Design of an equiripple FIR notch filter using a multiple exchange algorithm

Chien-Cheng Tseng^{a,*}, Soo-Chang Pei^b

^a*Department of Computer and Communication Engineering, National Kaohsiung First University of Science and Technology, Kaohsiung, Taiwan*

^b*Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan*

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Abstract

This paper presents a multiple exchange algorithm for designing linear phase equiripple FIR notch filters. The algorithm is first discussed for designing a single notch filter. Compared with conventional linear programming methods, the design speed can be improved significantly. Then, two modified algorithms are developed to reduce the ripple in the low-frequency passband because the energy of biomedical ECG signals is usually occupied in the low-frequency band. Next, we extend the proposed algorithm to design an equiripple multiple-notch filter. Finally, several design examples are presented to demonstrate its effectiveness. © 1999 Elsevier Science B.V. All rights reserved.

Zusammenfassung

Diese Arbeit stellt einen Mehrfachaustauschalgorithmus für den Tschebyscheff-Entwurf linearphasiger FIR-Notchfilter vor. Der Algorithmus wird zuerst für den Entwurf eines einzelnen Notchfilters diskutiert. Verglichen mit der konventionellen Methode der linearen Programmierung kann die Entwurfsgeschwindigkeit signifikant verbessert werden. Danach werden zwei modifizierte Algorithmen entwickelt, die die Schwankungen im nieder-frequenten Durchlaßbereich verringern, weil sich die Energie biomedizinischer ECG-Signale normalerweise im niedrigen Frequenzbereich befindet. Darauf folgend erweitern wir den vorgeschlagenen Algorithmus für den Tschebyscheff-Entwurf vielfacher Notchfilter. Am Ende werden Entwurfsbeispiele präsentiert, die die Effektivität belegen sollen. © 1999 Elsevier Science B.V. All rights reserved.

Résumé

Cet article présente un algorithme d'échanges multiples pour la conception de filtres en encoche RIF à ondulations égales. Nous présentons d'abord l'algorithme pour la conception de filtres en encoche simples. En comparaison avec une méthode de programmation linéaire conventionnelle, la vitesse de conception peut être améliorée significativement. Ensuite, deux algorithmes modifiés sont développés, pour réduire les ondulations dans la bande passante de basse fréquence, parce que l'énergie des signaux ECG biomédicaux est généralement comprise dans la bande de basse fréquence. Puis nous étendons l'algorithme proposé pour concevoir un filtre en encoche multiple à ondulations égales.

* Corresponding author. E-mail: tcc@ccms.nkfu.edu.tw

Finalement, plusieurs exemples de conception de filtres sont présentés pour démontrer son efficacité. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Notch filters; Multiple exchange algorithm

1. Introduction

Notch filters have been an effective means for eliminating narrowband or sinusoidal interferences in certain signal processing applications ranging from power line interference cancelation for electrocardiograms to multiple sinusoidal interference removal for corrupted images. For the one-dimensional (1-D) case, several methods for the design and performance analysis of IIR and FIR notch filters have been developed, see [1,4–6,8,11,14,17] among others. For the two-dimensional (2-D) case, [13,15] proposed the methods that reduce the design of a stable IIR 2-D notch filter to the designs of several simple 1-D IIR filters. Usually, the IIR notch filter is of non-linear phase, so it is meaningful to design a linear phase FIR notch filter for removing the phase distortion in notch filter output.

Conventionally, we often use the well-known McClellan–Parks–Rabiner (MPR) computer program and standard linear programming technique to design linear phase equiripple FIR filters according to the Chebyshev criterion which minimizes the maximum error in the frequency response [10,16]. The minimax designs usually give the designers a smallest length filter for a given specification. However, it is difficult to incorporate both time- and frequency-domain constraints into the MPR algorithm, so it cannot be applied to design an equiripple notch filter which is basically a minimax design with a linear constraint. Moreover, the linear programming technique can be used to design an equiripple notch filter, but it needs a large memory space and considerable computation time. Since linear programming is a single exchange method, it is significantly slower than the multiple exchange method [16].

The purpose of this paper is to develop a multiple exchange algorithm to design an equiripple FIR notch filter such that the design speed can be im-

proved. We call the presented methods as multiple exchange algorithms because a new set of extremal points is required to be searched for replacing the old set in each iteration of the design process. This operation is similar to what multiple exchange algorithms [3] do. The design algorithm is first discussed for single-notch filter design. Then, two modified algorithms are developed to reduce the ripple in the low-frequency passband because ECG is usually a low-frequency signal. Next, we extend the proposed algorithm to design an equiripple multiple notch filter. Finally, several design examples are presented to demonstrate its effectiveness.

