\ &\times \text{tr}\{(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\lambda A_\sigma - \partial_\sigma A_\lambda)(x)\}, \end{aligned} \quad (72)$$

where

$$A_\mu(x) = \int_p e^{ip \cdot x} \tilde{A}_\mu(p),$$

etc.

B. The axial anomaly at order g^3

Inserting the order g^3 terms of Eq. (17) into Eq. (15), we get³

$$\begin{aligned} \mathcal{A}_3(x) &= g^3 \int_p \int_q e^{i(p-q) \cdot x} \{J_a(p, q) + J_b(p, q)\} \\ &- g^3 \int_p \int_r \int_q e^{i(p-q) \cdot x} J_c(p, r, q), \end{aligned} \quad (73)$$

where

$$\begin{aligned} J_a(p, q) &= \int_k \text{tr}\{\gamma_5 e_0(p) d_1(p, k) e_0(k) d_2(k, q) e_0(q)\}, \\ J_b(p, q) &= \int_k \text{tr}\{\gamma_5 e_0(p) d_2(p, k) e_0(k) d_1(k, q) e_0(q)\}, \\ J_c(p, r, q) &= \int_k \text{tr}\{\gamma_5 e_0(p) d_1(p, r) e_0(r) d_1(r, k) e_0(k) \\ &\times d_1(k, q) e_0(q)\}, \end{aligned}$$

³The term $e_0 d_3 e_0$ has been dropped since $\text{tr}(\gamma_5 e_0 d_3 e_0) = 0$.

and e_0 and d_1 are given in Eqs. (33) and (35) respectively. It is easy to show that J_a and J_b vanish in the continuum limit, and only J_c has a nonzero contribution to the axial anomaly at the order g^3 . To proceed, we derive d_2 [Eq. (20)] by expanding f and t^μ in power series of the gauge coupling [Eqs. (23) and (24)] similar to the procedure of deriving d_1 in Sec. II A. First, we solve for f_2 from the higher order equations (27), then together with the formulas of t_n^μ [Eq. (A17)] and w_n [Eq. (A18)] derived in the Appendix, we obtain

$$\begin{aligned} d_2(p, k) &= g^2 \sum_{\mu} \tilde{A}_\mu^{(2)}(p-k) d_{2,\mu}(p, k) \\ &= g^2 \sum_{\mu, \nu} \tilde{A}_\mu^{(2)}(p-k) \gamma^\nu d_{2,\mu}^\nu(p, k), \end{aligned} \quad (74)$$

where $\tilde{A}_\mu^{(2)}(p-k)$ is defined in Eq. (A9), and the explicit expression of $d_{2,\mu}^\nu(p, k)$ is not required for our subsequent calculations. In general, to any order of g , we have

$$\begin{aligned} d_n(p, k) &= g^n \sum_{\mu} \tilde{A}_\mu^{(n)}(p-k) d_{n,\mu}(p, k) \\ &= g^n \sum_{\mu, \nu} \tilde{A}_\mu^{(n)}(p-k) \gamma^\nu d_{n,\mu}^\nu(p, k), \end{aligned} \quad (75)$$

which can be used for higher-order calculations.

Using Eqs. (35) and (74), and changing variables $p \rightarrow p+k$ and $q \rightarrow q+k$ in J_a and J_b , while $p \rightarrow p+r+k$, $r \rightarrow r+k$, and $q \rightarrow q+k$ in J_c , we can then rewrite Eq. (73) as

$$\begin{aligned} \mathcal{A}_3(x) &= g^3 \int_p \int_q e^{i(p-q) \cdot x} \{I_a(p, q) + I_b(p, q)\} \\ &\quad - g^3 \int_p \int_r \int_q e^{i(p+r-q) \cdot x} I_c(p, r, q), \end{aligned} \quad (76)$$

where

$$I_a(p, q) = \sum_{\mu, \nu} \text{tr}[\tilde{A}_\mu(p) \tilde{A}_\nu^{(2)}(-q)] G_a^{\mu\nu}(p, q), \quad (77)$$

$$I_b(p, q) = \sum_{\mu, \nu} \text{tr}[\tilde{A}_\mu^{(2)}(p) \tilde{A}_\nu(-q)] G_b^{\mu\nu}(p, q), \quad (78)$$

$$I_c(p, r, q) = \sum_{\mu, \nu, \lambda} \text{tr}[\tilde{A}_\mu(p) \tilde{A}_\nu(r) \tilde{A}_\lambda(-q)] G_c^{\mu\nu\lambda}(p, r, q), \quad (79)$$

and

$$\begin{aligned} G_a^{\mu\nu}(p, q) &= \int_k \text{tr}[\gamma_5 e_0(p+k) d_{1,\mu}(p+k, k) e_0(k) \\ &\quad \times d_{2,\nu}(k, q+k) e_0(q+k)], \end{aligned} \quad (80)$$

$$\begin{aligned} G_b^{\mu\nu}(p, q) &= \int_k \text{tr}[\gamma_5 e_0(p+k) d_{2,\mu}(p+k, k) e_0(k) \\ &\quad \times d_{1,\nu}(k, q+k) e_0(q+k)], \end{aligned} \quad (81)$$

$$\begin{aligned} G_c^{\mu\nu\lambda}(p, r, q) &= \int_k \text{tr}[\gamma_5 e_0(p+r+k) \\ &\quad \times d_{1,\mu}(p+r+k, r+k) e_0(r+k) \\ &\quad \times d_{1,\nu}(r+k, k) e_0(k) d_{1,\lambda}(k, q+k) \\ &\quad \times e_0(q+k)]. \end{aligned} \quad (82)$$

