Relations Between Gabor Transforms and Fractional Fourier Transforms and Their Applications for Signal Processing

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Abstract—Many useful relations between the Gabor transform (GT) and the fractional Fourier transform (FRFT) have been derived. First, we find that, like the Wigner distribution function (WDF), the FRFT is also equivalent to the rotation operation of the GT. Then, we show that performing the scaled inverse Fourier transform (IFT) along an oblique line of the GT of $f(t)$ can yield its FRFT. Since the GT is closely related to the FRFT, we can use it for analyzing the characteristics of the FRFT. Compared with the WDF, the GT does not have the cross-term problem. This advantage is important for the applications of filter design, sampling, and multiplexing in the FRFT domain. Moreover, we find that if the GT is combined with the WDF, the resultant operation [called the Gabor–Wigner transform (GWT)] also has rotation relation with the FRFT. We also derive the general form of the linear distribution that has rotation relation with the FRFT.

Index Terms—Fractional filter design, fractional Fourier transform (FRFT), fractional multiplexing, fractional sampling, Gabor transform (GT), Gabor–Wigner transform (GWT), Wigner distribution function (WDF).

I. INTRODUCTION

THE FRACTIONAL Fourier transform (FRFT) is a generalization of the Fourier transform (FT) [1], [2]

$$\begin{aligned} O_F^\alpha(f(t)) &= F_\alpha(u) \\ &= \sqrt{\frac{1-j\cot\alpha}{2\pi}} \int_{-\infty}^{\infty} e^{(j/2)u^2\cot\alpha - jut\csc\alpha + (j/2)t^2\cot\alpha} \\ &\quad \cdot f(t) dt. \end{aligned} \quad (1)$$

It has the additive property

$$O_F^\alpha \left[O_F^\beta(f(t)) \right] = O_F^{\alpha+\beta}(f(t)). \quad (2)$$

When $\alpha = 0$, the FRFT becomes the identity operation. When $\alpha = \pi/2$, the FRFT becomes the FT

$$FT[f(t)] = O_F^{\pi/2}(f(t)) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{-j\omega t} f(t) dt. \quad (3)$$

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The FRFT can extend the utilities of the FT and is very useful for filter design, signal sampling, pattern recognition, optics analysis, radar system analysis, and communication.

The FRFT is closely related to the Wigner distribution function (WDF) [3], which is defined as

$$W_f(t, \omega) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} f\left(\frac{t+\tau}{2}\right) \cdot f^*\left(\frac{t-\tau}{2}\right) e^{-j\omega\tau} \cdot d\tau. \quad (4)$$

If $W_f(t, \omega)$ is the WDF of a signal $f(t)$ and $W_{F_\alpha}(t, \omega)$ is the WDF of its FRFT $F_\alpha(u) = O_F^\alpha(f(t))$, then they have the following clockwise-rotation relation [4]–[7]:

$$W_{F_\alpha}(u, v) = W_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha). \quad (5)$$

Since the WDF and the FRFT have such a close relation, we often use the WDF to aid the FRFT for signal processing applications. For example, when using the FRFT for filter design in the FRFT domain, the WDF is helpful for estimating the optimal order α [6]. For multiplexing and modulation, the WDF is helpful for assigning the time–frequency slot [7]. The fractional WDF, which generalizes the WDF by the FRFT, was developed in [24].

However, there is a problem for the WDF, i.e., “the cross-term problem.” In other words, if $f(t) = s(t) + r(t)$ and $W_f(t, \omega)$, $W_s(t, \omega)$, and $W_r(t, \omega)$ are the WDFs of $f(t)$, $s(t)$, and $r(t)$, respectively, then

$$W_f(t, \omega) \neq W_s(t, \omega) + W_r(t, \omega). \quad (6)$$

This is because the WDF contains an autocorrelation term $f(t + \tau/2)f^*(t - \tau/2)$ (see (4)) and is not a linear operation. The cross-term problem of the WDF can be seen from Fig. 2(d) or Fig. 5(c). This problem makes it difficult to distinguish the signal part, the noise part, and the cross-term part from the WDF. Using the WDF to analyze the FRFT characters is improper for the cases where the signal consists of several time–frequency components.

In [8], Ozaktas *et al.* found that Cohen’s class distributions also have clockwise-rotation relation with the FRFT. However, as the WDF, Cohen’s class distributions also have the cross-term problem.

In this paper, we derive many useful relations between the Gabor transform (GT) [9]–[11] and the FRFT (see Section II). The definition of the GT used in this paper is

$$G_f(t, \omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{-((\tau-t)^2/2)} e^{-j\omega(\tau-(t/2))} f(\tau) d\tau. \quad (7)$$

Fig. 1. GT of $F_\alpha(u)$, where $F_\alpha(u)$ is the FRFT of $f(t)$, $f(t) = 1$ for $|t| \leq 3$, and $f(t) = 0$ otherwise. (a) $G_f(t, \omega)$, $\alpha = 0$. (b) $G_{F_\alpha}(t, \omega)$, $\alpha = \pi/6$. (c) $G_{F_\alpha}(t, \omega)$, $\alpha = 2\pi/6$. (d) $G_{F_\alpha}(t, \omega)$, $\alpha = \pi/2$.

We find that, like the WDF, the FRFT is also equivalent to the clockwise rotation of the GT (Section II-A). Thus, it is possible to use the GT instead of the WDF to perform signal processing in the FRFT domain. Compared with the WDF, the GT has the advantages of 1) avoiding the cross-term problem and 2) consuming less computation time.

In particular, because it can avoid the cross-term problem, the GT is more effective than the WDF for filter design, sampling, multiplexing, and signal compression in the FRFT domain (see Section IV). Without being misled by the cross term, the optimal parameter α of the FRFT can be more easily determined. (This problem has perplexed the researchers in the field of the FRFT for many years. Now, we can use the GT to solve it successfully.) Moreover, with the GT, the cutoff criterion and the location of time-frequency slot can also be obtained more easily. As the WDF, the GT can also be generalized into the fractional GT [25], [26]. However, the fractional GT does not satisfy the constraint in (37). A linear operation that does not satisfy the constraint will have no rotation relation with the FRFT.

Although sometimes the GT may not have very good clarity, we can combine the GT with the WDF, i.e., the Gabor–Wigner transform (GWT) (see Section III). The GWT also has rotation relation with the FRFT. Moreover, the GWT combines the advantages of the GT (no cross term) and the WDF (high clarity).

II. RELATIONS BETWEEN GABOR TRANSFORMS AND FRFTS

A. Clockwise-Rotation Relation

Theorem 1: If $F_\alpha(u)$ is the FRFT of $f(t)$, $G_f(t, \omega)$ is the GT of $f(t)$, and $G_{F_\alpha}(u, v)$ is the GT of $F_\alpha(u)$, then $G_f(t, \omega)$ and $G_{F_\alpha}(u, v)$ have the following relation:

$$G_{F_\alpha}(u, v) = G_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha). \quad (8)$$

That is, the FRFT with parameter α is equivalent to **rotating the GT in the clockwise direction with angle α** . Its proof is shown in the Appendix. We perform an experiment in Fig. 1 to show that the FRFT is equivalent to the rotation operation for the GT.

$$\text{In Fig. 1: } f(t) = 1 \text{ for } |t| \leq 3, \quad f(t) = 0 \text{ for } |t| > 3. \quad (9)$$

Note that the standard definition of the GT in [9]–[11] is

$$\tilde{G}_f(t, \omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{-(\tau-t)^2/2} e^{-j\omega\tau} f(\tau) d\tau. \quad (10)$$

Fig. 2. GTs and the WDFs of $s(t)$, $r(t)$, and $f(t) = s(t) + r(t)$. Note that the WDF has the “cross-term problem” but not the GT. (a) GT of $s(t)$. (b) GT of $r(t)$. (c) GT of $f(t) = s(t) + r(t)$. (d) WDF of $f(t)$.