2. Problem formulation

A causal N th-order FIR filter can be represented as

$$H(z) = \sum_{n=0}^N h(n)z^{-n}. \quad (1)$$

Depending on whether N is even or odd, and whether $h(n)$ is symmetric or antisymmetric, we obtain four types of real coefficient linear phase filters [12]. The amplitude response of these four types of filters can be expressed as

$$A(\omega) = \sum_{n=1}^M a_n \text{trig}(\omega, n). \quad (2)$$

where $\text{trig}(\omega, n)$ is an appropriate trigonometrical function, and coefficients a_n is related to the impulse response of the filter, whereas M is a function of the filter order N . Table 1 lists the relationship among them for four types of linear phase filters. Defining the column vector

$$\mathbf{a} = [a_1, a_2, \dots, a_M]^T \quad (3)$$

Table 1
Relationship between $h(n)$, N and $M, a_n, \text{trig}(\omega, n)$.

Type	$M, a_n, \text{trig}(\omega, n)$
$h(n)$: Symmetric	$M = \frac{1}{2}(N + 1)$
N : Even	$a_1 = h(\frac{1}{2}N)$ $a_n = 2h(\frac{1}{2}N - n + 1) \quad n = 2, \dots, M$ $\text{trig}(\omega, n) = \cos((n - 1)\omega)$
$h(n)$: Symmetric	$M = \frac{1}{2}(N + 1)$
N : Odd	$a_n = 2h(\frac{1}{2}(N + 1) - n) \quad n = 1, \dots, M$ $\text{trig}(\omega, n) = \cos((n - \frac{1}{2})\omega)$
$h(n)$: Antisymmetric	$M = \frac{1}{2}N$
N : Even	$a_n = 2h(\frac{1}{2}N - n) \quad n = 1, \dots, M$ $\text{trig}(\omega, n) = \sin(n\omega)$
$h(n)$: Antisymmetric	$M = \frac{1}{2}(N + 1)$
N : Odd	$a_n = 2h(\frac{1}{2}(N + 1) - n) \quad n = 1, \dots, M$ $\text{trig}(\omega, n) = \sin((n - \frac{1}{2})\omega)$

and

$$\mathbf{c}(\omega) = [\text{trig}(\omega, 1), \text{trig}(\omega, 2), \dots, \text{trig}(\omega, M)]^T, \quad (4)$$

then Eq. (2) can be rewritten as

$$A(\omega) = \mathbf{a}^T \mathbf{c}(\omega) = \mathbf{c}^T(\omega) \mathbf{a}. \quad (5)$$

Now, the problem is to find the filter coefficient vector $\mathbf{a}$ such that amplitude response $A(\omega)$ fits the following notch filter response $D(\omega)$ as close as possible:

$$D(\omega) = \begin{cases} 0 & \omega = \pm \omega_N, \\ 1 & \text{otherwise,} \end{cases} \quad (6)$$

where ω_N is the notch frequency. In this paper, we will develop a multiple exchange algorithm to solve this optimization problem.

3. Two conventional methods

In this section, two conventional methods to design an FIR notch filter will be reviewed. One is the Lagrange multiplier method which is optimal in the least-squared sense, the other is the linear programming method which is optimal in the minimax sense.

3.1. Lagrange multiplier method

In this method, the optimal filter coefficients are obtained by minimizing the following squared error:

$$\begin{aligned} J_1(\mathbf{a}) &= \int_0^\pi (1 - A(\omega))^2 d\omega \\ &= \mathbf{a}^T \mathbf{Q} \mathbf{a} - 2\mathbf{p}^T \mathbf{a} + \pi, \end{aligned} \quad (7)$$

where matrix $\mathbf{Q}$ and vector $\mathbf{p}$ are

$$\mathbf{Q} = \int_0^\pi \mathbf{c}(\omega) \mathbf{c}^T(\omega) d\omega,$$

$$\mathbf{p} = \int_0^\pi \mathbf{c}(\omega) d\omega.$$

In order to make the gain at notch frequency ω_N to be equal to zero exactly, the following constraint is incorporated into the minimization process in Eq. (7):

$$A(\omega_N) = \mathbf{a}^T \mathbf{e} = 0, \quad (8)$$

where $\mathbf{e} = \mathbf{c}(\omega_N)$. Using the Lagrange multiplier method, the optimal solution of this constrained problem is given by

$$\mathbf{a} = \mathbf{Q}^{-1} \mathbf{p} - \mathbf{Q}^{-1} \mathbf{e} (\mathbf{e}^T \mathbf{Q}^{-1} \mathbf{e})^{-1} [\mathbf{e}^T \mathbf{Q}^{-1} \mathbf{p}]. \quad (9)$$

In [5,11], several design examples have been presented to illustrate the performance of this method. In this paper, we will use the least-squares solution as the initial guess of the iterative multiple exchange algorithm.