In the following, we show that G_a and G_b vanish in the continuum limit. First, we expand Eq. (80) in powers series of p and q :

$$\begin{aligned} G_a^{\mu\nu}(p, q) &= G_a^{\mu\nu}(0, 0) + \sum_{\sigma} p^{\sigma} \frac{\partial}{\partial p^{\sigma}} G_a^{\mu\nu}(p, 0) \Big|_{p=0} \\ &\quad + \sum_{\sigma} q^{\sigma} \frac{\partial}{\partial q^{\sigma}} G_a^{\mu\nu}(0, q) \Big|_{q=0} + O(a); \end{aligned} \quad (83)$$

where the zeroth-order and first-order terms can be easily shown to be zero, by symmetry. Explicitly,

$$G_a^{\mu\nu}(0, 0) = \int_k \text{tr}[\gamma_5 e_0(k) d_{1,\mu}(k, k) e_0(k) d_{2,\nu}(k, k) e_0(k)] = 0,$$

since the integrand vanishes identically,

$$\begin{aligned} &\text{tr}[\gamma_5 e_0(k) \partial_{\mu} e_0^{-1}(k) e_0(k) d_{2,\nu}(k, k) e_0(k)] \\ &= -\text{tr}[\gamma_5 \partial_{\mu} e_0(k) d_{2,\nu}(k, k) e_0(k)] = 0, \end{aligned}$$

where Eqs. (33), (39), (60), and (44) have been used.

To evaluate the first-order terms in Eq. (83), we observe that⁴

$$\frac{\partial}{\partial p^{\sigma}} d_{1,\mu}(p+k, k) \Big|_{p=0} \equiv \sum_{\lambda} \gamma^{\lambda} h_{\sigma\mu}^{\lambda}(k) = \gamma \cdot h_{\sigma\mu}(k), \quad (84)$$

$$\frac{\partial}{\partial q^{\sigma}} d_{2,\nu}(k, q+k) \Big|_{q=0} \equiv \sum_{\lambda} \gamma^{\lambda} g_{\sigma\nu}^{\lambda}(k) = \gamma \cdot g_{\sigma\nu}(k), \quad (85)$$

which follow from Eqs. (19), (20), (36), and (74), where the explicit expressions of $h_{\sigma\mu}^{\lambda}(k)$ and $g_{\sigma\nu}^{\lambda}(k)$ are not required for our subsequent calculations. Then

⁴Note that each term of d_1 [Eq. (19)] and d_2 [Eq. (20)] has only one factor of the γ matrix.

$$\begin{aligned} & \left. \frac{\partial}{\partial p^\sigma} G_a^{\mu\nu}(p,0) \right|_{p=0} \\ &= \int_k \text{tr} \{ \gamma_5 [\partial_\sigma e_0(k) \partial_\mu d_0(k) + e_0(k) \gamma \cdot h_{\sigma\mu}(k)] \\ & \quad \times e_0(k) d_{2,\nu}(k, k) e_0(k) \} = 0, \end{aligned}$$

since its integrand is equal to

$$\begin{aligned} & \text{tr}(\mathbb{1}) \epsilon_{\alpha\beta\lambda\delta} \{ [\partial_\sigma b(k) \partial_\mu d_0^\alpha(k) + b(k) h_{\sigma\mu}^\alpha(k)] \\ & \quad \times c^\beta(k) c^\delta(k) d_{2,\nu}^\lambda(k) \} + \text{tr}(\mathbb{1}) (\epsilon_{\alpha\beta\lambda\delta} + \epsilon_{\alpha\beta\delta\lambda}) \\ & \quad \times \{ [\partial_\sigma c^\alpha(k) \partial_\mu d_0^\beta(k) + c^\alpha(k) h_{\sigma\mu}^\beta(k)] c^\lambda(k) d_{2,\nu}^\delta(k) \} = 0, \end{aligned}$$

where Eqs. (33), (34), (84), (44), (45), and (47) have been used. Similarly, we have

$$\begin{aligned} & \left. \frac{\partial}{\partial q^\sigma} G_a^{\mu\nu}(0,q) \right|_{q=0} = \int_k \text{tr} \{ \gamma_5 e_0(k) \partial_\mu d_0(k) e_0(k) \\ & \quad \times [\gamma \cdot g_{\sigma\nu}(k) e_0(k) \\ & \quad + d_{2,\nu}(k, k) \partial_\sigma e_0(k)] \} = 0. \end{aligned}$$

Therefore $G_a^{\mu\nu}(p, q) = O(a) \rightarrow 0$ in the limit $a \rightarrow 0$. By the same token, $G_b^{\mu\nu}(p, q) = 0$ in the continuum limit.