In this paper, we modify the definition slightly as in (7) for the consideration of rotation relation. When using the standard definition, the amplitude has rotation relation with the FRFT, but the phase does not.

$$\begin{aligned} \tilde{G}_{F_\alpha}(u, v) &= e^{j[-uv \sin^2 \alpha + (u^2 - v^2) \sin(2\alpha)/4]} \\ &\cdot \tilde{G}_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha) \\ &\neq \tilde{G}_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha). \end{aligned} \quad (11)$$

Thus, we use (7) instead of (10) as the definition of the GT in this paper. With the modification, the GT has rotation relation with FRFTs both in amplitude and in phase, as (8).

B. Comparison With Wigner Distribution Function

We have proven that the clockwise rotation of the GT is equivalent to performing the FRFT. Although the WDF also has such similar property, it has the problem of “cross term.” If we use the GT instead of the WDF, because the GT is **a linear operator** and need not calculate the autocorrelation $f(t + \tau/2)f^*(t - \tau/2)$, **the cross-term problem can be avoided**. That is, if $f(t) = s(t) + r(t)$ and $G_f(t, \omega)$, $G_s(t, \omega)$ and $G_r(t, \omega)$ are their GTs, then

$$G_f(t, \omega) = G_s(t, \omega) + G_r(t, \omega). \quad (12)$$

In Fig. 2, we show the GTs and the WDFs of $s(t)$, $r(t)$, and $f(t) = s(t) + r(t)$, where

$$\begin{aligned} s(t) &= e^{jt^2/10 - j3t} \text{ for } -9 \leq t \leq 1 \\ s(t) &= 0 \text{ otherwise} \\ r(t) &= e^{-jt^2/2 + j6t} e^{-(t-4)^2/10}. \end{aligned} \quad (13)$$

Obviously, the cross-term problem can be avoided if we use the GT instead of the WDF, see Fig. 2(c) and (d). This advantage is very important for filter design and other applications (Section IV).

Then, we discuss the implementation of the GT. Note that, although from (4) and (7), the ranges of integration for the WDF and the GT are both $(-\infty, \infty)$, from the fact that

$$\exp\left(\frac{-x^2}{2}\right) < 0.0001 \text{ when } |x| > 4.2919 \quad (14)$$

the GT in (7) can be well approximated by

$$G_f(t, \omega) \approx \sqrt{\frac{1}{2\pi}} \int_{t-4.2919}^{t+4.2919} e^{-(\tau-t)^2/2} e^{-j\omega(\tau-t/2)} f(\tau) d\tau \quad (15)$$

$$G_f(n\Delta_t, m\Delta_\omega) \approx \sqrt{\frac{1}{2\pi}} \Delta_t e^{j(\pi/N)mn} \sum_{k=n-K}^{n+K} e^{-j(2\pi/N)mk} \cdot \left[e^{((k-n)^2/2)\Delta_t^2} f(k\Delta_t) \right], \text{ where } \Delta_t \Delta_\omega = \frac{2\pi}{N} \quad (16)$$

$$K\Delta_t \approx 4.2919, \quad \Delta_t < \frac{\pi}{B} \quad (\text{to avoid the aliasing effect}) \quad (17)$$

and B is the bandwidth of $\exp[-(\tau-t)^2/2] \cdot f(\tau)$, i.e.,

$$|FT\{e^{-(\tau-t)^2/2} f(\tau)\}| = |G_f(t, \omega)| \approx 0 \text{ when } |\omega| > B. \quad (18)$$

If in (16), the range of n and m are $L_1 \leq n < L_1 + L$ and $\Omega_1 \leq m < \Omega_1 + \Omega$, then, to implement the GT, we need to perform the N -point DFTs

$$L \left\lfloor \frac{\Omega}{N} \right\rfloor \left\lfloor \frac{(2K+1)}{N} \right\rfloor \text{ times, where } \lfloor x \rfloor \text{ means the least integer is no smaller than } x. \quad (19)$$

In comparison, the digital implementation of the WDF is

$$G_f(n\Delta_t, m\Delta_\omega) \approx \left(\frac{\Delta_t}{\pi} \right) \sum_{k=K}^K e^{-j2\pi mk/N} \cdot [f((n+k)\Delta_t) f^*((n-k)\Delta_t)] \quad (20)$$

$$\text{where } \Delta_t \Delta_\omega = \frac{\pi}{N} \text{ and } \Delta_t < \frac{\pi}{2B}. \quad (21)$$

B is the bandwidth of $f(t+\tau)f^*(t-\tau)$. Note that, from (4), $FT[f(t+\tau)f^*(t-\tau)] = W_f(t, \omega/2) \approx G_f(t, \omega/2)$. Thus, from (18), the bandwidth of $f(t+\tau/2)f^*(t-\tau/2)$ is about twice the bandwidth of $e^{-(\tau-t)^2/2} f(\tau)$. Therefore, to avoid the aliasing effect, Δ_t of the WDF should be even smaller than that of the GT (about half). Moreover, although the WDF also needs $L[\Omega/N][\lfloor (2K+1)/N \rfloor]$ times of the N -point DFTs, as in (19), however, if $f(t)$ is not time limited, then in (20), K should be chosen as

$$K\Delta_t \rightarrow \infty \quad (22)$$

which is much larger than (17). Even when $f(t)$ is time limited but the time duration is many times larger than 4.2919, the value of K of the WDF is much larger than that of the GT.

Thus, because the integration range of the WDF is $(-\infty, \infty)$ and that of the GT can be reduced into $(t-4.2919, t+4.2919)$, as in (15), **the computation time of the GT will be much less than that of the WDF.** It is another advantage of the GT.

Then, what is the disadvantage of the GT? The disadvantage is that the GT may have worse clarity than the WDF for the case where $f(t)$ has the form of $\exp(jat^2 + jbt)$. However, the GT does not always have worse clarity. For example, when

Fig. 3. GT and the WDF for $f(t) = \exp(j0.15t^3)$. (a) GT of $\exp(j0.15t^3)$. (b) WDF of $\exp(j0.15t^3)$.

$f(t) = \exp(jt^3/10)$, the GT has better clarity than the WDF, as the example in Fig. 3. In fact, if

$$f(t) = \exp(ja_k t^k + ja_{k-1} t^{k-1} + ja_{k-2} t^{k-2} + \dots + ja_0), \quad a_k \neq 0 \quad (23)$$

when $k \leq 2$, the WDF has higher clarity. When $k \geq 3$, the GT will have better clarity.