3.2. Linear programming method

In this method, the optimal filter coefficients are obtained by minimizing the maximum error $|1 - A(\omega)|$ over the non-notch band, i.e., $\omega \in [0, \omega_N - \varepsilon] \cup [\omega_N + \varepsilon, \pi]$, where ε is a prescribed small positive number. Letting δ represent the maximum error, a set of linear inequalities can be written to describe this minimax problem, i.e.,

$$-\delta \leq 1 - A(\omega_i) \leq \delta, \quad \omega_i \in F, \quad (10)$$

where F is a dense grid of frequencies in the bands $[0, \omega_N - \varepsilon] \cup [\omega_N + \varepsilon, \pi]$ over which the approximation is being made. Substitute Eq. (5) into Eq. (10) and incorporate the zero constraint at the notch frequency ω_N , the design problem can be rewritten as the following standard linear programming problem:

Minimize δ

Subject to $\mathbf{a}^T \mathbf{c}(\omega_i) - \delta \leq 1,$

$$-\mathbf{a}^T \mathbf{c}(\omega_i) - \delta \leq -1, \quad \omega_i \in F,$$

$$\mathbf{a}^T \mathbf{c}(\omega_N) = 0.$$

The well-known linear programming techniques such as the simplex algorithm can be used to solve

the set of equations above. Since linear programming is basically a single exchange method, it is significantly slower than the multiple exchange method [16]. The purpose of this paper is to develop a multiple exchange algorithm to design an equiripple FIR notch filter such that the design speed can be improved. Finally, it is worth mentioning that the solution of linear programming method depends on the choice of the parameter ε . The larger ε is, the smaller maximum error δ will be.

4. Notch filter design using a multiple exchange algorithm

4.1. Design algorithm

The amplitude responses $A(\omega)$ of an equiripple FIR notch filter is shown in Fig. 1. Observe this response, it is clear that the amplitudes will be exactly equal to $1 + \delta$ at two extremal frequencies ω_{1l} and ω_{1r} which are adjacent to the notch frequency ω_N . And, the amplitudes at other extremal frequencies ω_{il} and ω_{ir} are equal to $1 - (-1)^i \delta$. Let us assume that there are N_l extrema in the left passband $[0, \omega_N - \varepsilon]$ and N_r extrema in the right passband $[\omega_N + \varepsilon, \pi]$, where ε is a prescribed small positive number. Because zero

Fig. 1. The amplitude response of an equiripple FIR notch filter.

constraint $A(\omega_N) = 0$ is imposed on the amplitude response, there is no alternation theorem to specify extreme number N_l and N_r such that the optimality is guaranteed [3]. However, a multiple exchange algorithm can still be developed to design such a filter. By selecting extremal frequency ω_{il} ($i = 1, \dots, N_l$) in the left passband as follows:

$$\omega_{1l} > \omega_{2l} > \dots > \omega_{N_l l}, \quad (11)$$

we want to find a set of filter coefficients that satisfy the conditions expressed as

$$A(\omega_{il}) = 1 - (-1)^i \delta, \quad i = 1, \dots, N_l. \quad (12)$$

Similarly, by selecting extremal frequency ω_{ir} ($i = 1, \dots, N_r$) in the right passband as follows:

$$\omega_{1r} < \omega_{2r} < \dots < \omega_{N_r r}, \quad (13)$$

we have the following equations:

$$A(\omega_{ir}) = 1 - (-1)^i \delta, \quad i = 1, \dots, N_r. \quad (14)$$

Now, our aim is to find a solution vector $\mathbf{a}$ that satisfies Eqs. (12), (14) and zero constraint Eq. (8) simultaneously. Substitute Eq. (2) into Eqs. (8), (12) and (14), we have the following matrix form:

$$\begin{bmatrix} \text{trig}(\omega_{N_l l}, 1) & \dots & \text{trig}(\omega_{N_l l}, M) \\ \vdots & \dots & \vdots \\ \text{trig}(\omega_{2l}, 1) & \dots & \text{trig}(\omega_{2l}, M) \\ \text{trig}(\omega_{1l}, 1) & \dots & \text{trig}(\omega_{1l}, M) \\ \text{trig}(\omega_N, 1) & \dots & \text{trig}(\omega_N, M) \\ \text{trig}(\omega_{1r}, 1) & \dots & \text{trig}(\omega_{1r}, M) \\ \text{trig}(\omega_{2r}, 1) & \dots & \text{trig}(\omega_{2r}, M) \\ \vdots & \dots & \vdots \\ \text{trig}(\omega_{N_r r}, 1) & \dots & \text{trig}(\omega_{N_r r}, M) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{bmatrix}$$

$$= \begin{bmatrix} 1 - (-1)^{N_l} \delta \\ \vdots \\ 1 - \delta \\ 1 + \delta \\ 0 \\ 1 + \delta \\ 1 - \delta \\ \vdots \\ 1 - (-1)^{N_r} \delta \end{bmatrix}. \quad (15)$$

Denote the matrix on the left-hand side by Φ and the vector on the right-hand side by $\mathbf{b}$, then it can be expressed as

$$\Phi \mathbf{a} = \mathbf{b}. \quad (16)$$

Assume that $N_l + N_r + 1$ is not less than M , then Eq. (15) is an overdetermined set of equations whose least-squares solution is given by

$$\mathbf{a} = (\Phi^T \Phi)^{-1} \Phi^T \mathbf{b}. \quad (17)$$

Based on the above description, we propose an iterative multiple exchange algorithm for obtaining the equiripple notch filter coefficient $\mathbf{a}$ as follows:

1. Read filter order N and notch frequency ω_N .
2. Use least squares solution in Eq. (9) as the initial guess and find the extremal frequencies Ω_{il} ($i = 1, 2, \dots, N_l$), Ω_{ir} ($i = 1, 2, \dots, N_r$) of $A(\omega)$.
- Repeat**
3. Set $\omega_{il} = \Omega_{il}$ ($i = 1, \dots, N_l$) and $\omega_{ir} = \Omega_{ir}$ ($i = 1, \dots, N_r$).
4. Compute average peak error

$$\delta = \frac{1}{N_l + N_r} \left(\sum_{i=1}^{N_l} |A(\omega_{il}) - 1| + \sum_{i=1}^{N_r} |A(\omega_{ir}) - 1| \right). \quad (18)$$

5. Compute the new filter coefficient vector $\mathbf{a}$ by solving Eq. (17).
6. Search for the extremal frequencies Ω_{il} , Ω_{ir} of $A(\omega)$ within the passband

Until they satisfy the following conditions for prescribed small constants ε_1 , ε_2 :

$$|\Omega_{il} - \omega_{il}| \leq \varepsilon_1, \quad i = 1, \dots, N_l \quad (19)$$

and

$$|\Omega_{ir} - \omega_{ir}| \leq \varepsilon_2, \quad i = 1, \dots, N_r. \quad (20)$$

Now, let us address the convergence issue as follows. Given the filter coefficients $\mathbf{a}_k$ at the k th iteration, the algorithm proposed above uses the following two steps to obtain the new coefficients $\mathbf{a}_{k+1}$:

Step 1. Compute the frequency response of the filter according to the coefficients $\mathbf{a}_k$, and find the extremal frequency set Ω and the average peak error δ .

Step 2. Substitute Ω , δ into Eq. (15), and solve it to find the new coefficients $\mathbf{a}_{k+1}$.

It should be pointed out that these two steps were motivated by several recent design methods for complex allpass filters [2,7] where a similar technique was used to develop an iterative design algorithm. Let T be the operator that maps $\mathbf{a}_k$ to $\mathbf{a}_{k+1}$, i.e.,

$$\mathbf{a}_{k+1} = T(\mathbf{a}_k), \quad (21)$$

then the equiripple solution $\mathbf{a}_f$ is the fixed point of mapping T , i.e., $\mathbf{a}_f = T(\mathbf{a}_f)$. Now, we present a sufficient condition for the convergence of the proposed algorithm. Define the ratio

$$\eta_k = \frac{\|\mathbf{a}_{k+1} - \mathbf{a}_k\|}{\|\mathbf{a}_k - \mathbf{a}_{k-1}\|}. \quad (22)$$

It can be shown that if η_k has a less-than-unity upper bound, i.e.,

$$\eta_k \leq \gamma < 1 \quad (23)$$

for $k \geq L$ where L is a positive integer, then sequence $\{\mathbf{a}_k\}$ converges to the fixed point $\mathbf{a}_f$ of the mapping T , i.e., an equiripple solution. As a matter of fact, this condition implies that

$$\|\mathbf{a}_{k+1} - \mathbf{a}_k\| \leq \gamma \|\mathbf{a}_k - \mathbf{a}_{k-1}\|. \quad (24)$$

So for sufficiently large m and n with $m > n \geq L$, we have

$$\|\mathbf{a}_m - \mathbf{a}_n\| \leq \frac{\gamma^{n-L+1} - \gamma^{m-L+1}}{1 - \gamma} \|\mathbf{a}_L - \mathbf{a}_{L-1}\|, \quad (25)$$

which approaches to zero when $m, n \rightarrow \infty$, and hence $\{\mathbf{a}_k\}$ is a Cauchy sequence in a finite-dimensional Euclidean space. Further notice that the above sufficient condition is equivalent to

$$\|T(\mathbf{a}_k) - T(\mathbf{a}_{k-1})\| \leq \gamma \|\mathbf{a}_k - \mathbf{a}_{k-1}\| \quad (26)$$

for a $\gamma \in (0, 1)$. In other words $\{\mathbf{a}_k\}$ converges to the fixed point $\mathbf{a}_f$ if T is a *contraction mapping*. This fact is based on the Banach fixed point theorem [9]. Although a rigorous proof is not available to date, the proposed algorithm is always successful in producing $\{\mathbf{a}_k\}$ with ratio $\eta_k \leq \gamma < 1$ in our extensive simulation study.

4.2. Design examples

Here, we present several design examples of the notch filter to demonstrate the effectiveness of the proposed method. The program is performed in an IBM PC compatible computer (with an Intel Pentium Pro CPU inside) by using MATLAB language. In the following examples, the type 1 FIR filter is used, i.e., $M = N/2 + 1$ and $\text{trig}(\omega, i) = \cos((i - 1)\omega)$.