Next we expand G_c in power series of p , q and r ,

$$\begin{aligned} G_c^{\mu\nu\lambda}(p, r, q) &= G_c^{\mu\nu\lambda}(0,0,0) + \sum_\sigma p^\sigma \frac{\partial}{\partial p^\sigma} G_c^{\mu\nu\lambda}(p,0,0) \Big|_{p=0} \\ & \quad + \sum_\sigma r^\sigma \frac{\partial}{\partial r^\sigma} G_c^{\mu\nu\lambda}(0,r,0) \Big|_{r=0} \\ & \quad + \sum_\sigma q^\sigma \frac{\partial}{\partial q^\sigma} G_c^{\mu\nu\lambda}(0,0,q) \Big|_{q=0} + O(a). \end{aligned} \quad (86)$$

It is easy to see that the zeroth-order term vanishes,

$$\begin{aligned} G_c^{\mu\nu\lambda}(0,0,0) &= \int_k \text{tr} [\gamma_5 e_0(k) d_{1,\mu}(k, k) e_0(k) d_{1,\nu}(k, k) e_0(k) \\ & \quad \times d_{1,\lambda}(k, k) e_0(k)] = 0, \end{aligned} \quad (87)$$

since its integrand vanishes,

$$\begin{aligned} & \text{tr} [\gamma_5 e_0(k) d_{1,\mu}(k, k) e_0(k) d_{1,\nu}(k, k) e_0(k) d_{1,\lambda}(k, k) e_0(k)] \\ &= \text{tr} [\gamma_5 e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) \partial_\nu e_0^{-1}(k) e_0(k) \\ & \quad \times \partial_\lambda e_0^{-1}(k) e_0(k)] \\ &= \text{tr} [\gamma_5 \partial_\mu e_0(k) \partial_\nu e_0^{-1}(k) \partial_\lambda e_0(k)] = 0, \end{aligned}$$

where Eqs. (39), (60), and (44) have been used.

Next we evaluate the first-order terms

$$\begin{aligned} & \left. \frac{\partial}{\partial p^\sigma} G_c^{\mu\nu\lambda}(p,0,0) \right|_{p=0} \\ &= - \int_k \text{tr} \{ \gamma_5 [\partial_\sigma e_0(k) \partial_\mu e_0^{-1}(k) + e_0(k) \gamma \cdot h_{\sigma\mu}(k)] \\ & \quad \times e_0(k) \partial_\nu e_0^{-1}(k) \partial_\lambda e_0(k) \} \\ &= - \int_k \text{tr} \{ \gamma_5 \partial_\sigma e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) \partial_\nu e_0^{-1}(k) \partial_\lambda e_0(k) \}, \end{aligned} \quad (88)$$

where Eqs. (39), (84) and (60) have been used, and the last equality is due to the vanishing of the second term [containing the factor $\gamma \cdot h_{\sigma\mu}(k)$] in the integrand,

$$\begin{aligned} & \text{tr} \{ \gamma_5 e_0(k) \gamma \cdot h_{\sigma\mu}(k) e_0(k) \partial_\nu e_0^{-1}(k) \partial_\lambda e_0(k) \} \\ &= \text{tr}(\mathbb{1}) \epsilon_{\alpha\beta\gamma\delta} \partial_\lambda b(k) c^\alpha(k) c^\gamma h_{\sigma\mu}^\beta(k) \partial_\nu d_0^\delta(k) \\ & \quad + \text{tr}(\mathbb{1}) (\epsilon_{\alpha\beta\gamma\delta} + \epsilon_{\beta\alpha\gamma\delta}) b(k) c^\alpha(k) \partial_\lambda c^\delta(k) h_{\sigma\mu}^\beta(k) \\ & \quad \times \partial_\nu d_0^\gamma(k) = 0. \end{aligned}$$

Comparing Eqs. (88) with (56), one immediately obtains⁵

$$\left. \frac{\partial}{\partial p^\sigma} G_c^{\mu\nu\lambda}(p,0,0) \right|_{p=0} = I_{\mu\nu\lambda\sigma} = - \frac{N_f}{8\pi^2} \epsilon_{\mu\nu\lambda\sigma}, \quad (89)$$

which has been evaluated in Eq. (70).

Similarly, we obtain

$$\left. \frac{\partial}{\partial q^\sigma} G_c^{\mu\nu\lambda}(0,0,q) \right|_{q=0} = - I_{\mu\nu\lambda\sigma} = \frac{N_f}{8\pi^2} \epsilon_{\mu\nu\lambda\sigma}. \quad (90)$$

Next we evaluate

$$\begin{aligned} & \left. \frac{\partial}{\partial r^\sigma} G_c^{\mu\nu\lambda}(0,r,0) \right|_{r=0} \\ &= \int_k \text{tr} [\gamma_5 \partial_\sigma e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) \partial_\nu e_0^{-1}(k) e_0(k) \\ & \quad \times \partial_\lambda e_0^{-1}(k) e_0(k)] \\ & \quad + \text{tr} [\gamma_5 e_0(k) \partial_\mu e_0^{-1}(k) \partial_\sigma e_0(k) \partial_\nu e_0^{-1}(k) \\ & \quad \times e_0(k) \partial_\lambda e_0^{-1}(k) e_0(k)] \\ & \quad + \text{tr} [\gamma_5 e_0(k) \partial_\sigma \partial_\mu d_0(k) e_0(k) \partial_\nu e_0^{-1}(k) e_0(k) \\ & \quad \times \partial_\lambda e_0^{-1}(k) e_0(k)] + \text{tr} \{ \gamma_5 e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) \\ & \quad \times [\gamma \cdot h_{\sigma\nu}(k)] e_0(k) \partial_\lambda e_0^{-1}(k) e_0(k) \} \}, \end{aligned} \quad (91)$$

$$= 0, \quad (92)$$

⁵Note that $\partial_\mu e_0^{-1}(k) = \partial_\mu d_0(k)$, from Eq. (33).

where in the integrand of Eq. (91), the first two terms cancel each other, and the last two terms vanish respectively, after the (by now familiar) manipulations using Eq. (33), (39), (60), (61), (44), (45), and (47).