In summary, compared with the WDF, the GT has the advantages of 1) avoiding the cross-term problem (due to linearity, $a \cdot s(t) + b \cdot r(t) \rightarrow a \cdot G_s(t, \omega) + b \cdot G_r(t, \omega)$), 2) less computation time for digital implementation (K in (16) is smaller than that in (20)), and 3) suitable for analyzing the signal $f(t) = \exp(jat^k + \text{remained terms})$, where $k \geq 3$.

The disadvantage of the GT is blurring. However, this problem can be solved by combining the GT with the WDF. The new distribution can also avoid the cross-term problem, and its clarity is as good as that of the WDF. See Section III-A.

C. Other Important Relations Between GTs and FRFTs

We then derive other relations between the GT and the FRFT in Table I. We can compare them with those of the WDF [6], [19]. The new materials here are not limited in deriving the relations between the GT and the FRFT. Since, from what we know, many relations, such as the 1) recovery, 2) projection, and 3) power integration relations in Table I, between GTs and the original function $f(t)$ shown here have not been reported in the literature.

The proofs of recovery, power integration, and power-decayed relations are in the Appendix. From the “recovery relation,” we can obtain $F_\alpha(u)$ from the GT of $f(t)$ if we perform the half-scaled inverse Fourier transform (IFT) along the direction of

$$(-\sin \alpha, \cos \alpha). \quad (24)$$

From the “power integration relation,” the integration of $|G_f(t, \omega)|^2$ along the oblique line of

$$u(\cos \alpha, \sin \alpha) + v(-\sin \alpha, \cos \alpha), \quad \text{where } v \text{ varies from } -\infty \text{ to } \infty \quad (25)$$

is equal to **the local energy of $F_\alpha(\tau)$** around $\tau = u$. The “energy sum property” is a generalization of that in [10]. From the “power-decayed relation,” if $F_\alpha(u) = 0$ for $u > u_0$, the average power of $G_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha)$ decays with u when $u > u_0$ and the decayed rate is faster than $\exp(-(u - u_0)^2)$.

TABLE I
RELATIONS BETWEEN THE GABOR TRANSFORM AND THE FRFT

(1) Recovery Relation	$\sqrt{1/2\pi} \int_{-\infty}^{\infty} G_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha) e^{j\omega v/2} dv = F_{\alpha}(u)$ <p>Specially, $\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G_f(t, \omega) e^{\frac{j\omega t}{2}} d\omega = f(t)$ and $\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G_f(t, \omega) e^{\frac{j\omega t}{2}} dt = F(\omega)$.</p>
(2) Projection Relation	$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha) e^{jku v} dv = e^{\frac{(k-1/2)^2 u^2}{2}} F_{\alpha}((k+1/2)u)$ <p>When $k = 1/2$, it becomes the recovery property. When $k = 0$,</p> $\sqrt{1/2\pi} \int_{-\infty}^{\infty} G_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha) dv = e^{-u^2/8} F_{\alpha}(u/2)$
(3) Power Integration relation	$\int_{-\infty}^{\infty} G_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha) ^2 dv = \int_{-\infty}^{\infty} e^{-(\tau-u)^2} F_{\alpha}(\tau) ^2 d\tau$ $\approx \int_{-4.292}^{4.292} e^{-(\tau-u)^2} F_{\alpha}(\tau) ^2 d\tau$
(4) Energy Sum Relation	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{F_{\beta}}(t, \omega) G_{H_{\beta}}^*(t, \omega) d\omega dt = \sqrt{\pi} \int_{-\infty}^{\infty} F_{\alpha}(\tau) H_{\alpha}^*(\tau) d\tau \quad \text{for any } \alpha, \beta,$ <p>Specially, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_f(t, \omega) ^2 d\omega dt = \sqrt{\pi} \int_{-\infty}^{\infty} f(\tau) ^2 d\tau$ (for $\alpha=\beta=0$ and $f(t)=h(t)$).</p>
(5) Power Decayed Relation	<p>If $F_{\alpha}(u) = 0$ for $u > u_0$, then when $u > u_0$,</p> $\text{average of } G_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha) ^2$ <p>(fix u, vary v)</p> $< e^{-(u-u_0)^2} \times \text{average of } G_f(u_0 \cos \alpha - v \sin \alpha, u_0 \sin \alpha + v \cos \alpha) ^2$ <p>(fix u_0, vary v)</p>
(6) Shifting relation in the FRFT Domain	<p>If $H_{\alpha}(u) = F_{\alpha}(u - u_0)$, then</p> $G_h(t, \omega) = G_f(t - u_0 \cos \alpha, \omega - u_0 \sin \alpha) e^{j(t \sin \alpha - \omega \cos \alpha) u_0 / 2}.$ <p>In particular, if $h(t) = f(t - t_0)$, then $G_h(t, \omega) = G_f(t - t_0, \omega) e^{-j\omega t_0 / 2}$.</p> <p>If $h(t) = f(t) \exp(j\omega_0 t)$, then $G_h(t, \omega) = G_f(t, \omega - \omega_0) e^{j\omega_0 t / 2}$.</p>

III. OTHER OPERATIONS THAT HAVE ROTATION RELATIONS WITH FRFT

A. Combination of Gabor Transforms and WDFs

We have compared the performances of the WDF and the GT in Section II. The most important advantage of the WDF is its **high clarity**, but it has the cross-term problem. In contrast, the most significant merit of the GT is its ability to **avoid the cross-term problem**, but its clarity is not as good as that of the WDF. Then, one may ask how to achieve the goals of i) higher clarity and ii) avoiding the cross-term problem at the same time. This problem can be solved by the GWT.

Theorem 2: Gabor-Wigner Transform: If we define a new time frequency $C_f(t, \omega)$ (called the GWT) that has the following relation with the GT $G_f(t, \omega)$ and the WDF $W_f(t, \omega)$:

$$C_f(t, \omega) = p(G_f(t, \omega), W_f(t, \omega)), \text{ where } p(x, y) \text{ is any function with two variables} \quad (26)$$

then $C_f(t, \omega)$ also has **rotation relation** with the FRFT:

$$C_{F_{\alpha}}(u, v) = C_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha) \quad (27)$$

where $C_f(t, \omega)$ and $C_{F_{\alpha}}(u, v)$ are GWTs of $f(t)$ and its FRFT, $F_{\alpha}(u)$, respectively. Equation (26) can be further generalized as

$$C_f(t, \omega) = p(G_f(t, \omega), W_f(t, \omega), q_1(r), q_2(r), \dots, q_N(r)) \quad (28)$$

where $p(x, y, z_1, z_2, \dots, z_N)$ is **any** function with $N + 2$ variables, and $q_n(r)$ are any rotation-symmetry functions that depend only on $r = (t^2 + \omega^2)^{1/2}$. The GWT defined in (28) also satisfies (27) and has **rotation relation** with the FRFT.

Note that, in (26) and (28), there are infinite ways to choose $p(x, y)$ and $p(x, y, z_1, z_2, \dots, z_N)$. For example, if $p(x, y) = xy$, then $C_f(t, \omega) = G_f(t, \omega)W_f(t, \omega)$. If $p(x, y) = x + y$, then

Fig. 4. GWT of $f(t)$ (defined the same as that in Fig. 2) and the GWT we use are defined in (29)–(31).