Example 1. In this example, let us consider the following specification for the notch filter: order $N = 60$ and notch frequency $\omega_N = 0.5\pi$. When the multiple exchange algorithm with $\varepsilon_1 = \varepsilon_2 = 10^{-4}$ is used to design this filter, the final value of δ is 0.0701. The magnitude response of this filter is shown in Fig. 2(a). It is clear that the specification is well satisfied. To illustrate the convergence of the algorithm, the average peak error δ is listed in Table 2 as a function of the number of iterations. It is seen that the errors do not change significantly past three iterations. Thus, convergence speed of the proposed algorithm is very fast. Now, the linear programming method with $\varepsilon = 0.035$ (see Section 3) is used to design a notch filter with the same specification. The resultant magnitude response is shown in Fig. 2(b) and peak error δ is 0.0698 which is almost the same as the peak error of the multiple exchange method. In this example, the CPU time which the linear programming method takes is 32.24s, and the multiple exchange method only takes 0.66s which includes the time to compute the initial guess by using the Lagrange multiplier

Fig. 2. (a) The amplitude response of an equiripple FIR notch filter designed by the multiple exchange algorithm. (b) The amplitude response of an equiripple FIR notch filter designed by the linear programming method.

method. Thus, the presented algorithm outperforms the linear programming method in design speed.

From this example, two remarks are made as follows. First, the numbers of extrema in the left

Table 2

Average peak error δ in Example 1

Number of iteration	Peak error δ
1	0.06404840869397
2	0.07010266515967
3	0.07017783226188
4	0.07017783226188
5	0.07017783226188

Table 3

The 3 dB rejection bandwidth for various filter order N

Order N	Bandwidth
40	0.079 π
60	0.055 π
80	0.043 π
100	0.035 π
120	0.029 π
140	0.025 π
160	0.023 π
180	0.021 π
200	0.019 π

and right passbands are $N_l = 15$ and $N_r = 15$. Because the parameter $M = N/2 + 1 = 31$, the condition

$$N_l + N_r + 1 \geq M \quad (27)$$

is satisfied and the iterative process can be run. Second, the 3 dB rejection bandwidth of the notch filter is determined automatically by the proposed multiple exchange algorithm. The larger filter order N is, the narrower bandwidth will be. In order to illustrate this fact, Table 3 lists 3 dB rejection bandwidths for notch frequency $\omega_N = 0.5\pi$ and various orders N . Thus, when bandwidth is a prescribed design parameter, we can choose a suitably large filter order N to meet the specification.

Example 2. In this example, we utilize the FIR notch filter designed by a multiple exchange algorithm to remove the 60 Hz power line interference in the measured ECG signal. The samples used here have 8 bits and the sampling rate is 400 Hz. Fig. 3(a) shows the input waveform which is an ECG signal corrupted by the 60 Hz power line interference. And, the spectrum of this input signal is also shown

Fig. 3. (a) Input waveform of the FIR notch filter. (b) The spectrum of input signal in (a). (c) Output waveform of the FIR notch filter.

in Fig. 3(b). It is clear that ECG is a low-frequency signal. The specification of the notch filter is chosen as

$$D(\omega) = \begin{cases} 0 & \omega = \pm \frac{3}{10}\pi, \\ 1 & \text{otherwise} \end{cases} \quad (28)$$

Now, an FIR notch filter with order 48 is designed to remove the powerline interference. Fig. 3(c) shows the waveform of the notch filter output. From this result, it is obvious that the interference has been removed by our designed notch filter.

4.3. Discussions

In the above multiple exchange algorithm, the parameter δ is updated by averaging the peak errors at all extracted extremal frequencies. When the design algorithm terminates, the peak errors at non-notch passband are all the same. This means that all frequencies in the passband are equally important. However, from Example 2, we see that the ECG signal is a low-frequency signal, so it is desired to let the peak error at the low-frequency

passband be smaller than one in the high frequency passband. To achieve this purpose, two modified multiple exchange algorithms are presented as follows. The first method is to incorporate the frequency weighting function $W(\omega)$ into the proposed design algorithm. The larger weights are assigned to, the smaller errors are obtained. In this case, the design algorithm is the same as the original algorithm except that the matrix form in Eq. (15) is modified by

$$\begin{bmatrix} \text{trig}(\omega_{N_1l}, 1) & \cdots & \text{trig}(\omega_{N_1l}, M) \\ \vdots & \cdots & \vdots \\ \text{trig}(\omega_{21}, 1) & \cdots & \text{trig}(\omega_{21}, M) \\ \text{trig}(\omega_{11}, 1) & \cdots & \text{trig}(\omega_{11}, M) \\ \text{trig}(\omega_N, 1) & \cdots & \text{trig}(\omega_N, M) \\ \text{trig}(\omega_{1r}, 1) & \cdots & \text{trig}(\omega_{1r}, M) \\ \text{trig}(\omega_{2r}, 1) & \cdots & \text{trig}(\omega_{2r}, M) \\ \vdots & \cdots & \vdots \\ \text{trig}(\omega_{N_r}, 1) & \cdots & \text{trig}(\omega_{N_r}, M) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{bmatrix}$$