Substituting Eqs. (87), (89), (90), and (92) into Eq. (86), we obtain

$$G_c^{\mu\nu\lambda}(p, r, q) = -\frac{N_f}{8\pi^2} \sum_{\sigma} (p_{\sigma} - q_{\sigma}) \epsilon_{\mu\nu\lambda\sigma} + O(a). \quad (93)$$

Therefore, in the continuum limit, the axial anomaly (76) at $O(g^3)$ is

$$\begin{aligned} \mathcal{A}_3(x) &= \frac{g^3 N_f}{8\pi^2} \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \int_p \int_r \int_q e^{i(p+r-q)\cdot x} (p_{\sigma} - q_{\sigma}) \\ &\quad \times \text{tr}[\tilde{A}_{\mu}(p) \tilde{A}_{\nu}(r) \tilde{A}_{\lambda}(-q)] \\ &= -i \frac{g^3 N_f}{32\pi^2} \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \text{tr}\{[\partial_{\sigma} A_{\mu}(x) - \partial_{\mu} A_{\sigma}(x)] \\ &\quad \times [A_{\nu}(x), A_{\lambda}(x)] + [A_{\mu}(x), A_{\nu}(x)] \\ &\quad \times [\partial_{\sigma} A_{\lambda}(x) - \partial_{\lambda} A_{\sigma}(x)]\}, \end{aligned} \quad (94)$$

where

$$A_{\mu}(x) = \int_p e^{ip\cdot x} \tilde{A}_{\mu}(p),$$

etc.

C. The axial anomaly up to order g^4

Inserting the order g^4 terms of Eq. (17) into Eq. (15), we have⁶

$$\begin{aligned} \mathcal{A}_4(x) &= g^4 \int_p \int_q e^{i(p-q)\cdot x} \{J_1(p, q) + J_2(p, q) + J_3(p, q)\} \\ &\quad - g^4 \int_p \int_r \int_q e^{i(p-q)\cdot x} \{J_4(p, r, q) + J_5(p, r, q) \\ &\quad + J_6(p, r, q)\} + g^4 \int_p \int_r \int_s \int_q e^{i(p-q)\cdot x} J_7(p, r, s, q), \end{aligned} \quad (95)$$

where

$$J_1(p, q) = \int_k \text{tr}[\gamma_5 e_0(p) d_1(p, k) e_0(k) d_3(k, q) e_0(q)],$$

$$J_2(p, q) = \int_k \text{tr}[\gamma_5 e_0(p) d_2(p, k) e_0(k) d_2(k, q) e_0(q)],$$

$$J_3(p, q) = \int_k \text{tr}[\gamma_5 e_0(p) d_3(p, k) e_0(k) d_1(k, q) e_0(q)],$$

$$J_4(p, r, q) = \int_k \text{tr}[\gamma_5 e_0(p) d_1(p, r) e_0(r) d_1(r, k) e_0(k) \\ \times d_2(k, q) e_0(q)],$$

$$J_5(p, r, q) = \int_k \text{tr}[\gamma_5 e_0(p) d_1(p, r) e_0(r) d_2(r, k) e_0(k) \\ \times d_1(k, q) e_0(q)],$$

$$J_6(p, r, q) = \int_k \text{tr}[\gamma_5 e_0(p) d_2(p, r) e_0(r) d_1(r, k) e_0(k) \\ \times d_1(k, q) e_0(q)],$$

$$J_7(p, r, s, q) = \int_k \text{tr}[\gamma_5 e_0(p) d_1(p, r) e_0(r) d_1(r, s) e_0(s) \\ \times d_1(s, k) e_0(k) d_1(k, q) e_0(q)].$$

Now we change the variables as follows:

- (i) $p \rightarrow p+k$ and $q \rightarrow q+k$ in J_1, J_2 , and J_3 ;
- (ii) $p \rightarrow p+r+k$, $r \rightarrow r+k$, and $q \rightarrow q+k$ in J_4, J_5 , and J_6 ;
- (iii) $p \rightarrow p+r+s+k$, $r \rightarrow r+s+k$, $s \rightarrow s+k$, and $q \rightarrow q+k$ in J_7 . Then Eq. (95) becomes

$$\begin{aligned} \mathcal{A}_4(x) &= g^4 \int_p \int_q e^{i(p-q)\cdot x} \{I_1(p, q) + I_2(p, q) + I_3(p, q)\} \\ &\quad - g^4 \int_p \int_r \int_q e^{i(p+r-q)\cdot x} \{I_4(p, r, q) + I_5(p, r, q) \\ &\quad + I_6(p, r, q)\} \\ &\quad + g^4 \int_p \int_r \int_s \int_q e^{i(p+r+s-q)\cdot x} I_7(p, r, s, q), \end{aligned} \quad (96)$$

where

$$I_1(p, q) = \sum_{\mu, \nu} \text{tr}[\tilde{A}_{\mu}(p) \tilde{A}_{\nu}^{(3)}(-q)] G_1^{\mu\nu}(p, q)$$