$C_f(t, \omega) = G_f(t, \omega) + W_f(t, \omega)$. We can even choose $p(x, y)$ as the logic operation of $(x \geq \Delta_1)$ AND $(y \geq \Delta_2)$ and set $C_f(t, \omega)$ as $C_f(t, \omega) = [G_f(t, \omega) \geq \Delta_1] \text{ AND } [W_f(t, \omega) \geq \Delta_2]$. From Theorem 2, all the GWTs $C_f(t, \omega)$ shown above have rotation relation [see (27)] with the FRFT.

Moreover, if $p(x, y)$ in (26) or $p(x, y, z_1, z_2, \dots, z_N)$ in (28) is chosen properly, the resultant GWT can avoid the cross-term problem while maintaining the clarity as good as that of the WDF. That is, **the GWT can combine the advantages of the GT and the WDF** and will be a powerful tool for analyzing the character of a signal in the FRFT domain.

In Fig. 4, we perform several experiments. For each of the subfigures, the GWT we choose are

$$\text{in Fig. 4(a): } C_f(t, \omega) = G_f(t, \omega)W_f(t, \omega)$$

$$\text{in Fig. 4(b): } C_f(t, \omega) = \min\{|G_f(t, \omega)|^2, |W_f(t, \omega)|\} \quad (29)$$

$$\text{in Fig. 4(c): } C_f(t, \omega) = W_f(t, \omega)\{|G_f(t, \omega)| > 0.25\} \quad (30)$$

$$\text{in Fig. 4(d): } C_f(t, \omega) = G_f^{2.6}(t, \omega)W_f^{0.6}(t, \omega). \quad (31)$$

Comparing Fig. 4 with Fig. 2, we find that, like the GT, the GWT can also avoid the cross-term problem. Moreover, the clarity of the GWT is obviously better than that of the GT. Thus, the GWT combines the advantages of the GT and the WDF. In particular, using the GWT defined in (31) will achieve better performance than other types of GWT.

Corollary 1: Real Gabor Transform and Real-GWT: To process a real signal, in (26), we can choose $p(x, y) = \text{real}(x)$. That is,

$$\begin{aligned} C_f(t, \omega) &= \text{real}(G_f(t, \omega)) \\ &= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{-(\tau-t)^2/2} \cos\left(\omega\left(\frac{\tau-t}{2}\right)\right) f(\tau) d\tau. \end{aligned} \quad (32)$$

We can call it the real GT. From Theorem 2, it has rotation relation with the FRFT. It has the property of real-input–real-output and are suitable for processing a real signal. Generally,

we can choose $p(x, y) = \text{real}(h(x, y))$, and the resultant GWT will be suitable for processing real signals.

Corollary 2: From (28), if $G_f(t, \omega)$ is the GT of $f(t)$, then

a) $G_f(t, \omega) + A$ (A is a constant) also has rotation relation with the FRFT; it is a special case of (28)

$$\text{where } N = 1, \quad q_1(r) = A,$$

$$\text{and } p(G_f(t, \omega), W_f(t, \omega), A) = G_f(t, \omega) + A; \quad (33)$$

b) $G_f(t, \omega) + h(r)$ and $h(r)G_f(t, \omega)$ also have rotation relation with the FRFT, where $h(r)$ depends only on $r = (t^2 + \omega^2)^{1/2}$.

B. General Form of Linear Distribution With Rotation Relation

We have known that the GT, the WDF, and their combination (the GWT) all have rotation relations with the FRFT. The GWT can avoid the cross-term problem if $p(x, y, z_1, z_2, \dots, z_N)$ is chosen properly. However, *the GWT is not always a linear operation*. The following question is raised: *Is there any other time–frequency distribution that a) has rotation relation with the FRFT and b) is a linear operation in addition to the GT?* Then, we try to answer this question.

Any linear time–frequency distribution can be expressed as the following form:

$$A_f(t, \omega) = \int_{-\infty}^{\infty} K(t, \omega, \tau) f(\tau) d\tau. \quad (34)$$

If we replace $f(t)$ by $F_\alpha(u)$ (the FRFT of $f(t)$), then, from (1)

$$\begin{aligned} A_{F_\alpha}(t, \omega) &= \int_{-\infty}^{\infty} K(t, \omega, u) \sqrt{\frac{1 - j \cot \alpha}{2\pi}} \\ &\cdot \int_{-\infty}^{\infty} e^{(j/2)u^2 \cot \alpha - ju\tau \csc \alpha + (j/2)\tau^2 \cot \alpha} f(\tau) d\tau du. \end{aligned} \quad (35)$$

Theorem 3: General Form of Linear Distribution That has Rotation Relation With the FRFT: For a linear time–frequency distribution defined in (34), if it has rotation relation with the FRFT, i.e.,

$$A_{F_\alpha}(u, v) = A_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha) \quad (36)$$

then, from (35), the kernel $K(t, \omega, \tau)$ should satisfy

$$\begin{aligned} &K(t \cos \alpha - \omega \sin \alpha, t \sin \alpha + \omega \cos \alpha, \tau) \\ &= \sqrt{\frac{1 - j \cot \alpha}{2\phi}} \int_{-\infty}^{\infty} e^{(j/2)u^2 \cot \alpha - ju\tau \csc \alpha + (j/2)\tau^2 \cot \alpha} \\ &\cdot f(\tau) d\tau du. \end{aligned} \quad (37)$$

Moreover, because $A_f(t, \omega)$ is a linear operation, **the cross-term problem is avoided**, as in the GT. The general form of quadratic distributions with rotation relation is discussed in [8]. Here, we derive the general form of **linear** distributions that have rotation relation with the FRFT.

Theorem 3 is helpful for finding new linear distributions that can analyze the character of a signal in the FRFT domain. For example, suppose that $K(t, \omega, \tau)$ has the form of

$$K(t, \omega, \tau) = e^{k_1(t, \omega)\tau^2 + k_2(t, \omega)\tau + k_3(t, \omega)}. \quad (38)$$

After some calculation (see the Appendix), we obtain the following.

Theorem 4: If the linear time–frequency distribution $A_f(t, \omega)$ has the form (39), shown at the bottom of the page, where $r = (t^2 + \omega^2)^{1/2}$ and $\theta = \arg(t + j\omega)$, **then** $A_f(t, \omega)$ **has rotation relation with the FRFT**, as in (36). Moreover, from $t = r \cos \theta$ and $\omega = r \sin \theta$, (39) can be rewritten as

$$A_f(t, \omega) = \int_{-\infty}^{\infty} e^{-\tau^2/2 + (t-j\omega)C_2(r)\tau + C_3(r) + (\omega^2 + jt\omega)C_2(r)/2} f(\tau) d\tau \quad (40)$$

where $C_2(r) = B_2(r)/r$ and $C_3(r) = B_3(r)$. Equation (40) is the **general form** of the second-order exponential distribution that has rotation-symmetry relation with the FRFT. The proof is shown in the Appendix.