$$= \begin{bmatrix} 1 - (-1)^{N_l} \frac{\delta}{W(\omega_{N_l})} \\ \vdots \\ 1 - \frac{\delta}{W(\omega_{2l})} \\ 1 + \frac{\delta}{W(\omega_{1l})} \\ 0 \\ 1 + \frac{\delta}{W(\omega_{1r})} \\ 1 - \frac{\delta}{W(\omega_{2r})} \\ \vdots \\ 1 - (-1)^{N_r} \frac{\delta}{W(\omega_{N_r})} \end{bmatrix} \quad (29)$$

and the computation of average peak error in step (4) of design algorithm is changed into the following form:

$$\delta = \frac{1}{N_l + N_r} \left(\sum_{i=1}^{N_l} W(\omega_{il}) |A(\omega_{il}) - 1| + \sum_{i=1}^{N_r} W(\omega_{ir}) |A(\omega_{ir}) - 1| \right). \quad (30)$$

Now, an example is used to illustrate the performance of this modified multiple exchange algorithm.

Example 3. In this example, let us consider the following specification for notch filter: order $N = 60$ and notch frequency $\omega_N = 0.2\pi$. Type 1 FIR filter is used, i.e., $M = N/2 + 1$ and $\text{trig}(\omega, i) = \cos((i-1)\omega)$. The weighting function $W(\omega)$ is chosen as

$$W(\omega) = \begin{cases} 2 & \omega \in [0, \omega_N - \varepsilon], \\ 1 & \omega \in [\omega_N + \varepsilon, \pi], \end{cases} \quad (31)$$

where ε is a prescribed small positive number. Here, we choose $\varepsilon = 0.01$. The magnitude response of this filter is shown in Fig. 4(a,b). It is clear that the response is well satisfied. The peak error in the left passband is 0.0436

and the peak error in the right passband is 0.0872.

The second method is to find optimal filter coefficients such that the peak error in the lower-frequency band is given and the peak error in the high-frequency passband is minimized. Since the prescribed peak error can be assigned to any small value, the peak error in the low-frequency passband will be smaller than one in the high-frequency passband. Without loss of generality, we only concentrate on the equiripple notch filter design whose peak error in the left passband is given by a prescribed value δ_p . In this case, the design algorithm is the same as the original algorithm except that matrix form in Eq. (15) is modified to

$$\begin{bmatrix} \text{trig}(\omega_{N_l}, 1) & \cdots & \text{trig}(\omega_{N_l}, M) \\ \vdots & \cdots & \vdots \\ \text{trig}(\omega_{2l}, 1) & \cdots & \text{trig}(\omega_{2l}, M) \\ \text{trig}(\omega_{1l}, 1) & \cdots & \text{trig}(\omega_{1l}, M) \\ \text{trig}(\omega_N, 1) & \cdots & \text{trig}(\omega_N, M) \\ \text{trig}(\omega_{1r}, 1) & \cdots & \text{trig}(\omega_{1r}, M) \\ \text{trig}(\omega_{2r}, 1) & \cdots & \text{trig}(\omega_{2r}, M) \\ \vdots & \cdots & \vdots \\ \text{trig}(\omega_{N_r}, 1) & \cdots & \text{trig}(\omega_{N_r}, M) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{bmatrix}$$

$$= \begin{bmatrix} 1 - (-1)^{N_l} \delta_p \\ \vdots \\ 1 - \delta_p \\ 1 + \delta_p \\ 0 \\ 1 + \delta \\ 1 - \delta \\ \vdots \\ 1 - (-1)^{N_r} \delta \end{bmatrix} \quad (32)$$

and step (1) and (4) of the original design algorithm must be changed into the following form:

1. Read filter order N , notch frequency ω_N and peak error δ_p in the low-frequency band.

Fig. 4. (a) The amplitude response of an equiripple FIR notch filter designed by a weighted multiple exchange algorithm. (b) Zoom of the non-notch passband in (a).

4. Compute average peak error

$$\delta = \frac{1}{N_r} \sum_{i=1}^{N_r} |A(\omega_{ir}) - 1|. \quad (33)$$

Now, an example is used to illustrate the performance of the second modified multiple exchange algorithm.

Fig. 5. (a) The amplitude response of an equiripple FIR notch filter designed by the modified multiple exchange algorithm. (b) Zoom of the non-notch passband in (a).

Example 4. In this example, let us consider the following specification for the notch filter: order $N = 60$ and notch frequency $\omega_N = 0.3\pi$. Type 1 FIR filter is used, i.e., $M = N/2 + 1$ and $\text{trig}(\omega, i) = \cos((i - 1)\omega)$. The prescribed peak error δ_p is chosen as 0.05. The amplitude response of this filter is shown in Fig. 5(a,b). It is clear that the

response is well satisfied. The peak error in the left passband is the prescribed value 0.05 which is smaller than the peak error 0.0692 in the right passband.