$$I_2(p, q) = \sum_{\mu, \nu} \text{tr}[\tilde{A}_{\mu}^{(2)}(p) \tilde{A}_{\nu}^{(2)}(-q)] G_2^{\mu\nu}(p, q)$$

$$I_3(p, q) = \sum_{\mu, \nu} \text{tr}[\tilde{A}_{\mu}^{(3)}(p) \tilde{A}_{\nu}(-q)] G_3^{\mu\nu}(p, q)$$

$$I_4(p, r, q) = \sum_{\mu, \nu, \lambda} \text{tr}[\tilde{A}_{\mu}(p) \tilde{A}_{\nu}(r) \tilde{A}_{\lambda}^{(2)}(-q)] G_4^{\mu\nu\lambda}(p, r, q)$$

$$I_5(p, r, q) = \sum_{\mu, \nu, \lambda} \text{tr}[\tilde{A}_{\mu}(p) \tilde{A}_{\nu}^{(2)}(r) \tilde{A}_{\lambda}(-q)] G_5^{\mu\nu\lambda}(p, r, q)$$

⁶The term $e_0 d_4 e_0$ has been dropped since $\text{tr}(\gamma_5 e_0 d_4 e_0) = 0$.

$$I_6(p, r, q) = \sum_{\mu, \nu, \lambda} \text{tr}[\tilde{A}_\mu^{(2)}(p) \tilde{A}_\nu(r) \tilde{A}_\lambda(-q)] G_6^{\mu\nu\lambda}(p, r, q)$$

$$I_7(p, r, s, q) = \sum_{\mu, \nu, \lambda, \sigma} \text{tr}[\tilde{A}_\mu(p) \tilde{A}_\nu(r) \tilde{A}_\lambda(s) \tilde{A}_\sigma(-q)] \\ \times G_7^{\mu\nu\lambda\sigma}(p, r, s, q)$$

and

$$G_1^{\mu\nu}(p, q) = \int_k \text{tr}[\gamma_5 e_0(p+k) d_{1,\mu}(p+k, k) e_0(k) \\ \times d_{3,\nu}(k, q+k) e_0(q+k)],$$

$$G_2^{\mu\nu}(p, q) = \int_k \text{tr}[\gamma_5 e_0(p+k) d_{2,\mu}(p+k, k) e_0(k) \\ \times d_{2,\nu}(k, q+k) e_0(q+k)],$$

$$G_3^{\mu\nu}(p, q) = \int_k \text{tr}[\gamma_5 e_0(p+k) d_{3,\mu}(p+k, k) e_0(k) \\ \times d_{1,\nu}(k, q+k) e_0(q+k)],$$

$$G_4^{\mu\nu\lambda}(p, r, q) = \int_k \text{tr}[\gamma_5 e_0(p+r+k) d_{1,\mu}(p+r+k, r+k) \\ \times e_0(r+k) d_{1,\nu}(r+k, k) e_0(k) \\ \times d_{2,\lambda}(k, q+k) e_0(q+k)],$$

$$G_5^{\mu\nu\lambda}(p, r, q) = \int_k \text{tr}[\gamma_5 e_0(p+r+k) d_{1,\mu}(p+r+k, r+k) \\ \times e_0(r+k) d_{2,\nu}(r+k, k) e_0(k) \\ \times d_{1,\lambda}(k, q+k) e_0(q+k)],$$

$$G_6^{\mu\nu\lambda}(p, r, q) = \int_k \text{tr}[\gamma_5 e_0(p+r+k) d_{2,\mu}(p+r+k, r+k) \\ \times e_0(r+k) d_{1,\nu}(r+k, k) e_0(k) \\ \times d_{1,\lambda}(k, q+k) e_0(q+k)],$$

$$G_7^{\mu\nu\lambda\sigma}(p, r, s, q) = \int_k \text{tr}[\gamma_5 e_0(p+r+s+k) \\ \times d_{1,\mu}(p+r+s+k, r+s+k) \\ \times e_0(r+s+k) d_{1,\nu}(r+s+k, s+k) \\ \times e_0(s+k) d_{1,\lambda}(s+k, k) e_0(k) \\ \times d_{1,\sigma}(k, q+k) e_0(q+k)],$$

where Eqs. (35), (74), and (75) (for $n=3$) have been used, and $\tilde{A}_\mu^{(n)}(p)$, etc., are defined in Eq. (A9).

In the following, we show that all G 's vanish in the continuum limit except G_7 , by expanding each one of them in power series of p , q , r and s . First consider

$$G_1^{\mu\nu}(p, q) = G_1^{\mu\nu}(0, 0) + O(a), \quad (97)$$

where

$$G_1^{\mu\nu}(0, 0) = \int_k \text{tr}[\gamma_5 e_0(k) d_{1,\mu}(k, k) e_0(k) d_{3,\nu}(k, k) e_0(k)] = 0.$$