Note that the GT is the special case of (40) where

$$C_2(r) = 1 \text{ and } C_3(r) = \frac{-r^2}{2}. \quad (41)$$

IV. APPLICATIONS FOR SIGNAL PROCESSING IN THE FRFT DOMAIN

Since the GT and the GWT are closely related to the FRFT, we can use it as an assistant tool for signal processing in the FRFT domain. Compared with the WDF, the GT and the GWT can avoid the cross-term problem. This advantage is very important for the applications that are related to **multiple component analysis**, such as filter design, sampling, multiplexing, multipath problem analysis, communication for multiple users, and system modeling.

Suppose that the signal $f(t)$ is composed of many time–frequency components

$$f(t) = f_1(t) + f_2(t) + f_3(t) + \cdots + f_N(t). \quad (42)$$

In many applications, we should decompose $f(t)$ into $f_n(t)$ ($n = 1, 2, \dots, N$), i.e., perform multiple component analysis. However, this is hardly achieved by the WDF. Due to cross terms, the WDFs of $f_1(t), f_2(t), \dots$, and $f_N(t)$ usually mix together. In contrast, when using the GT (or the GWT), without the interference of the cross term, multiple component analysis will be easier to do.

A. Fractional Filter Design

It is known that we can use FRFTs instead of FTs for filter design [1], [13], [14], [20]–[23], as follows:

$$r(t) = O_F^{-\alpha} \{O_F^{\alpha} [f(t)] H(u)\} \quad (43)$$

where $f(t)$ and $r(t)$ are the input and the output of the filter, and $H(u)$ is the transfer function. Equation (43) can be generalized into the multiple-output form, i.e., trying more than one α [20]–[22]:

$$\begin{aligned} f_1(t) &= O_F^{-\alpha_1} \{O_F^{\alpha_1} [f(t)] H_1(u)\} \\ f_2(t) &= O_F^{-\alpha_2} \{O_F^{\alpha_2} [f_1(t)] H_2(u)\}, \dots, \\ f_{n-1}(t) &= O_F^{-\alpha_{n-1}} \{O_F^{\alpha_{n-1}} [f_{n-2}(t)] H_{n-1}(u)\} \\ r(t) &= O_F^{-\alpha_n} \{O_F^{\alpha_n} [f_{n-1}(t)] H_n(u)\}. \end{aligned} \quad (44)$$

When using the FRFT for filter design, if α_n 's are chosen properly, many noises that are difficult to be removed by the filter designed by the FT will be eliminated successfully, especially for the noises whose instantaneous frequencies vary with time.

However, there are two important issues regarding the filter design in the FRFT domain:

- A) how to choose the parameter α properly;
- B) how to choose the cutoff criteria when designing the low-pass, high-pass, or bandpass filter.

Although the method for searching the optimal transfer function $H(u)$ has been developed when the original signal $f(t)$ is a random process and we have known some statistical properties of $f(t)$ [13], [14], there is no efficient way for determining the optimal order α . Although we can use an iterative method to determine α , it is cumbersome. Moreover, it is not suitable for the case where we should try more than one α to filter out the noises [i.e., the case in (44)]. In addition, we still need a way to determine cutoff criteria when $f(t)$ is deterministic and $H(u)$ is the form of low-pass, high-pass, or bandpass filter.

It seems that the WDF is helpful for determining α and the cutoff line. However, due to cross terms, using the WDF may not be suitable when $h(t)$ consists of several time–frequency components.

In this paper, we find that the GT and the GWT also have rotation relations with the FRFT. This hints that we can use them instead of the WDF for fractional filter design. Moreover, since the cross term can be avoided, the signals and the noises are easy to separate by the GT and the GWT. We can apply the following theorem.

Theorem 5: The cutoff line on the t – ω plane of the GT (or the GWT) is equivalent to the low-pass, high-pass, or bandpass filter in the FRFT domain. In fact, the following FRFT filter:

$$f_o(t) = O_F^{-\alpha} \{O_F^{\alpha} [f_i(t)] H(u)\}, \quad \text{where } H(u) = 1 \text{ for } u < \tau, \quad H(u) = 0 \text{ for } u > \tau \quad (45)$$

(or $H(u) = 0$ for $u < \tau$, $H(u) = 1$ for $u > \tau$, dependent on which part is the noise), is equivalent to the following cutoff line in the t – ω plane of the GT (or the GWT):

$$\mathbf{L} : \tau(\cos \alpha, \sin \alpha) + k(-\sin \alpha, \cos \alpha). \quad (46)$$

$$A_f(t, \omega) = \int_{-\infty}^{\infty} e^{-\tau^2/2 + (\cos \theta - j \sin \theta) B_2(r) \tau + B_3(r) + (\sin^2 \theta + j \sin \theta \cos \theta) B_2^2(r)/2} f(\tau) d\tau \quad (39)$$

After applying the FRFT filter, the time–frequency components on one side of $\mathbf{L}$ is removed, and those on the other side are preserved.

Theorem 5 provides an efficient way for designing the filter in the FRFT domain. With it, we can use the following process to design the FRFT filter by the GT (or the GWT).

Step 1) First, we perform the GT (or the GWT) for the received signal.

Step 2) Then, we determine the **cutoff lines** on the t – ω plane of the GT (or the GWT) that can well separate the noise region from the signal region. The cutoff lines can be determined by the method of “clustering” and “segmentation,” which are usually used in image processing [27]. We first use clustering to segment the t – ω plane into the signal regions, the noise regions, and the regions that are neither signal nor noise. Then, we try to find the straight lines that can well separate the noise regions from the signal regions. These straight lines are the cutoff lines we want. If the noise region is not adjacent to the signal region, the cutoff line can be placed between the two regions. If the noise region is adjacent to the signal region, the cutoff line should approximate to the boundaries of the two regions.

Step 3) Then, we determine the orders of the FRFT and the transfer functions by the cutoff lines.

- i) The **slope** of the cutoff line determines the order of the FRFT.
- ii) The **distance** between the cutoff line and the origin determines the criterion of the transfer function. See (45) and (46) in Theorem 5.

Step 4) Then, using (44) and applying the parameters determined by Step 3), we can filter the noises successfully by no more than $K + 1$ times of the FRFTs (K is the number of cutoff lines).

We give an example in Fig. 5. The input signal is

$$s(t) = 2 \cos(5t) \exp\left(\frac{-t^2}{10}\right) \quad [\text{show in Fig. 5(a)}]. \quad (47)$$

It is interfered by the following three noises:

$$n(t) = 0.5e^{j0.23t^2} + 0.5e^{j0.3t^2 + j8.5t} + 0.5e^{j0.46t^2 - j9.6t} \quad (48)$$

and $f(t) = s(t) + n(t)$ is plotted in Fig. 5(b). We want to recover $s(t)$ from $f(t)$ by the FRFT filters.

The WDFs of $f(t) = s(t) + n(t)$ are plotted in Fig. 5(c). In Fig. 5(c), the signals, the noises, and cross terms are mixed together. It is hard to know how to separate the signals from the noises after observing Fig. 5(c). In contrast, when we perform the GT, the cross-term problem can be avoided. In Fig. 5(d), the signals and the noises are separable. Moreover, we can use the GWT to further improve the clarity. The GWT [applying the definition in (30)] of $f(t)$ is shown in Fig. 5(e), which shows the signal parts and the noise parts are very clearly separable.