5. Extension to multiple notch filter design

So far, single-notch filter design has been investigated in detail. In this section, we will extend the multiple exchange algorithm to design multiple notch filters. Without loss of generality, we only consider double notch filters whose ideal amplitude response is given by

$$D(\omega) = \begin{cases} 0 & \omega = \pm \omega_{1N} \text{ or } \omega = \pm \omega_{2N}, \\ 1 & \text{otherwise,} \end{cases} \quad (34)$$

where ω_{1N} and ω_{2N} are the notch frequencies. An equiripple approximation to this ideal response is depicted in Fig. 6. Let us assume that there are N_l extrema in the left passband $[0, \omega_{1N} - \varepsilon]$, N_m extrema in the middle passband $[\omega_{1N} + \varepsilon, \omega_{2N} - \varepsilon]$, and N_r extrema in the right passband $[\omega_{2N} + \varepsilon, \pi]$, where ε is a small positive number. By selecting extremal frequencies in the three passbands as follows:

$$\begin{aligned} \omega_{11} &> \omega_{21} > \cdots > \omega_{N_l}, \\ \omega_{1m} &> \omega_{2m} > \cdots > \omega_{N_m}, \\ \omega_{1r} &< \omega_{2r} < \cdots < \omega_{N_r}, \end{aligned} \quad (35)$$

we want to find a set of filter coefficients that satisfy the conditions expressed as

$$\begin{aligned} A(\omega_{il}) &= 1 - (-1)^i \delta, & i &= 1, \dots, N_l, \\ A(\omega_{im}) &= 1 - (-1)^i \delta, & i &= 1, \dots, N_m, \\ A(\omega_{ir}) &= 1 - (-1)^i \delta, & i &= 1, \dots, N_r. \end{aligned} \quad (36)$$

Since extremal frequency ω_{N_m} is adjacent to notch frequency ω_{1N} , we have $A(\omega_{N_m}) = 1 + \delta$. Now, our aim is to find a solution vector $\mathbf{a}$ that satisfies Eq. (36) and zero constraints at two notch frequencies:

$$A(\omega_{1N}) = 0 \text{ and } A(\omega_{2N}) = 0 \quad (37)$$

simultaneously. Substitute Eq. (2) into Eqs. (36) and (37), we have the following matrix form:

$$\begin{bmatrix} \text{trig}(\omega_{N_l}, 1) & \cdots & \text{trig}(\omega_{N_l}, M) \\ \vdots & \cdots & \vdots \\ \text{trig}(\omega_{21}, 1) & \cdots & \text{trig}(\omega_{21}, M) \\ \text{trig}(\omega_{11}, 1) & \cdots & \text{trig}(\omega_{11}, M) \\ \text{trig}(\omega_{1N}, 1) & \cdots & \text{trig}(\omega_{1N}, M) \\ \text{trig}(\omega_{N_m}, 1) & \cdots & \text{trig}(\omega_{N_m}, M) \\ \vdots & \cdots & \vdots \\ \text{trig}(\omega_{2m}, 1) & \cdots & \text{trig}(\omega_{2m}, M) \\ \text{trig}(\omega_{1m}, 1) & \cdots & \text{trig}(\omega_{1m}, M) \\ \text{trig}(\omega_{2N}, 1) & \cdots & \text{trig}(\omega_{2N}, M) \\ \text{trig}(\omega_{1r}, 1) & \cdots & \text{trig}(\omega_{1r}, M) \\ \text{trig}(\omega_{2r}, 1) & \cdots & \text{trig}(\omega_{2r}, M) \\ \vdots & \cdots & \vdots \\ \text{trig}(\omega_{N_r}, 1) & \cdots & \text{trig}(\omega_{N_r}, M) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{bmatrix} = \begin{bmatrix} 1 - (-1)^{N_l} \delta \\ \vdots \\ 1 - \delta \\ 1 + \delta \\ 0 \\ 1 + \delta \\ \vdots \\ 1 - \delta \\ 1 + \delta \\ 0 \\ 1 + \delta \\ 1 - \delta \\ \vdots \\ 1 - (-1)^{N_r} \delta \end{bmatrix}. \quad (38)$$

Denote the matrix on the left-hand side by Ψ and the vector on the right-hand side by $\mathbf{d}$, then it can be expressed as

$$\Psi \mathbf{a} = \mathbf{d}. \quad (39)$$

Assume that $N_l + N_m + N_r + 2$ is not less than M , then Eq. (38) is an overdetermined set of equations whose least-squares solution is given by

$$\mathbf{a} = (\Psi^T \Psi)^{-1} \Psi^T \mathbf{d}. \quad (40)$$

Moreover, from the vector on the right-hand side of Eq. (38), it can be observed that the magnitude responses at the extremal frequencies of middle passband alternate from $1 + \delta$ to $1 - \delta$. Thus, the number of extrema in the middle passband is odd,