Since its integrand vanishes,

$$\text{tr}[\gamma_5 e_0(k) d_{1,\mu}(k, k) e_0(k) d_{3,\nu}(k, k) e_0(k)] \\ = \text{tr}(1) \epsilon_{\alpha\beta\gamma\delta} b(k) c^\alpha(k) c^\gamma(k) \partial_\mu d_0^\beta(k) d_{3,\nu}^\delta(k) = 0,$$

where Eqs. (33), (39), (75), (44), (45), and (47) have been used. Similarly, we obtain

$$G_2^{\mu\nu}(0, 0) = G_3^{\mu\nu}(0, 0) = 0.$$

Next consider

$$G_4^{\mu\nu\lambda}(p, r, q) = G_4^{\mu\nu\lambda}(0, 0, 0) + O(a), \quad (98)$$

where

$$G_4^{\mu\nu\lambda}(0, 0, 0) = \int_k \text{tr}[\gamma_5 e_0(k) d_{1,\mu}(k, k) e_0(k) d_{1,\nu}(k, k) \\ \times e_0(k) d_{2,\lambda}(k, k) e_0(k)] = 0, \quad (99)$$

Since its integrand vanishes,

$$\text{tr}[\gamma_5 e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) \partial_\nu d_0(k) e_0(k) d_{2,\lambda}(k, k) e_0(k)] \\ = -\text{tr}[\gamma_5 \partial_\mu e_0(k) \partial_\nu d_0(k) e_0(k) d_{2,\lambda}(k, k) e_0(k)] \\ = -\text{tr}(1) \{ \epsilon_{\alpha\beta\gamma\delta} \partial_\mu b(k) c^\beta(k) c^\delta(k) \partial_\nu d_0^\alpha(k) d_{2,\lambda}^\gamma(k) \} \\ + \text{tr}(1) \{ (\epsilon_{\alpha\beta\gamma\delta} + \epsilon_{\alpha\beta\delta\gamma}) b(k) c^\gamma(k) \partial_\mu c^\alpha(k) \partial_\nu d_0^\beta(k) \\ \times d_{2,\lambda}^\delta(k) \} = 0, \quad (100)$$

where Eqs. (33), (39), (60), (74), (44), (45), and (47) have been used. By the same token, we have

$$G_5^{\mu\nu\lambda}(0, 0, 0) = G_6^{\mu\nu\lambda}(0, 0, 0) = 0. \quad (101)$$

Finally, we consider

$$G_7^{\mu\nu\lambda\sigma}(p, r, s, q) = G_7^{\mu\nu\lambda\sigma}(0, 0, 0, 0) + O(a), \quad (102)$$

where

$$\begin{aligned}
G_7^{\mu\nu\lambda\sigma}(0,0,0,0) &= \int_k \text{tr} [\gamma_5 e_0(k) d_{1,\mu}(k, k) e_0(k) d_{1,\nu}(k, k) \\
&\quad \times e_0(k) d_{1,\lambda}(k, k) e_0(k) d_{1,\sigma}(k, k) e_0(k)] \\
&= \int_k \text{tr} [\gamma_5 e_0(k) \partial_\mu e_0^{-1}(k) e_0(k) \partial_\nu e_0^{-1}(k) \\
&\quad \times e_0(k) \partial_\lambda e_0^{-1}(k) e_0(k) \partial_\sigma e_0^{-1}(k) e_0(k)] \\
&= \int_k \text{tr} [\gamma_5 \partial_\mu e_0(k) \partial_\nu e_0^{-1}(k) e_0(k) \partial_\lambda e_0^{-1}(k) \\
&\quad \times \partial_\sigma e_0(k)]. \tag{103}
\end{aligned}$$

Here we have used Eqs. (39) and (60) to simplify the integrand. Comparing Eqs. (103) with (56), one immediately obtains

$$G_7^{\mu\nu\lambda\sigma}(0,0,0,0) = I_{\mu\nu\lambda\sigma} = -\frac{N_f}{8\pi^2} \epsilon_{\mu\nu\lambda\sigma}, \tag{104}$$

which has been evaluated in Eq. (70).

Thus, in the continuum limit, only G_7 has a nonzero contribution to the axial anomaly. Substituting Eq. (102) into Eq. (96), we obtain the axial anomaly at the order g^4 ,

$$\begin{aligned}
\mathcal{A}_4(x) &= -\frac{g^4 N_f}{8\pi^2} \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \int_p \int_r \int_s \int_q e^{i(p+r+s-q)\cdot x} \\
&\quad \times \tilde{A}_\mu(p) \tilde{A}_\nu(r) \tilde{A}_\lambda(s) \tilde{A}_\sigma(-q) \\
&= -\frac{g^4 N_f}{8\pi^2} \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \text{tr} \{A_\mu(x) A_\nu(x) A_\lambda(x) A_\sigma(x)\} \\
&= -\frac{g^4 N_f}{32\pi^2} \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \text{tr} \{[A_\mu(x), A_\nu(x)] \\
&\quad \times [A_\lambda(x), A_\sigma(x)]\}. \tag{105}
\end{aligned}$$

Finally, adding Eqs. (72), (94), and (105) gives the axial anomaly (15) in the continuum limit,

$$\begin{aligned}
\mathcal{A}(x) &= \text{tr} [\gamma_5 (1 - arD)(x, x)] \\
&= \frac{g^2 N_f}{32\pi^2} \sum_{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} \text{tr} \{F_{\mu\nu}(x) F_{\lambda\sigma}(x)\}, \tag{106}
\end{aligned}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]. \tag{107}$$

Note that the axial anomaly (106) is *invariant* for any $r > 0$, since the anomaly coefficient (70) is independent of r . This completes our perturbation calculation of the axial anomaly of the Ginsparg-Wilson lattice Dirac operator proposed in Ref. [1].