Fig. 5. Using the FRFT together with the GT for filter design. (a) Signal $s(t)$. (b) $f(t) = s(t) + \text{noise}$. (c) WDF of $f(t)$. (d) GT of $f(t)$. (e) GWT of $f(t)$. (f) Cutoff lines on the GWT. (g) FRFT of $f(t) = s(t) + n(t)$ ($\alpha = -1.1397$). (h) High-pass filter. (i) Applying high-pass filter for (g). (j) GWT of (g). (k) Cutoff lines corresponding to the high-pass filter. (l) GWT of (i). (m) Recovered signal. (n) Recovered signal (real part) and the original signal.

From Fig. 5(f), we can use the following four cutoff lines to separate the noises and the signals, as follows:

$$\begin{aligned} \mathbf{L1}: & -0.65(\cos \alpha, \sin \alpha) + k(-\sin \alpha, \cos \alpha), \quad \alpha = -1.1397, \\ & k \in (-\infty, \infty), \\ \mathbf{L2}: & 0.65(\cos \alpha, \sin \alpha) + k(-\sin \alpha, \cos \alpha), \\ \mathbf{L3}: & -6.8(\cos \beta, \sin \beta) + k(-\sin \beta, \cos \beta), \quad \beta = -1.0304, \\ \mathbf{L4}: & 6.75(\cos \gamma, \sin \gamma) + k(-\sin \gamma, \cos \gamma), \quad \gamma = -0.827. \end{aligned} \quad (49)$$

We can use them together with Theorem 5 to design the FRFT filters and remove the noises. For example, for cutoff lines $\mathbf{L1}$

and **L2**, $\alpha = -1.1397$. From (45) and (46), it means that we should perform the FRFT with order -1.1397 for $f(t)$. The result is shown in Fig. 5(g) and its GWT is shown in Fig. 5(j). Then, in (49), since the values of τ for **L1** and **L2** are -0.65 , and 0.65 , from (46), to remove the component between **L1** and **L2**, we must multiply Fig. 5(g) by the high-pass filter in Fig. 5(h) ($H(u) = 0$ for $-0.65 < u < 0.65$, $H(u) = 1$ otherwise). The result is shown in Fig. 5(i). The GWT of Fig. 5(i) is shown in Fig. 5(l), which shows that the noise between **L1** and **L2** is filtered successfully.

Therefore, from (49) and Theorem 5, we can use the following process to filter noises and recover the original signal $s(t)$ from $f(t)$:

- (a) $F_\alpha(u) = O_F^\alpha[f(t)]$, $\alpha = -1.14$
- (b) $F_{1,\alpha}(u) = F_\alpha(u)$ for $|u| \geq 0.65$
 $F_{1,\alpha}(u) = 0$ for $|u| < 0.65$
- (c) $F_{1,\beta}(u) = O_F^{\beta-\alpha}[F_{1,\alpha}(t)]$, $\beta = -1.03$
- (d) $F_{2,\beta}(u) = F_{1,\beta}(u)$ for $u \geq -6.8$
 $F_{2,\beta}(u) = 0$ for $u < -6.8$,
- (e) $F_{2,\gamma}(u) = O_F^{\gamma-\beta}[F_{2,\beta}(t)]$, $\gamma = -0.827$
- (f) $F_{3,\gamma}(u) = F_{2,\gamma}(u)$ for $u \geq 6.75$
 $F_{3,\gamma}(u) = 0$ for $u < 6.75$
- (g) $s(t) \approx O_F^\gamma[F_{3,\gamma}(u)]$. (50)

(a) and (b) come from **L1** and **L2**, (c) and (d) come from **L3**, and (e) and (f) come from **L4** in Fig. 5(f). The recovered signal is plotted in Fig. 5(m). It is very close to the original signal $s(t)$ [The recovered signal and $s(t)$ are plotted together in Fig. 5(n).] The mean-square error (MSE) is only

$$\text{MSE} = 0.1128\% \quad (51)$$

$$\text{where MSE} = \int |r(t) - s(t)|^2 dt \left(\int |s(t)|^2 dt \right)^{-1},$$

and $r(t)$: recovered signal. (52)

In comparison, when using the GT, the MSE is 0.2105%. When using the WDF, only two of the three chirp noises are possible to be removed. In Fig. 5(c), due to the cross term, the central chirp noise part cannot be detected and removed. We can only remove the outer two chirp noises, and the MSE is as high as 63.18%. Therefore, when we use the GT and the GWT for filter design, the performance gains (compared with the case of the WDF) are

$$\begin{aligned} \text{for Gabor transform : } & \text{performance gain} = 300.14 \\ \text{for the GWT : } & \text{performance gain} = 560.11. \end{aligned} \quad (53)$$

Thus, using the GT and the GWT for the filter design in the FRFT domain has the following two significant advantages.

- 1) *The design process becomes easier.* The orders of the FRFTs and the criteria can be easily determined by the cutoff lines and the iterative process that often used in the literature can be avoided.

- 2) *The ability of noise removing is much improved.* In addition to the examples in Fig. 5 and (47)–(53), we also perform many other experiments and show that the GT and the GWT always have higher ability than the WDF for filtering the noise.

In fact, we can conclude as in Theorem 6.

Theorem 6: Using the FT can only filter the noises that do not overlap with the signals **in the frequency domain**. In contrast, using the FRFT can filter the noises that do not overlap with the signals **on the t - ω plane of the GT or the GWT**.

Note that the frequency domain is 1-D but the t - ω plane is 2-D. **Two objects are easier to separate on a 2-D plane than on a 1-D axis.** Thus, with the aid of the GT or the GWT, many noises that cannot be removed by the conventional filter will be filtered by the FRFT successfully.

B. Fractional Sampling

In [15], Xia introduced a way that used the FRFT instead of the FT for signal sampling. When using FRFTs for signal sampling, we first try to find α such that the supporting of $|F_\alpha(u)|$ is minimal, as follows:

$$\begin{aligned} \text{supporting } \Omega_\alpha : & |F_\alpha(u)| \approx 0 \text{ if } u \notin \Omega_\alpha; \\ \text{optimal } \alpha : & \frac{\text{width}(\Omega_\alpha)}{|\sin \alpha|} \text{ is minimal.} \end{aligned} \quad (54)$$

Then, we can sample $f(t)$ as

$$x[n] = f(n\Delta) \text{ where } \Delta \leq \frac{2\pi}{B}, \quad B = \frac{\text{width}(\Omega_\alpha)}{|\sin \alpha|}. \quad (55)$$

To recover $f(t)$ from $x[n]$, we can apply

$$f(t) = e^{-(j/2)t^2 \cot \alpha} \sum_n x[n] e^{(j/2)n^2 \Delta^2 \cot \alpha} \cdot \text{sinc} \left(\frac{t}{\Delta - n} \right) e^{j(u_0 + B/2) \csc \alpha |t - n\Delta|}. \quad (56)$$

Some signals have narrower bandwidth in the FRFT domain than that in the frequency domain, as follows:

$$\frac{\text{width}(\Omega_\alpha)}{|\sin \alpha|} < \text{width} \left(\frac{\Omega_\pi}{2} \right). \quad (57)$$

In this case, if we use the FRFT with parameter α instead of the FT to do signal sampling, the sampling interval in (55) can be larger, which means that the number of sampling points can be reduced. Alternatively, if the number of sampling points remains the same, the approximating error is reduced.