Fig. 6. The amplitude response of an equiripple multiple notch filter.

i.e., N_m is odd. Based on the above description, we propose an iterative multiple exchange algorithm for obtaining the equiripple multiple notch filter coefficient $\mathbf{a}$ as follows:

1. Read filter order N and notch frequencies ω_{1N}, ω_{2N} .
2. Use the Lagrange multiplier method to find the least-squares solution as the initial guess. The Lagrange multiplier method in the double-notch filter design is the same as that of the single-notch filter design (see Section 3) except zero constraint $\mathbf{e} = \mathbf{c}(\omega_N)$ is changed into the following form:

$$\mathbf{e} = [\mathbf{c}(\omega_{1N}) \quad \mathbf{c}(\omega_{2N})]. \quad (41)$$

Extract extremal frequencies Ω_{il} ($i = 1, 2, \dots, N_l$), Ω_{im} ($i = 1, 2, \dots, N_m$), Ω_{ir} ($i = 1, 2, \dots, N_r$) of response $A(\omega)$.

Repeat

3. Set $\omega_{il} = \Omega_{il}$ ($i = 1, \dots, N_l$), $\omega_{im} = \Omega_{im}$ ($i = 1, \dots, N_m$) and $\omega_{ir} = \Omega_{ir}$ ($i = 1, \dots, N_r$).
4. Compute average peak error

$$\delta = \frac{1}{N_l + N_m + N_r} \left(\sum_{i=1}^{N_l} |A(\omega_{il}) - 1| + \sum_{i=1}^{N_m} |A(\omega_{im}) - 1| + \sum_{i=1}^{N_r} |A(\omega_{ir}) - 1| \right). \quad (42)$$

5. Compute the new filter coefficient vector $\mathbf{a}$ by solving Eq. (40).
6. Search for the extremal frequencies Ω_{il} , Ω_{im} , Ω_{ir} of $A(\omega)$ within passband.

Until they satisfy the following conditions for prescribed small constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$:

$$|\Omega_{il} - \omega_{il}| \leq \varepsilon_1, \quad i = 1, \dots, N_l, \quad (43)$$

$$|\Omega_{im} - \omega_{im}| \leq \varepsilon_2, \quad i = 1, \dots, N_m, \quad (44)$$

and

$$|\Omega_{ir} - \omega_{ir}| \leq \varepsilon_3, \quad i = 1, \dots, N_r. \quad (45)$$

Now, an example is used to illustrate the performance of proposed design algorithm.

Example 5. In this example, let us consider the following specification for the notch filter: order $N = 60$ and notch frequencies $\omega_{1N} = 0.2\pi$, $\omega_{2N} = 0.8\pi$. The type 1 FIR filter is used, i.e., $M = N/2 + 1$ and $\text{trig}(\omega, i) = \cos((i - 1)\omega)$. When the multiple exchange algorithm with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 10^{-4}$ is used to design this filter, the final value of δ is 0.0942. The amplitude response of this filter is shown in Fig. 7. It is clear that the specification is well satisfied. Moreover, the convergence speed of proposed algorithm is also very fast in this case. The CPU time in this design example takes 0.76 s.

From this example, two remarks are made as follows. First, the numbers of extrema in left, middle and right passbands are $N_l = 6$, $N_m = 17$ and $N_r = 6$. Because the parameter $M = N/2 + 1 = 31$, the condition

$$N_l + N_m + N_r + 2 \geq M \quad (46)$$

Fig. 7. The amplitude response of an equiripple FIR multiple notch filter designed by multiple exchange algorithm.

is fulfilled and the iterative process can be performed. Second, when notch frequencies $\omega_{1N} = 0.2\pi$, $\omega_{2N} = 0.3\pi$ and filter order $N = 60$, the multiple exchange algorithm will fail to converge. However, when $\omega_{1N} = 0.2\pi$, $\omega_{2N} = 0.3\pi$ and $N = 200$, the multiple exchange algorithm will converge and provide a satisfactory solution. Thus, when notch frequencies ω_{1N} and ω_{2N} are very close, the filter order N must be large enough to guarantee the convergence of the design algorithm. For guaranteeing convergence, the relation between order N and the difference $|\omega_{1N} - \omega_{2N}|$ should be derived. However, this topic is not easy and will be investigated in the future.

6. Conclusions

In this paper, a multiple exchange algorithm has been presented to design an equiripple FIR notch filter. The algorithm is first discussed for designing a single-notch filter. Then, two modified algorithms are developed to reduce the ripple in the low-frequency passband because ECG is usually a low-frequency signal. Next, we extend the proposed algorithm to design equiripple multiple notch filter. Finally, several design examples are presented to

demonstrate its effectiveness. However, only a one-dimensional (1-D) FIR notch filter is considered here. Thus it is interesting to extend this method to design 2-D equiripple FIR notch filters in the future.

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