TABLE I. The ratio of the integral I_{1234} [Eq. (62)] to $1/8\pi^2$, for lattices of sizes 16^4 , 32^4 , 64^4 , 128^4 , 256^4 , as well as for $r = 0.5, 1.0, 2.0, 4.0$, respectively. The lattice spacing a is set to one, and the parameter c [Eqs. (3) and (8)] is fixed at 1.0.

L	r			
	0.5	1.0	2.0	4.0
16	0.9204	0.9800	0.9848	0.7617
32	0.9810	0.9952	0.9988	0.9907
64	0.9953	0.9988	0.9997	0.9999
128	0.9988	0.9997	0.9999	1.0000
256	0.9997	0.9999	1.0000	1.0000

III. CONCLUDING REMARKS

Several remarks are as follows. It is obvious that Eq. (106) also holds for other T_μ [Eq. (3)] such that D is doubler-free, is exponentially local, and has correct continuum behavior. In particular, Eq. (106) holds for

$$f = \left(\frac{2c}{\sqrt{t^2 + w^2} + w} \right)^\alpha, \quad \alpha \geq \frac{1}{2}, \quad 0 < c < 2, \tag{108}$$

as proposed in Ref. [1].

It is instructive to examine how well the continuum axial anomaly can be recovered on a finite lattice, by evaluating the integral I_{1234} [Eq. (62)] as a numerical sum over the discrete momenta on a finite lattice. In Table I, the ratios of I_{1234} to $1/8\pi^2$ are listed for several lattice sizes (L^4), as well as for a range of r , respectively. Here the lattice spacing a is set to one, and the value of c [see Eqs. (3) and (8)] is fixed at 1.0. Evidently, the integral I_{1234} tends to the continuum value $1/8\pi^2$ as $L \rightarrow \infty$, independent of the parameter r . This also provides a verification of our ansatz to substitute $d_0(K)$ by $d_0^{-1}(k)$ in evaluating $I_{\mu\nu\lambda\sigma}$ [Eq. (64)] in the continuum limit.

Next we repeat the same calculations of I_{1234} for another f , namely, replacing Eq. (3) by

$$f = \frac{2c}{\sqrt{t^2 + w^2} + w}. \tag{109}$$

The results are listed in Table II, which show that the integral I_{1234} (as a function of lattice size L^4) approaches the con-

TABLE II. The ratio of the integral I_{1234} [Eq. (62)] to $1/8\pi^2$, for $f = 2c/(\sqrt{t^2 + w^2} + w)$. Other parameters are the same as those in Table I.

L	r			
	0.5	1.0	2.0	4.0
16	1.0000	1.0000	0.9911	0.7565
32	1.0000	1.0000	1.0000	0.9914
64	1.0000	1.0000	1.0000	1.0000
128	1.0000	1.0000	1.0000	1.0000
256	1.0000	1.0000	1.0000	1.0000

tinuum value $1/8\pi^2$ much faster than that in Table I. This indicates that Eq. (109) may be a better choice than Eq. (3), especially for small lattices.

In general, T_μ [Eq. (2)] may not be in the form $T_\mu = ft_\mu f$; then our present perturbation calculation may not go through without modifications. We refer to our former derivation [6] for the general case, though some of our intermediate steps need further clarifications. If one assumes that the coefficient of the axial anomaly, $g^2/32\pi^2$, is the same for $U(1)$ and $SU(n)$ background gauge fields, then it can also be determined [8] by imposing a gauge configuration with constant field tensors to Lüscher's formula [7] for the axial anomaly of Ginsparg-Wilson lattice Dirac fermions in a $U(1)$ background gauge field, provided that D is *topologically proper*. However, for a *topologically trivial* Ginsparg-Wilson lattice Dirac operator such as Eq. (1), it seems to be necessary to perform an explicit calculation in order to determine its axial anomaly. In passing, we also refer to other axial anomaly calculations [9–12] for the overlap Dirac operator [13,14], as well as that in the original Ginsparg-Wilson paper [2]. So far, a *nonperturbative analytic* calculation of the axial anomaly for any Ginsparg-Wilson lattice Dirac operator, on a *finite* lattice, is still lacking.

In summary, we have shown that the lattice Dirac operator (1) reproduces the continuum axial anomaly for smooth gauge configurations, even though it does *not* possess any topological zero modes in topologically nontrivial gauge backgrounds. If one insists that the topologically zero modes of a lattice Dirac operator are crucial for lattice QCD to reproduce low energy hadron phenomenology, then one should ensure that a Ginsparg-Wilson lattice Dirac operator is indeed *topologically proper*, before it could be employed for lattice QCD computations. However, so far, there does not seem to be compelling experimental evidence that these topological zero modes are physically relevant, unlike the axial anomaly in the trivial gauge sector, which accounts for the decay rate of the neutral pion. So there might be a very slight possibility that lattice QCD with *topologically trivial* quarks could reproduce low energy hadron phenomenology. These issues seem to deserve further study.

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APPENDIX: BASIC FORMULAS IN WEAK COUPLING PERTURBATION THEORY

In this appendix, we set up our notation by deriving some basic formulas in weak coupling perturbation theory. These formulas are used in our anomaly calculation in Sec. II.