When $f(t)$ has only one time–frequency (T–F) component, we can use the WDF to estimate the optimal value of α . However, when $f(t)$ has two or more T–F components, as Fig. 2(d), since the “orient” for each of the T–F component is different, it is proper to separate $f(t)$ into several T–F components and determine the optimal α for each of the components. Owing to cross terms, it is difficult for the WDF to achieve this. For example, in Fig. 2(d), it is difficult to conclude whether the central region around $(0, 0)$ is “the third T–F component” of $f(t)$ or just the cross term of the left and right parts.

In contrast, when using the GT, since there is no cross-term problem, we can easily determine the number of the time–fre-

Fig. 6. Using the FRFT together with time–frequency component decomposition by the GT to sample the signal $f(t)$ [see (13) and Fig. 2] in the FRFT domain.

quency components of $f(t)$ and decompose $f(t)$ into the summation of these components. For the example in Fig. 2, we can do the following.

- First, we decompose $G_f(t, \omega)$ in Fig. 2(c) into the left part (near to $s(t)$) and right part (near to $r(t)$).
- Then, we use the power integration relation (see Table I) to estimate the optimal order α [defined by (54)] for the left and right parts, which are

$$\alpha_{\text{left}} = -1.373, \quad \alpha_{\text{right}} = 0.785. \quad (58)$$

- Then, we can use the process in Fig. 6 to perform fractional sampling for $f(t)$. When the number of sampling points is constrained to 10, the reconstruction errors are as in (59) and (60), shown at the bottom of the page.

C. Modulation and Multiplexing in FRFT Domain

The conventional modulation and multiplexing are

$$g(t) = e^{j\omega_1 t} f_1(t) + e^{j\omega_2 t} f_2(t) + e^{j\omega_3 t} f_3(t) + \dots + e^{j\omega_N t} f_N(t). \quad (61)$$

In [7], it was generalized into the fractional modulation and multiplexing, which modulates the FRFT of $f_n(t)$ instead of $f_n(t)$:

$$F_n(u) = e^{j\omega_0 u} \tilde{F}_n(u - u_0) \text{ where } \tilde{F}_n(u) = O_F^{\alpha_n}[f_n(t)], \quad n = 1, 2, 3, \dots, N \quad (62)$$

$$G(u) = F_1(u) + F_2(u) + F_3(u) + \dots + F_N(u). \quad (63)$$

The conventional multiplexing consider only whether there is unfilled space in the frequency domain. However, when using the FRFT for multiplexing, if there is some unfilled slot in the t – ω plane, we can place a new signal in this slot and multiplex it with $G(u)$. Using the fractional multiplexing can improve the efficiency of time–frequency slot usage and has higher capacity for data transmission.

Fig. 7. Experiments using the FRFT together with the GWT for modulation and multiplexing. (a) $G(u)$, consisted of seven components. (b) WDF of $G(u)$. (c) GWT of $G(u)$. (d) GWT after $f(t)$ is multiplexed.

However, the cross-term problem makes it difficult to know which T–F slot is spared by the WDF. Thus, to perform signal modulation and multiplexing in the FRFT domain, it is better to use the GT or the GWT to find the spare T–F slots.

We show an example in Fig. 7. In Fig. 7(a), $G(u)$ consists of nine components. We want to multiplex the scaled Gaussian signal $f(t) = \exp(-0.01t^2)$ into $G(u)$. When using the WDF, as in Fig. 7(b), due to the interference of the cross term, it seems that there is no spare T–F slot for placing $f(t)$. When using the GWT, the spare slot can be detected, as the region surrounded by the dashed lines in Fig. 7(c). We then multiplex $f(t)$ into the unfilled T–F slot in Fig. 7(c) by shifting, modulation, and the FRFT:

$$\tilde{F}(u) = O_F^{1.335}[f(t)], \quad F(u) = e^{j0.5u} F_n(u + 11). \quad (64)$$

Remember that in the t – ω plane of the GWT, the FRFT has the effect of rotation, shifting has the effect of horizontal displacement, and modulation has the effect of vertical displacement. Then, we add $G(u)$ by $F(u)$. The GWT of the result of multiplexing is shown in Fig. 7(d). Note that, in Fig. 7(d), $f(t)$ is successfully placed into the unfilled T–F slot in Fig. 7(c).

D. Other Applications

The relations between the GT (or the GWT) and the FRFT are also useful for fractional matching pursuit [16], super-res-

- Conventional method: err = 2.763%

- Sampling in the FRFT domain: err = 1.306%

- Sampling in the FRFT domain and using the Gabor transform to do signal decomposition: err = 0.061%.

(59)

(60)

olution [17], fractional wavelet transform [18], data compression, and spread-spectrum signal analyzing in the FRFT domain, since these applications are closely related to the time–frequency character of a signal, and the character can be effectively analyzed by the GT or the GWT.

V. CONCLUSION

We have derived several interesting relations between the FRFT and the GT, including rotation relation, recovery relation, and power integration relation. Moreover, because the GT can avoid the cross term, which is a serious problem of the WDF, we can use the GT instead of the WDF to perform signal processing in the FRFT domain, such as fractional sampling and fractional filter design. In addition, we have also introduced the GWT, i.e., the combination of the GT and the WDF. The GWT combines the advantages of the GT (no cross term) and the WDF (high clarity) and also has rotation relation with the FRFT.

APPENDIX

Proof of Clockwise-Rotation Relation in Theorem 1:

$$\begin{aligned}
 G_{F_\alpha}(u, v) &= \frac{\sqrt{1-j\cot\alpha}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(\tau-u)^2/2} \\
 &\quad \cdot e^{-jv(\tau-(u/2))} e^{(j/2)\tau^2 \cot\alpha} e^{-j\tau x \csc\alpha} \\
 &\quad \cdot e^{(j/2)x^2 \cot\alpha} f(x) dx d\tau \\
 &\quad (\text{Applying (7) and (1)}) \\
 &= \frac{\sqrt{1-j\cot\alpha}}{2\pi} \\
 &\quad \cdot \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{\tau^2(-(1/2)+(j/2)\cot\alpha)} \right. \\
 &\quad \cdot e^{\tau(u-jv-jx \csc\alpha)} d\tau \left. \right] \\
 &\quad \cdot e^{-(u^2/2)+(juv/2)+(j/2)x^2 \cot\alpha} f(x) dx. \quad (65)
 \end{aligned}$$

Then, applying the fact that (from [12])

$$\int_{-\infty}^{\infty} e^{-(a\tau^2+b\tau)} d\tau = \sqrt{\frac{\pi}{a}} \cdot e^{b^2/4a} \quad (66)$$

we can rewrite (65) as (67), shown at the bottom of the page. ■

Proof of Theorem 2: From (28) and (8)

$$\begin{aligned}
 C_{F_\alpha}(u, v) &= p(G_{F_\alpha}(u, v), W_{F_\alpha}(u, v), q_1(r), q_2(r), \dots, q_N(r)) \\
 &= p\left(G_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha) \right. \\
 &\quad \left. W_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha) \right. \\
 &\quad \left. q_1(r), q_2(r), \dots, q_N(r)\right) \\
 &= C_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha).
 \end{aligned}$$

#

Proof of Recovery Relation in Table I: From (7) and $\int_{-\infty}^{\infty} e^{-j\omega(\tau-t)} d\omega = \sqrt{2\pi} \delta(\tau-t)$, we obtain

$$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G_f(t, \omega) e^{(j\omega t/2)} d\omega = f(t). \quad (68)$$

Then, we generalize it into the case of the FRFT. If we replace $f(t)$ by $F_\alpha(u)$, then

$$\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G_{F_\alpha}(u, v) e^{(juv/2)} dv = F_\alpha(u). \quad (69)$$

After applying (8), we obtain the recovery relation. ■

Proof of Power Integration Relation in Table I: If we integrate $|G_f(t, \omega)|^2$ along the ω axis, then

$$\begin{aligned}
 &\int_{-\infty}^{\infty} |G_f(t, \omega)|^2 d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-((\tau-t)^2/2)} e^{-((\rho-t)^2/2)} \\
 &\quad \cdot \left[\int_{-\infty}^{\infty} e^{-j\omega(\tau-(t/2))} e^{j\omega(\rho-(t/2))} d\omega \right] f(\tau) f^*(\rho) d\tau d\rho \\
 &\quad (70)
 \end{aligned}$$

$$\begin{aligned}
 G_{F_\alpha}(u, v) &= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{((-u+jv+jx \csc\alpha)^2/2(1-j\cot\alpha))} \\
 &\quad \cdot e^{-(u^2/2)+(juv/2)+(j/2)x^2 \cot\alpha} f(x) dx \\
 G_{F_\alpha}(u, v) &= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \\
 &\quad \cdot e^{((u^2-v^2-2vx \csc\alpha-x^2 \csc^2\alpha-j2uv-j2ux \csc\alpha)/2(1-j\cot\alpha)) \cdot ((1+j\cot\alpha)/(1+j\cot\alpha))} \\
 &\quad \cdot e^{-(u^2/2)+(juv/2)+(j/2)x^2 \cot\alpha} f(x) dx \\
 &= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-((u \cos \alpha - \omega \sin \alpha)^2/2) + ux \cos \alpha - vx \sin \alpha - (x^2/2)} \\
 &\quad \cdot e^{j(uv(\cos^2 - \sin^2)/2) + (j(u^2 - v^2) \sin \alpha \cos \alpha/2) - jx(u \sin \alpha + \omega \cos \alpha)} dx \\
 &= G_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha). \quad (67)
 \end{aligned}$$

$$e^{k_1(t \cos \alpha - \omega \sin \alpha, t \sin \alpha + \omega \cos \alpha) \tau^2 (t \cos \alpha - \omega \sin \alpha, t \sin \alpha + \omega \cos \alpha) \tau + k_3(t \cos \alpha - \omega \sin \alpha, t \sin \alpha + \omega \cos \alpha)} \\ = \sqrt{\frac{1 - j \cot \alpha}{2\pi}} \sqrt{\frac{\pi}{\frac{-k_1(t, \omega) - j \cot \alpha}{2}}} e^{-(j\tau \csc \alpha - k_2(t, \omega))^2 / (4k_1(t, \omega) + j2 \cot \alpha)} e^{j\tau^2 \cot \alpha + k_3(t, \omega)/2}. \quad (74)$$

$$e^{-\tau^2/2 + k_2(t \cos \alpha - \omega \sin \alpha, t \sin \alpha + \omega \cos \alpha) \tau + k_3(t \cos \alpha - \omega \sin \alpha, t \sin \alpha + \omega \cos \alpha)} \\ = e^{(j\tau \csc \alpha - k_2(t, \omega))^2 / (2 - j2 \cot \alpha)} e^{j\tau^2 \cot \alpha / 2 + k_3(t, \omega)} \\ = e^{-\tau^2/2} e^{(-j \sin \alpha + \cos \alpha) k_2(t, \omega) \tau} e^{k_2^2(t, \omega) \sin \alpha (\sin \alpha + j \cos \alpha) / 2 + k_3(t, \omega)}. \quad (75)$$

$$\int_{-\infty}^{\infty} |G_f(t, \omega)|^2 d\omega \\ = \int_{-\infty}^{\infty} e^{-(\tau-t)^2/2} f(\tau) e^{-(\tau-t)^2/2} f^*(\tau) d\tau \\ = \int_{-\infty}^{\infty} e^{-(\tau-t)^2} |f(\tau)|^2 d\tau. \quad (71)$$

After replacing $f(t)$ by $F_\alpha(u)$ and applying (8) and (14), we obtain the power integration relation. ■

Proof of Power-Decayed Relation: From the fact that

$$(u - \tau)^2 > (u - u_0)^2 + (u_0 - \tau)^2 \text{ if } \tau \leq u_0 < u \quad (72)$$

we obtain

$$\int_{-\infty}^{\infty} e^{-(\tau-u)^2} |F_\alpha(\tau)|^2 d\tau \\ < e^{-(u-u_0)^2} \int_{-\infty}^{\infty} e^{-(\tau-u_0)^2} |F_\alpha(\tau)|^2 d\tau \text{ when } u > u_0. \quad (73)$$

Then, together with the power integration relation in Table I, we obtain

$$\int_{-\infty}^{\infty} |G_f(u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha)|^2 dv < e^{-(u-u_0)^2} \\ \times \int_{-\infty}^{\infty} |G_f(u_0 \cos \alpha - v \sin \alpha, u_0 \sin \alpha + v \cos \alpha)|^2 dv.$$

It is equivalent to the power-decayed relation in Table I. ■

Proof of Theorem 4: From (38), (37) can be rewritten as (74), shown at the top of the page. From (74), to eliminate the two square root terms, $k_1(t, \omega)$ should be $-1/2$. Then, (74) is simplified as (75), shown at the top of the page. Comparing the coefficients of τ in (75), we obtain

$$k_2(t \cos \alpha - \omega \sin \alpha, t \sin \alpha + \omega \cos \alpha) \\ = (-j \sin \alpha + \cos \alpha) k_2(t, \omega). \\ \text{i.e., } k_2(r, \theta) = e^{-j\theta} B_2(r), \text{ where } r = \sqrt{t^2 + \omega^2}, \\ \theta = \arg(t + j\omega). \quad (76)$$

Moreover, comparing the constant terms in (75), we obtain

$$k_3(t \cos \alpha - \omega \sin \alpha, t \sin \alpha + \omega \cos \alpha) \\ = \frac{k_2^2(t, \omega) \sin \alpha (\sin \alpha + j \cos \alpha)}{2} + k_3(t, \omega) \quad (77)$$

$$k_3(r, \theta + \alpha) \\ = k_3(r, \theta) + (\sin^2 \alpha + j \sin \alpha \cos \alpha) e^{-j2\theta} \frac{B_2^2(r)}{2}. \quad (78)$$

Thus, we can define $k_3(r, \theta)$ as the following rotation symmetry function:

$$k_3(r, \theta) = B_3(r) + (\sin^2 \theta + j \sin \theta \cos \theta) \frac{B_2^2(r)}{2}. \quad (79)$$

We can show that, no matter how we choose $B_2(r)$ and $B_3(r)$ as, (78) will be satisfied for any r, θ , and α . Therefore, we obtain Theorem 4. ■

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