Consider the forward and backward difference operators

$$\nabla_\mu^+(x, y) \equiv U_\mu(x) \delta_{x+a\hat{\mu}, y} - \delta_{x, y}, \quad (\text{A1})$$

$$\nabla_\mu^-(x, y) \equiv \delta_{x, y} - U_\mu^\dagger(x - a\hat{\mu}) \delta_{x-a\hat{\mu}, y}, \quad (\text{A2})$$

where the link variables [for the $SU(n)$ gauge group] are defined as

$$U_\mu(x) = \exp \left[iagA_\mu \left(x + \frac{a}{2}\hat{\mu} \right) \right], \quad (\text{A3})$$

$$U_\mu^\dagger(x - a\hat{\mu}) = \exp \left[-ia gA_\mu \left(x - \frac{a}{2}\hat{\mu} \right) \right]. \quad (\text{A4})$$

Expanding $\nabla^\pm$ in power series of the gauge coupling g ,

$$\nabla_\mu^\pm = (\nabla_\mu^\pm)_0 + (\nabla_\mu^\pm)_1 + (\nabla_\mu^\pm)_2 + \cdots, \quad (\text{A5})$$

and performing the Fourier transform

$$\sum_{x, y} e^{-i(p \cdot x - q \cdot y)} (\nabla_\mu^\pm)_n(x, y) \equiv (\nabla_\mu^\pm)_n(p, q), \quad (\text{A6})$$

we obtain

$$(\nabla_\mu^+)_0(p, q) = (e^{ip_\mu a} - 1) \delta^4(p - q) \equiv (\nabla_\mu^+)_0(p) \delta^4(p - q) \quad (\text{A7})$$

and

$$\begin{aligned} (\nabla_\mu^+)_n(p, q) &= \sum_{x, y} e^{-i(p \cdot x - q \cdot y)} \frac{1}{n!} (ia g)^n \\ &\quad \times \left[A_\mu \left(x + \frac{a}{2}\hat{\mu} \right) \right]^n \delta_{x+a\hat{\mu}, y} \\ &= \sum_x e^{-i(p-q) \cdot (x+a\hat{\mu}/2)} \frac{1}{n!} (ia g)^n \\ &\quad \times \left[A_\mu \left(x + \frac{a}{2}\hat{\mu} \right) \right]^n e^{i(p_\mu + q_\mu)a/2} \\ &= \frac{(ia g)^n}{n!} \tilde{A}_\mu^{(n)}(p - q) e^{i(p_\mu + q_\mu)a/2} \\ &= \frac{g^n}{n!} \tilde{A}_\mu^{(n)}(p - q) \partial_\mu^n (\nabla_\mu^+)_0 \left(\frac{p+q}{2} \right), \end{aligned} \quad (\text{A8})$$

where

$$\tilde{A}_\mu^{(n)}(p - q) = \sum_x e^{-i(p-q) \cdot (x+a\hat{\mu}/2)} \left[A_\mu \left(x + \frac{a}{2}\hat{\mu} \right) \right]^n, \quad (\text{A9})$$

$$(\nabla_\mu^+)_0 \left(\frac{p+q}{2} \right) = e^{i(p_\mu + q_\mu)a/2} - 1. \quad (\text{A10})$$

Similarly, we have

$$(\nabla_\mu^-)_0(p, q) = (1 - e^{-ip_\mu a}) \delta^4(p - q) \equiv (\nabla_\mu^-)_0(p) \delta^4(p - q), \quad (\text{A11})$$

$$(\nabla_{\mu}^{-})_n(p, q) = \frac{g^n}{n!} \tilde{A}_{\mu}^{(n)}(p - q) \partial_{\mu}^n (\nabla_{\mu}^{-})_0 \left(\frac{p + q}{2} \right). \quad (\text{A12})$$

Then we obtain the weak coupling perturbation series of t^{μ} [Eq. (4)] and w [Eq. (8)] in the momentum space:

$$\begin{aligned} t^{\mu}(p, q) &= \frac{1}{2} [\nabla_{\mu}^{+}(p, q) + \nabla_{\mu}^{-}(p, q)] \\ &= t_0^{\mu}(p) \delta^4(p - q) + t_1^{\mu}(p, q) + t_2^{\mu}(p, q) + \cdots, \end{aligned} \quad (\text{A13})$$

$$\begin{aligned} w(p, q) &= c - \frac{1}{2} \sum_{\mu} [\nabla_{\mu}^{+}(p, q) + \nabla_{\mu}^{-}(p, q)] \\ &= w_0(p) \delta^4(p - q) + w_1(p, q) + w_2(p, q) + \cdots, \end{aligned} \quad (\text{A14})$$

where

$$t_0^{\mu}(p) = i \sin(p_{\mu} a), \quad (\text{A15})$$

$$w_0(p) = c - \sum_{\mu} [1 - \cos(p_{\mu} a)], \quad (\text{A16})$$

$$t_n^{\mu}(p, q) = \frac{g^n}{n!} \tilde{A}_{\mu}^{(n)}(p - q) \partial_{\mu}^n t_0^{\mu} \left(\frac{p + q}{2} \right), \quad (\text{A17})$$

$$w_n(p, q) = \frac{g^n}{n!} \sum_{\mu} \tilde{A}_{\mu}^{(n)}(p - q) \partial_{\mu}^n w_0 \left(\frac{p + q}{2} \right). \quad (\text{A18})$$

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